Table 1

i	x_i	$\epsilon_{i+1}/\epsilon_{i-1}\epsilon_{i-2}$	$\epsilon_{i+1}/(\epsilon_i\epsilon_{i-1} + \epsilon_i\epsilon_{i-2} + \epsilon_{i-1}\epsilon_{i-2})$
0	2·0		
1	1·0		
2	0·5		
3	0·5162	0·2581	0·1475
4	0·2681	0·5362	0·2103
5	0·1366	0·5291	0·2574
6	$0\cdot6978\times10^{-1}$	0·5042	0·2842
7	$0\cdot2053\times10^{-1}$	0·5607	0·3166
8	$0\cdot4547\times10^{-2}$	0·4772	0·3303
9	$0\cdot6154\times10^{-3}$	0·4296	0·3339
10	$0\cdot3627\times10^{-4}$	0·3885	0·3333
11	$0\cdot9435\times10^{-6}$	0·3372	0·3333

function and **Table 1** shows the location at $x = 0$ of the minimum value of the polynomial $y=3x^4+4x^3+6x^2+8$ using the initial estimates $x_0 = 2$, $x_1 = 1$, $x_2 = 0\cdot5$. For this function, the asymptotic error constant, $c_3/2c_2$, has the value 1/3 and the approach of $\epsilon_{i+1}/\epsilon_{i-1}\epsilon_{i-2}$ to this figure can be noted from column 3 of the table. The behaviour of the more accurate error relation (2.11) is shown in column 4.

5. Acceleration of convergence

In cases where the evaluation of f is lengthy, it may be worth-while to accelerate convergence by applying an appropriate device to the sequence of iterates x_i. This will be particularly useful in cases where high accuracy is required. However, since asymptotic convergence is not geometric but superlinear, there is nothing to be gained from using Aitken's δ^2 method and a more suitable technique must be devised from the error relations (2.11) or (2.12). Thus, eliminating K from two applications of $\epsilon_{i+1} = K\epsilon_{i-1}\epsilon_{i-2}$, substituting $x_j - \theta$ for ϵ_j, $j = i+1, i, i-1, i-3$, and solving for θ, we find

$$\theta = \frac{x_{i+1}x_{i-3} - x_i x_{i-1}}{x_{i+1} - x_i - x_{i-1} + x_{i-3}},$$

which may be written more suitably for numerical work as

$$\theta = x_{i+1} - \frac{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})}{x_{i+1} - x_i - x_{i-1} + x_{i-3}}. \quad (5.1)$$

In practice (5.1) will not give the limit exactly, as (2.12) will be only approximately true. Hence (5.1) can be thought of as defining an $\bar{x}$ which replaces x_{i+1}, the normal iteration then proceeding as before with the next acceleration occurring as soon as $\bar{x}$ has been discarded.

A second formula for accelerating convergence may be obtained by eliminating K from

$$\epsilon_{i+1} = K(\epsilon_i\epsilon_{i-1} + \epsilon_i\epsilon_{i-2} + \epsilon_{i-1}\epsilon_{i-2}),$$

and this approach is probably preferable since (2.11) will be applicable at an earlier stage in the iteration. In order to obtain an incremental formula of the same type as (5.1), we define $\theta = x_{i+1} + \delta$, $\phi_j = x_j - x_{i+1}$, $j = i+1, i, i-1, i-2, i-3$ and by proceeding as before we find that δ is given by the solution of the quadratic equation $p\delta^2 + q\delta + r = 0$ where

$$p = 5\phi_i - 2\phi_{i-3},\ q = (\phi_{i-3} - 3\phi_i)(\phi_{i-1} + \phi_{i-2})$$

and $r = \phi_i(\phi_i\phi_{i-1} + \phi_i\phi_{i-2} + \phi_{i-1}\phi_{i-2})$.

Since we require the root of smaller modulus, severe cancellation is bound to occur if we use

$$\delta = \frac{-q + (q^2 - 4pr)^{1/2}}{2p}$$

and for computational purposes it is better to rewrite this as

$$\delta = \frac{-2r}{q + (q^2 - 4pr)^{1/2}}.$$

This last technique was applied to the calculation shown in Table 1 after the tenth iteration and gave the value $\theta = 0\cdot5425 \times 10^{-7}$. The improvement in this case is clear.

The iterative technique described in this paper has been tested on a good number of practical problems and has been found to work extremely well. It should prove valuable in any problem where f' is difficult to obtain.

Erratum

An extension of block design methods and an application in the construction of redundant fault reducing circuits for computers by R. J. Ord-Smith, University of Bradford, this *Journal*, Vol. 8, No. 1, April 1965.

There are a few errors in the above paper. Claims made for Table 5 are false. It and reference to it should be omitted. A second design mentioned in 2.2. does not constitute another automorphic design as stated. The geometrical analogy of this section should refer to a projective plane of order 2.