These equations are used in the design of a VHF aerial feeder system. The x_j are the lengths associated with the coaxial line connectors and the β_i are constants dependent on the carrier frequency. Of the many techniques tried, the mixed method was found to be the most efficient. Equations (5.3) were solved for $n = 6$. Parameter a was introduced such that

$$\beta_i = \beta_0 + a(\Delta\beta_i) \quad a^0 = 0, \quad a^f = 1$$

where β_0 and $\Delta\beta_i$ are constants.

For comparison, the solutions furnished by the method of Fletcher and Powell (1963) are included. Both sets of results are shown in **Table 2**. The same starting vectors were used. The mixed method was about 6 times faster than Fletcher and Powell's method. This seems to indicate that a method which has access to each residual independently will be more efficient than one which minimizes the sum of squared residuals.

References

DAVIDENKO, D. F. (1953). "On a new method of numerical solution of systems of non-linear equations", *Mathematical Reviews*, Vol. 14, p. 906.

FLETCHER, R., and POWELL, M. J. D. (1963). "A rapidly convergent descent method for minimization", *The Computer Journal*, Vol. 6, p. 163.

RALSTON, A., and WILF, S. (1960). *Mathematical Methods for Digital Computers*, New York: Wiley.

An iterative method for locating turning points

By P. Jarratt*

A method for calculating turning points is given which is shown to possess superlinear convergence. The iterative formula is applied to a numerical example and the problem of accelerating convergence is discussed.

1. Introduction

The problem of computing a value θ for which a function f has a turning point occurs frequently in scientific work and is usually solved by applying an appropriate root-finder to the derivative f'. In many cases of practical interest, however, an analytic form for f' is unavailable or difficult to obtain and alternative techniques must therefore be sought. One method which suggests itself is to compute new approximations to θ by the use of a polynomial which interpolates f. Thus let $x_i, x_{i-1}, \ldots, x_{i-n}$ be $n + 1$ approximations to a turning point θ of f and let $P_n(t)$ be the interpolatory polynomial of degree n such that

$$P_n(x_{i-j}) = f(x_{i-j}), \quad j = 0, 1, \ldots, n.$$

Define a new approximation to θ by

$$P_n'(x_{i+1}) = 0, \tag{1.1}$$

and then repeat the procedure for $x_{i+1}, x_i \ldots, x_{i-n+1}$, and so on. It is clear that this approach presents a number of problems. Firstly, since (1.1) is a polynomial of degree $n - 1$, a polynomial equation must be solved at each step of the iteration, and additionally x_{i+1} will not in general be uniquely specified. Some rule must therefore be formulated whereby x_{i+1} is chosen uniquely as one of the zeros of the polynomial. Secondly, it is not even certain that (1.1) has a real root in the region of θ. These objections can be met, however, if we restrict ourselves to a formulation in which (1.1) is linear in x_{i+1}, corresponding to interpolation by the quadratic $P_2(t)$. In this paper the properties of the corresponding iteration function are investigated and its behaviour is illustrated by a numerical example.

2. Formulation

Following the previous discussion, we fit the quadratic

$$y = a + bt + ct^2 \tag{2.1}$$

to three points (x_{i-j}, f_{i-j}), $j = 0, 1, 2$, and then predict x_{i+1} by imposing the condition $y'_{i+1} = 0$. This leads to the system

$$\begin{aligned} f_{i-j} &= a + bx_{i-j} + cx_{i-j}^2, \quad j = 0, 1, 2 \\ 0 &= \quad\; b \quad + 2cx_{i+1}, \end{aligned} \tag{2.2}$$

and these four equations in the three parameters a, b, c, will be consistent provided that the determinantal condition

$$\begin{vmatrix} 2x_{i+1} & 1 & 0 & 0 \\ x_i^2 & x_i & 1 & f_i \\ x_{i-1}^2 & x_{i-1} & 1 & f_{i-1} \\ x_{i-2}^2 & x_{i-2} & 1 & f_{i-2} \end{vmatrix} = 0, \tag{2.3}$$

is satisfied. We now use (2.3) to examine the convergence of the method. First we define the errors in the

* *University of Bradford, Bradford, Yorks.*

approximations by $x_{i-j+1} = \epsilon_{i-j+1} + \theta$, $j = 0, 1, 2, 3$, and substitution in (2.3) and simplification gives

$$\begin{vmatrix} 2\epsilon_{i+1} & 1 & 0 & 0 \\ \epsilon_i^2 & \epsilon_i & 1 & f_i \\ \epsilon_{i-1}^2 & \epsilon_{i-1} & 1 & f_{i-1} \\ \epsilon_{i-2}^2 & \epsilon_{i-2} & 1 & f_{i-2} \end{vmatrix} = 0. \quad (2.4)$$

ϵ_{i+1} may be obtained from (2.4) as the ratio of two determinants and, using an obvious compact notation, we have

$$\epsilon_{i+1} = \frac{1}{2} \frac{|\epsilon^2, \ 1, \ f|_{i,i-2}}{|\epsilon, \ 1, \ f|_{i,i-2}}. \quad (2.5)$$

We must now expand f in order to develop (2.5) further.

Accordingly we write

$$f_{i-j} = \sum_{r=0}^{\infty} c_r \epsilon_{i-j}^r, \ j = 0, 1, 2 \quad (2.6)$$

where $c_r = f^{(r)}(\theta)/r!$ and we note that $c_0 = f(\theta) = M$, the maximum or minimum value, and $c_1 = f'(\theta) = 0$. Using (2.6) we can now expand the numerator and denominator of (2.5) as infinite series of determinants. Thus

$$|\epsilon^2, 1, f|_{i,i-2} = |\epsilon^2, 1, \sum_{r=0}^{\infty} c_r \epsilon^r|_{i,i-2}$$

$$= \sum_{j=3}^{\infty} c_j |\epsilon^2, 1, \epsilon^j|_{i,i-2} \quad (2.7)$$

since the first three determinants of the expansion all vanish. Similarly for the denominator we find

$$|\epsilon, 1, f|_{i,i-2} = \sum_{j=2}^{\infty} c_j |\epsilon, 1, \epsilon^j|_{i,i-2}. \quad (2.8)$$

We now write $\hat{\epsilon}_{i,i-2} = \text{Max}\{|\epsilon_i|, |\epsilon_{i-1}|, |\epsilon_{i-2}|\}$ and noting that $|\epsilon, 1, \epsilon^2|_{i,i-2}$ is a factor of each term of (2.7) we obtain

$$|\epsilon^2, 1, f|_{i,i-2} = |\epsilon, 1, \epsilon^2|_{i,i-2} \times \{c_3(\epsilon_i\epsilon_{i-1} + \epsilon_i\epsilon_{i-2} + \epsilon_{i-1}\epsilon_{i-2}) + O(\hat{\epsilon}_{i,i-2}^2)\}. \quad (2.9)$$

For the denominator we have similarly

$$|\epsilon, 1, f|_{i,i-2} = |\epsilon, 1, \epsilon^2|_{i,i-2}\{c_2 + O(\hat{\epsilon}_{i,i-2})\}, \quad (2.10)$$

assuming $c_2 \neq 0$, and combining (2.5), (2.9) and (2.10) results in

$$\epsilon_{i+1} = \frac{c_3}{2c_2}(\epsilon_i\epsilon_{i-1} + \epsilon_i\epsilon_{i-2} + \epsilon_{i-1}\epsilon_{i-2}) + O(\hat{\epsilon}_{i,i-2}^2). \quad (2.11)$$

It is clear from (2.11) that convergence will be assured provided the initial values x_0, x_1 and x_2 are chosen sufficiently close to θ. In order to examine the asymptotic behaviour of (2.11) we neglect $O(\hat{\epsilon}_{i,i-2}^2)$ compared with $O(\hat{\epsilon}_{i,i-2})$ and we assume $\frac{\epsilon_{i+1}}{\epsilon_i} \to 0$, which implies $\epsilon_{i+1} \to 0$. We can now write (2.11) as

$$\epsilon_{i+1} = \frac{c_3}{2c_2} \epsilon_{i-1}\epsilon_{i-2}\left(1 + \frac{\epsilon_i}{\epsilon_{i-2}} + \frac{\epsilon_i}{\epsilon_{i-1}}\right)$$

and hence asymptotically

$$\epsilon_{i+1} \sim \frac{c_3}{2c_2} \epsilon_{i-1}\epsilon_{i-2}. \quad (2.12)$$

(2.12) may be linearized by taking logarithms of both sides. The resulting linear equation has the solution $\eta_i = Ad_1^i + Bd_2^i + Cd_3^i - \log \frac{c_3}{2c_2}$ where $\eta_i = \log |\epsilon_i|$ and d_1, d_2 and d_3 are the roots of the equation $d^3 - d - 1 = 0$. We find $d_1 = 1{\cdot}325$, d_2, $d_3 = -0{\cdot}6624 \pm 0{\cdot}5623i$ and hence for large i, $\eta_i \sim Ad_1^i - \log K$ where $K = c_3/2c_2$. It is easy to show from this that $\epsilon_{i+1} \sim K^{d_1-1}\epsilon_i^{d_1}$ and we therefore have a process of order d_1. We see that asymptotic convergence, although not rapid, is appreciably better than first order.

3. The iterative formula

In order to obtain the numerically most accurate representation of the iterative formula, we change our origin to x_i by writing $\phi = t - x_i$ and fit the parabola $y = \alpha + \beta\phi + \gamma\phi^2$ through the three points (ϕ_{i-j}, f_{i-j}), $j = 0, 1, 2$. The consistency condition corresponding to (2.3) gives

$$\begin{vmatrix} 2\phi_{i+1} & 1 & 0 & 0 \\ 0 & 0 & 1 & f_i \\ \phi_{i-1}^2 & \phi_{i-1} & 1 & f_{i-1} \\ \phi_{i-2}^2 & \phi_{i-2} & 1 & f_{i-2} \end{vmatrix} = 0$$

and we may solve for ϕ_{i+1}, the result being

$$\phi_{i+1} = \frac{1}{2}\left(\frac{\phi_{i-1}^2(f_i - f_{i-2}) + \phi_{i-2}^2(f_{i-1} - f_i)}{\phi_{i-1}(f_i - f_{i-2}) + \phi_{i-2}(f_{i-1} - f_i)}\right). \quad (3.1)$$

Equation (3.1) can be written in terms of the original co-ordinates as

$$x_{i+1} = x_i + \frac{1}{2}\left[\frac{(x_{i-1} - x_i)^2(f_i - f_{i-2}) + (x_{i-2} - x_i)^2(f_{i-1} - f_i)}{(x_{i-1} - x_i)(f_i - f_{i-2}) + (x_{i-2} - x_i)(f_{i-1} - f_i)}\right]. \quad (3.2)$$

It is interesting to note that if the estimates straddle the turning point in such a way that $f_i = f_{i-1}$, then (3.2) predicts the next approximation x_{i+1} at the mean of x_i and x_{i-1}. A similar behaviour occurs if $f_i = f_{i-2}$.

4. Numerical illustration

We now give a numerical example which serves the dual purpose of checking the theoretical prediction and also providing an illustration of the iterative method in practice. It is sufficient for our purposes to use a simple

Table 1

i	x_i	$\epsilon_{i+1}/\epsilon_{i-1}\epsilon_{i-2}$	$\epsilon_{i+1}/(\epsilon_i\epsilon_{i-1} + \epsilon_i\epsilon_{i-2} + \epsilon_{i-1}\epsilon_{i-2})$
0	2·0		
1	1·0		
2	0·5		
3	0·5162	0·2581	0·1475
4	0·2681	0·5362	0·2103
5	0·1366	0·5291	0·2574
6	$0{\cdot}6978 \times 10^{-1}$	0·5042	0·2842
7	$0{\cdot}2053 \times 10^{-1}$	0·5607	0·3166
8	$0{\cdot}4547 \times 10^{-2}$	0·4772	0·3303
9	$0{\cdot}6154 \times 10^{-3}$	0·4296	0·3339
10	$0{\cdot}3627 \times 10^{-4}$	0·3885	0·3333
11	$0{\cdot}9435 \times 10^{-6}$	0·3372	0·3333

function and **Table 1** shows the location at $x = 0$ of the minimum value of the polynomial $y=3x^4+4x^3+6x^2+8$ using the initial estimates $x_0 = 2$, $x_1 = 1$, $x_2 = 0{\cdot}5$. For this function, the asymptotic error constant, $c_3/2c_2$, has the value 1/3 and the approach of $\epsilon_{i+1}/\epsilon_{i-1}\epsilon_{i-2}$ to this figure can be noted from column 3 of the table. The behaviour of the more accurate error relation (2.11) is shown in column 4.

5. Acceleration of convergence

In cases where the evaluation of f is lengthy, it may be worth-while to accelerate convergence by applying an appropriate device to the sequence of iterates x_i. This will be particularly useful in cases where high accuracy is required. However, since asymptotic convergence is not geometric but superlinear, there is nothing to be gained from using Aitken's δ^2 method and a more suitable technique must be devised from the error relations (2.11) or (2.12). Thus, eliminating K from two applications of $\epsilon_{i+1} = K\epsilon_{i-1}\epsilon_{i-2}$, substituting $x_j - \theta$ for ϵ_j, $j = i + 1$, i, $i - 1$, $i - 3$, and solving for θ, we find

$$\theta = \frac{x_{i+1}x_{i-3} - x_i x_{i-1}}{x_{i+1} - x_i - x_{i-1} + x_{i-3}},$$

which may be written more suitably for numerical work as

$$\theta = x_{i+1} - \frac{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})}{x_{i+1} - x_i - x_{i-1} + x_{i-3}}. \quad (5.1)$$

In practice (5.1) will not give the limit exactly, as (2.12) will be only approximately true. Hence (5.1) can be thought of as defining an $\bar{x}$ which replaces x_{i+1}, the normal iteration then proceeding as before with the next acceleration occurring as soon as $\bar{x}$ has been discarded.

A second formula for accelerating convergence may be obtained by eliminating K from

$$\epsilon_{i+1} = K(\epsilon_i\epsilon_{i-1} + \epsilon_i\epsilon_{i-2} + \epsilon_{i-1}\epsilon_{i-2}),$$

and this approach is probably preferable since (2.11) will be applicable at an earlier stage in the iteration. In order to obtain an incremental formula of the same type as (5.1), we define $\theta = x_{i+1} + \delta$, $\phi_j = x_j - x_{i+1}$, $j = i + 1$, i, $i - 1$, $i - 2$, $i - 3$ and by proceeding as before we find that δ is given by the solution of the quadratic equation $p\delta^2 + q\delta + r = 0$ where

$$p = 5\phi_i - 2\phi_{i-3},\ q = (\phi_{i-3} - 3\phi_i)(\phi_{i-1} + \phi_{i-2})$$

and $r = \phi_i(\phi_i\phi_{i-1} + \phi_i\phi_{i-2} + \phi_{i-1}\phi_{i-2})$.

Since we require the root of smaller modulus, severe cancellation is bound to occur if we use

$$\delta = \frac{-q + (q^2 - 4pr)^{1/2}}{2p}$$

and for computational purposes it is better to rewrite this as

$$\delta = \frac{-2r}{q + (q^2 - 4pr)^{1/2}}.$$

This last technique was applied to the calculation shown in Table 1 after the tenth iteration and gave the value $\theta = 0{\cdot}5425 \times 10^{-7}$. The improvement in this case is clear.

The iterative technique described in this paper has been tested on a good number of practical problems and has been found to work extremely well. It should prove valuable in any problem where f' is difficult to obtain.

Erratum

An extension of block design methods and an application in the construction of redundant fault reducing circuits for computers by R. J. Ord-Smith, University of Bradford, this *Journal*, Vol. 8, No. 1, April 1965.

There are a few errors in the above paper. Claims made for Table 5 are false. It and reference to it should be omitted. A second design mentioned in 2.2. does not constitute another automorphic design as stated. The geometrical analogy of this section should refer to a projective plane of order 2.