

On the best linear Chebyshev approximation

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The principal part of this paper is devoted to reworking the equivalence of the Stiefel exchange algorithm for Chebyshev approximation with the simplex algorithm applied to the dual of the linear programming formulation of the same problem. Our main concern has been the provision of algorithms free from the major restrictions of classical approximation theory, and it appears that these restrictions can be relaxed almost entirely.

1. Introduction

The aim of this paper is to discover how the restrictions of classical linear approximation theory affect techniques for computing best approximations to functions. We give an account of this classical theory with a view to underlining the main assumptions and to presenting a unified approach to both discrete and continuous problems. Also we rework the equivalence of the Stiefel exchange algorithm which was developed largely within the framework of classical approximation theory with an algorithm based on a linear programming approach. The point here is that the linear programming approach should be free from the restrictions usually imposed on the Stiefel exchange algorithm so that it should provide a useful tool for gauging the limitations of the classical theory.

The main conclusion is that the restrictions of the classical theory can be relaxed almost entirely. Specifically, algorithms can be provided which give a best approximation in very general circumstances. However, the classical assumptions cannot be weakened without permitting the possibility of the non-uniqueness of the best approximation (see, for example, Cheney (1966), pp. 80–82).

We mention some details concerning notation. We write $\rho_i(A)$ and $\kappa_j(A)$ to refer to the i th row and the j th column respectively of the matrix A . The unit vector e_j is defined as usual to be a vector with 1 in the j th place and zeros elsewhere. We write e for the vector each element of which is 1. The appropriate dimension of a vector should be clear from the context. We will have some occasion to use partitioned vectors—for example $\begin{bmatrix} \lambda \\ \tau \end{bmatrix}$ is a column vector λ extended by a scalar τ . A similar notation will be used for row vectors—for example $\begin{bmatrix} \lambda \\ \tau \end{bmatrix}^T$ is written $[\lambda^T, \tau]$.

2. A survey of the classical theory

The classical results concerning the minimax solution of n linear equations in $p(<n)$ variables (also called the discrete T problem) are based on the assumption that any $p \times p$ submatrix of the set of equations is non-singular. This assumption is usually called the *Haar condition*.

To explain the significance of this condition it is convenient to introduce first the following definitions.

1. Any set of $p + 1$ equations is called a *reference*. The corresponding submatrix is written A_p .
2. By the Haar condition the rank of A_p is p . Therefore there is a unique vector (up to a scalar multiplier) satisfying the equation

$$\lambda^T A_p = 0. \quad (2.1)$$

This vector is called the λ -vector. Note that all components of λ are different from zero, if the Haar condition is satisfied.

3. Let the original set of equations be written

$$A x = b - r. \quad (2.2)$$

Then r is called the residual vector. The components of b and r associated with a reference form vectors which are written b_p and r_p respectively.

4. The vector x is called a *reference vector* if either

$$\text{sgn}(r_p)_i = \text{sgn}(\lambda_i), \quad i = 1, \dots, p + 1,$$

$$\text{or} \quad \text{sgn}(r_p)_i = -\text{sgn}(\lambda_i), \quad i = 1, \dots, p + 1.$$

5. Let g be the vector defined by $g_i = \text{sgn}(\lambda_i)$. Then

the matrix $(A_p | g)$ is nonsingular so that the vector $\begin{bmatrix} x \\ h \end{bmatrix}$ is uniquely defined by the equations

$$A_p x = b_p - h g. \quad (2.3)$$

In this case x is called the *levelled reference vector* and h is called the *reference deviation*.

It may be noted that

$$\lambda^T b_p = h \sum_{i=1}^{p+1} |\lambda_i| \quad (2.4)$$

and that for any reference vector

$$0 = \lambda^T b_p \pm \sum_{i=1}^{p+1} |\lambda_i| |(r_p)_i| \quad (2.5)$$

so that $|h|$ lies between the greatest and the least of the $|(r_p)_i|$. Thus the levelled reference vector solves the discrete T problem for the given reference. From this observation the following theorem follows readily.

Fundamental Theorem (De la Vallée Poussin). The minimax solution to equation (2.2)—that is the solution for which $\max |r_i|$ is a minimum—is a levelled reference

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vector for some reference. Further, $\max |r_i| = |h|$ where h is the reference deviation for this reference.

The Haar condition is sufficient for the validity of this theorem. It is also sufficient for the validity of the following result which provides a basis for the computation of the minimax solution.

Exchange Theorem (Stiefel (1959)). Given any reference and a corresponding reference vector then it is possible to add to the reference any other equation and to drop an appropriate equation from the reference so that the given vector is also a reference vector for the new reference. A proof of this result will be given in the next section.

In an actual computation the chosen vector would be the levelled reference vector for the given reference, and the equation to be added would be that associated with the component of maximum modulus of r . If this equation is in the reference then the computation is completed. It is readily shown that the magnitude of the reference deviation rises monotonically. Let the indices of the equations in the reference be $\sigma_1, \sigma_2, \dots, \sigma_{p+1}$, let j be the index of the equation to be added, and σ_j the index of the equation to be dropped, then, by equations (2.4) and (2.5),

$$|h^{(n)}| = \frac{\sum_{i \neq j} |\lambda_i^{(n)}| |h^{(o)}| + |\lambda_j^{(n)}| |r_j|}{\sum_{i=1}^p |\lambda_i^{(n)}|} \quad (2.6)$$

$$> |h^{(o)}|$$

provided $|r_j| > |h^{(o)}|$. The superfixes n and o refer to the new and old references respectively. As there are only a finite number of references the solution is found in a finite number of steps. This algorithm is called the *Stiefel exchange algorithm*.

Closely related to the discrete T problem is the continuous T problem. In this case it is required to find numbers x_1, x_2, \dots, x_p such that the maximum deviation in the approximation to the continuous functions $f(z)$

by linear compounds of the form $\sum_{i=1}^p x_i \phi_i(z)$ is a minimum in $a \leq z \leq b$. Again the classical results are based on a special assumption. Here it is that no linear combination of the $\phi_i(z)$ has more than $(p-1)$ zeros on $[a, b]$. In this case the $\phi_i(z)$ are said to form a Chebyshev set.

In the continuous T problem it is convenient to define a *reference set* (z_i) to be any set of $(p+1)$ points $z_i, a \leq z_1 < z_2 < \dots < z_{p+1} \leq b$. On this reference set the discrete T problem

$$\sum_{i=1}^p x_i \phi_i(z_s) = f(z_s) - r(z_s)$$

$$s = 1, 2, \dots, p+1 \quad (2.7)$$

can be defined, and the matrix of this problem forms a reference. Two important results can now be derived in the case that the $\phi_i(z)$ form a Chebyshev set.

Lemma 2.1. The reference determined by equation (2.7) satisfies the Haar condition.

This is immediate, for if any $p \times p$ minor is singular then there is a linear combination of the ϕ_i which vanishes at p points z_s .

Remark. This is equivalent to the possibility of constructing an interpolation to $f(z)$ by linear combinations of the $\phi_i(z)$ on any set of p distinct points in $[a, b]$.

Lemma 2.2. The components of the λ -vector alternate in sign for any reference based on a reference set (z_i) .

Proof. Let $\phi(z) = \sum_{i=1}^p x_i \phi_i(z)$ vanish at the points $z_1, z_2, \dots, z_{s-1}, z_{s+2}, \dots, z_{p+1}$. Such a function always exists. Then

$$\sum_{j=1}^{p+1} \lambda_j \phi(z_j) = 0$$

by definition of the λ -vector,

$$= \lambda_s \phi(z_s) + \lambda_{s+1} \phi(z_{s+1})$$

by the construction of ϕ .

Now $\phi(z_s)$ and $\phi(z_{s+1})$ have the same sign as otherwise ϕ would vanish between them giving a linear combination of the ϕ_i with at least p zeros. Thus λ_s and λ_{s+1} have opposite sign.

The fundamental theorem for the continuous T problem can be stated in terms that reflect its close connection with the theorems already stated for discrete T approximations.

The Chebyshev Theorem. Let functions $\phi_1, \phi_2, \dots, \phi_p$ form a Chebyshev set on $[a, b]$. Then the minimax approximation to the continuous function $f(x)$ on $[a, b]$ by linear combinations of the ϕ_i is characterized by the existence of a reference set (z_i) . The coefficients of the best approximation are the components of the levelled reference vector for this reference, and the maximum modulus of the error function $r(z)$ is equal to the reference deviation. By Lemma 2.2 the extrema of $r(z)$ alternate in sign on the points of this reference set.

The wording of this theorem suggests that the exchange algorithm for the discrete T problem might be modified to solve also the continuous T problem. In the modified algorithm a search is made for the point at which $|r(z)|$ is a maximum. This point is added to the reference set and an appropriate point dropped (here this is a trivial decision because use can be made of Lemma 2.2). Again the reference deviation is increased in magnitude. Further, it is bounded (it can be seen readily that $|h| \leq \max |f(z)|, a \leq z \leq b$), so that the procedure is convergent.

But an even more elaborate exchange is possible in this case, for it is possible to select $(p+1)$ points of extremal deviation of the error function for the current approximation that are alternately $+$ and $-$. Thus the current approximation forms a reference function with respect to these points as a reference set. This should be made clear by Fig. 1. where the end points are points of the original reference and remain as points in the new reference. This is the usual case.

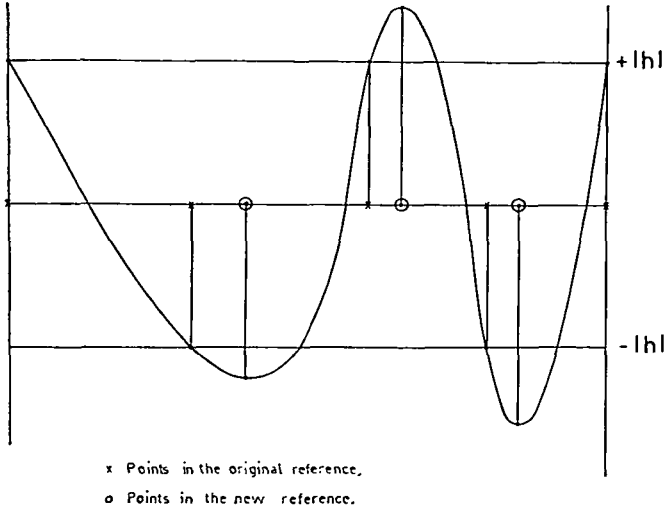


Fig. 1

This second exchange has proved popular in computing polynomial approximations to functions. It has been shown to have second order convergence (Veidinger, 1960). In this algorithm Lemma 2.2 plays an essential part.

3. The Stiefel exchange

This section is devoted to giving a proof of the Stiefel exchange theorem quoted in Section 2.

If $\rho_k(A)$ is the row to be added to the current reference then, by the Haar condition, the matrix

$$M = \begin{bmatrix} A_p \\ \rho_k(A) \end{bmatrix} \quad (3.1)$$

has rank p . There are therefore two linearly independent vectors v_1 and v_2 such that

$$v^T M = 0. \quad (3.2)$$

One such vector is $[\lambda^{(o)T}, 0]$ where $\lambda^{(o)}$ is the reference vector for the current reference, so that any vector satisfying (3.2) must be expressible in the form (to within a scalar multiplier)

$$u = \gamma \begin{bmatrix} \lambda^{(o)} \\ 0 \end{bmatrix} + v \quad (3.3)$$

where v^T is any solution of equation (3.2) which is not a scalar multiple of $[\lambda^{(o)T}, 0]$. A choice which is convenient for computation satisfies

$$\text{sgn}(r_k) \rho_k(A) + \sum_{i=1}^{p+1} v_i \rho_i(A_p) = 0 \quad (3.4)$$

but it could, for example, be chosen to be orthogonal to $[\lambda^{(o)T}, 0]$.

Now let x be a reference vector for the current reference. Then

$$Mx = \begin{bmatrix} b_p \\ b_k \end{bmatrix} - \begin{bmatrix} r_p \\ r_k \end{bmatrix} \quad (3.5)$$

and it can be arranged that (i) $\lambda_i^{(o)}(r_p)_i > 0$, $i = 1$,

$2, \dots, p+1$; (ii) $v_{p+2} r_k > 0$ (see for example equation (3.4)). If, for some value of j , we choose

$$\gamma = -v_j / \lambda_j^{(o)} \quad (3.6)$$

then $u_j = 0$, and the remaining components of u form the λ -vector for the reference obtained by deleting the j th row of M . If x is to be a reference vector for this new reference then we must have

$$u_i(r_p)_i = ((r_p)_i \lambda_i^{(o)})(\gamma + v_i / \lambda_i^{(o)}) > 0, \quad i \neq j, i = 1, 2, \dots, p+1. \quad (3.7)$$

This inequality is satisfied provided j is the index of the algebraically least among the quotients $v_i / \lambda_i^{(o)}$, $i = 1, \dots, p+1$, and this proves the theorem. The Haar condition ensures that these quotients are finite.

This is essentially the derivation given in Stiefel (1959) for the case $p = 2$. For convenience in making comparisons with the results of the next section note that if u is defined by

$$u = \begin{bmatrix} \lambda^{(o)} \\ 0 \end{bmatrix} - \gamma v \quad (3.8)$$

then the appropriate value of γ is equal to the least in modulus among those quotients $\lambda_i^{(o)} / v_i$ which are negative. This is obviously a non-empty set when v is orthogonal to $[\lambda^{(o)T}, 0]$. It will follow from the results of the next section that it is non-empty also with v defined by equation (3.4).

4. Linear programming and the Stiefel exchange algorithm

In this section and the next it will be necessary to make use of standard results from linear programming theory. The text we have followed is Hadley (1962), and references to this will be cited where appropriate in the form (H. page no.).

It is well known that the discrete T problem can be posed as a linear programming problem (Stiefel, 1960). This is done by introducing a new variable $x_{p+1} \geq 0$ which is to be minimized subject to the constraints

$$x_{p+1} - r_i \geq 0, x_{p+1} + r_i \geq 0, i = 1, 2, \dots, n. \quad (4.1)$$

The matrix form of the linear programming problem is

$$\text{minimize } Z = e_{p+1}^T \begin{bmatrix} x \\ x_{p+1} \end{bmatrix} \quad (4.2)$$

subject to $x_{p+1} \geq 0$, and (from equation (2.2))

$$\begin{bmatrix} A & e \\ -A & e \end{bmatrix} \begin{bmatrix} x \\ x_{p+1} \end{bmatrix} \geq \begin{bmatrix} b \\ -b \end{bmatrix} \quad (4.3)$$

where e is a vector each component of which is 1.

In this form it is not particularly suitable for the application of standard techniques because the matrix of constraints is $2n \times (p+1)$ so that $2n$ slack variables would be required, and because the components of x are not constrained to be positive. However, both these

difficulties are overcome by going to the dual program because the dual constraints corresponding to columns of the primal associated with the unconstrained variables are equalities (H. p. 236), and all the dual variables are constrained to be positive. The advantage to be gained by using the dual of the linear programming formulation of the approximation problem seems to have been pointed out first by Kelley (1958) who gives an application to curve fitting. He notes that the solution of the linear programming problem implies the theorem of De la Vallée Poussin.

The form of the dual program is

$$\begin{bmatrix} A^T & -A^T \\ e^T & e^T \end{bmatrix} w = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (4.4)$$

$$w \geq 0 \quad (4.5)$$

$$\text{maximize} \quad z = [b^T, -b^T]w. \quad (4.6)$$

Only one slack variable has to be added to (4.4) to make all constraints into equalities (although an additional p artificial variables are required to set up the usual simplex tableau), and it is shown in Lemma 4.1 that even this can be ignored.

Lemma 4.1. An optimal solution to the system (4.4)–(4.6) with a non-zero slack variable is possible only if $w = 0$.

Proof. With the addition of the slack the constraints (4.4) become

$$\begin{bmatrix} A^T & -A^T & 0 \\ e^T & e^T & 1 \end{bmatrix} \begin{bmatrix} w \\ w_s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (4.7)$$

Assume $\begin{bmatrix} w \\ w_s \end{bmatrix}$ is an optimal solution with $w \neq 0$, $w_s \neq 0$, and define

$$\begin{bmatrix} w^* \\ 0 \end{bmatrix} = \frac{1}{\sum_{i=1}^{2n} w_i} \begin{bmatrix} w \\ 0 \end{bmatrix}. \quad (4.8)$$

This vector satisfied all the constraints on the problem and gives a value of the objective function

$$z = \frac{1}{\sum_{i=1}^{2n} w_i} [b^T, -b^T, 0] \begin{bmatrix} w \\ 0 \end{bmatrix} > [b^T, -b^T, 0] \begin{bmatrix} w \\ w_s \end{bmatrix}$$

because $\sum_{i=1}^{2n} w_i < 1$ if $w_s \neq 0$ by the last equation of (4.7). This contradicts the hypothesis that $\begin{bmatrix} w \\ w_s \end{bmatrix}$ is an optimal solution.

Remark. If $w = 0$ is an optimal solution then

$$x_{p+1} = Z = z = 0$$

so that, by (4.1), $r_i = 0$, $i = 1, 2, \dots, n$. This implies the existence of an exact solution to the original set of equations. We specifically exclude this case from consideration. We assume $w_s = 0$ in (4.7) and drop the last column of matrix.

Lemma 4.2. $-\max |b_i| \leq z \leq \max |b_i|$, $i = 1, 2, \dots, n$

Proof. This is a direct consequence of

$$[e^T, e^T]w = 1,$$

$$w \geq 0,$$

and

$$z = [b^T, -b^T]w.$$

Remark. This shows that the feasible region is bounded.

Lemma 4.3. If in applying the simplex algorithm to the system (4.4)–(4.6) any column of A^T and the corresponding column of $-A^T$ appear together in the basis then the current value of $z \leq 0$.

Proof. Consider the T problem obtained by deleting all equations except those relating to columns in the dual basis. The set which remains contains at most p equations. Let this problem be solved by applying the simplex algorithm to the dual program, then the optimal basis for the reduced problem also contains a column of A^T and the corresponding column of $-A^T$. Now if a column is in the optimal basis for the dual the corresponding equation in the primal is an equality (H. p. 239). Therefore, for at least one i ,

$$x_{p+1} + r_i = x_{p+1} - r_i = 0$$

whence $x_{p+1} = 0$. Therefore the optimum for the reduced problem is $z = 0$ whence $z \leq 0$ for the initial basis.

Remark. We have excluded from consideration the case where $x_{p+1} = 0$ for the optimal solution. Therefore progress can only be made towards a solution by dropping one of the duplicated columns from the basis, and a stage must be reached when they are absent. We will assume that no basis contains columns duplicated in this way.

It will now be demonstrated that the simplex algorithm applied to the dual linear program is equivalent to the Stiefel exchange algorithm. This result is contained in Stiefel (1960). However, Stiefel eliminates the unconstrained variables from the primal before proceeding to the dual. Also his argument is largely geometric and is carried out for a small number of variables. We have already shown that the elimination of the unconstrained variables is unnecessary, and we demonstrate the algebraic equivalence of the two methods.

A basic feasible solution to the linear programming problem has at most $p + 1$ non-zero components. By the remark following Lemma 4.3 the columns of equation (4.7) which make up the basis correspond to a reference for the original T problem. In the simplex algorithm each basis matrix must be nonsingular, and for this it is both necessary and sufficient for the matrix A_p associated with the corresponding reference to have rank p . Note that the first p equations of (4.7) express a relation of linear dependence between the rows of the reference while the last equation can be interpreted as imposing a scale. As A_p has rank p this relation of linear dependence is unique, and this proves

Lemma 4.4. The non-zero components of a basic feasible solution w are equal in modulus to the appro-

priate components of the λ -vector for the corresponding reference. The λ -vector is scaled so that the sum of the moduli of its components is one.

Remark. The Haar condition is sufficient for the existence of a basis for the simplex algorithm but it is clearly much stronger than necessary.

Remark. To each reference there corresponds two distinct bases for the dual program. Each basis determines the same basic feasible solution, but the corresponding values of z are opposite in sign. As the optimum value of z is positive this fixes the basis of interest. This corresponds to choosing the λ -vector so that $\lambda_i r_i \geq 0$.

Lemma 4.5. The value of z given by the basic feasible solution is equal to the levelled reference deviation for the corresponding reference.

Proof.

$$\begin{aligned} z &= [b^T, -b^T]w \\ &= |b^T \lambda| / \sum_{i=1}^{p+1} |\lambda_i| \end{aligned}$$

by the previous Lemma,

$$= |h|$$

by equation (2.4).

Remark. Lemmas 4.4 and 4.5 show that the current basic feasible solution provides a solution to the T problem for the current reference.

Lemma 4.6. The choice of the vector to enter the basis in the simplex algorithm is equivalent to choosing the appropriate column corresponding to that equation of (2.2) which has the residual of greatest modulus.

Proof. Adopting Hadley's notation let

$$c^T = [b^T, -b^T],$$

let B be the matrix of the basis vectors, and c_B be the vector obtained from c by deleting the components corresponding to the nonbasic vectors. Let

$$z_s = c_B^T B^{-1} \kappa_s \left(\begin{bmatrix} A^T & -A^T \\ e^T & e^T \end{bmatrix} \right) \quad s = 1, 2, \dots, 2n. \quad (4.10)$$

Now if a column is in the optimal dual basis then the corresponding equation in the primal is an equality. By the previous remark the current basic feasible solution solves the T problem for the corresponding reference and hence is optimal for this restricted problem.

$$\therefore (B^{-1})^T c_B = \begin{bmatrix} x \\ h \end{bmatrix} \quad (4.11)$$

where x is the levelled reference vector and h the levelled reference deviation for the current reference.

$$\therefore z_s = h \pm \sum_{q=1}^p a_{jq} x_q \quad (4.12)$$

where $j = s$ if $s \leq n$ and the $+$ sign is appropriate, otherwise $j = s - n$ and the $-$ sign is taken. By equation (2.2) this is equivalent to

$$z_s = h \mp r_j + c_s. \quad (4.13)$$

The new column to enter the basis is that associated with the largest positive value of $c_s - z_s$ (H. p. 111).

Now

$$c_s - z_s = \pm r_j - h \quad (4.14)$$

so that the value of r_j of maximum modulus determines s .

We turn now to the question of determining the column to leave the basis.

From equation (4.14) the column to be added to the basis is

$$\begin{bmatrix} \text{sgn}(r_j) \kappa_j(A^T) \\ 1 \end{bmatrix}$$

and this is expressible in the form

$$\begin{bmatrix} \text{sgn}(r_j) \kappa_j(A^T) \\ 1 \end{bmatrix} = \sum_{i=1}^p y_i \kappa_i(B). \quad (4.15)$$

Assume that the k th column of B is to be deleted from the basis. Labelling the new basis matrix B_k we have

$$B_k = B(I + (y - e_k)e_k^T) \quad (4.16)$$

so that

$$B_k^{-1} = \left(I - \frac{1}{y_k} (y - e_k)e_k^T \right) B^{-1}. \quad (4.17)$$

Let $w^{(n)}$ denote the basic feasible solution for the new basis. Then

$$w_B^{(n)} = \left(I - \frac{1}{y_k} (y - e_k)e_k^T \right) w_B \quad (4.18)$$

so that

$$\begin{aligned} (w_B^{(n)})_i &= (w_B)_i - \frac{y_i}{y_k} (w_B)_k, \quad i \neq k \\ &= (w_B)_k / y_k, \quad i = k. \end{aligned} \quad (4.19)$$

The column to be dropped from the basis is chosen so that $w_B^{(n)} \geq 0$. Clearly it is sufficient to take

$$k: (w_B)_k / y_k = \min (w_B)_i / y_i \text{ for all } i \text{ such that } y_i > 0. \quad (4.20)$$

If the Haar condition does not hold then degeneracy permits the possibility of cycling in the simplex algorithm. However, this can always be resolved (see H. p. 174–196, also Desclos (1961)).

It is a standard result that there exists a k if the linear programming problem has a bounded optimum (H. p. 93).

Lemma 4.7. The equation to be dropped from the reference in the Stiefel exchange algorithm corresponds to the column to be dropped from the basis in the simplex algorithm.

Proof. We show that the test given above for the simplex algorithm is equivalent to the second test given in Section 3 for the Stiefel exchange.

We have

$$\begin{bmatrix} \text{sgn}(r_j) \kappa_j(A^T) \\ 1 \end{bmatrix} = B y = \begin{bmatrix} A_p^T \\ g^T \end{bmatrix} G y \quad (4.21)$$

where G is a diagonal matrix and $G_{ii} = g_i$ (g was defined in Section 2; but notice also the second remark following Lemma 4.4) so that

$$M^T \begin{bmatrix} -G y \\ \text{sgn}(r_j) \end{bmatrix} = 0. \quad (4.22)$$

By equation (3.4) it follows that

$$v = \begin{bmatrix} -Gy \\ \text{sgn}(r_j) \end{bmatrix}. \quad (4.23)$$

Now the column to leave the basis is given by

$$\begin{aligned} j: (w_B)_j/y_j &= \min (w_B)_i/y_i \text{ for all } i \text{ such that } y_i > 0 \\ &= \max (w_B)_i/(-y_i) \\ &= \max g_i(w_B)_i/(-g_i y_i) \\ &= \max \lambda_i^{(0)}/v_i \end{aligned}$$

and this is the second test given in Section 3.

Theorem 4.1. The Stiefel exchange algorithm is exactly equivalent to the simplex algorithm applied to the asymmetric dual of the linear programming formulation of the discrete T problem.

Proof. This is an immediate consequence of Lemmas 4.5-4.7 which itemize the major aspects of the equivalence.

5. Discussion

The Haar condition was not used in Section 4. All that was required was the nonsingularity of the successive basis matrices. The condition for this is that the matrix of the constraints (4.4) has rank $p+1$, and for this it is sufficient that A has rank p . This is much weaker than the Haar condition, and it is remarkable that in at least $p+1$ of the equations (2.2) the residuals are equal in modulus to $z = x_{p+1}$. This follows from the result used before (H. p. 239) that if a column is in the dual basis then the corresponding equation in the primal is an equality. Thus the classical theorem of De la Vallée Poussin can be restated in the following more general form.

Theorem 5.1. Let the matrix of the discrete T problem have rank p . Then there exists a solution to this problem for which the residual of maximum modulus is a minimum. Further there is a reference on which the residuals

are equal in modulus to this maximum value, and there is a λ -vector for this reference such that either

$$\begin{aligned} \lambda_i r_i &\geq 0 & i = 1, 2, \dots, p+1. \\ \text{or} & & \lambda_i r_i \leq 0 \end{aligned}$$

The rank of the matrix of the optimal reference is p .

Surprisingly little is lost by relaxing the Haar condition almost completely. An immediate corollary to this is the linear case of a theorem due to Curtis and Powell (1966).

Corollary. An optimal solution to the discrete T problem for which the maximum residual is obtained at less than $(p+1)$ points is possible only if the rank of A is less than p . This result implies that for the continuous problem also, because it then states a property of a continuous system when sampled on a discrete set of points.

It is interesting that the two-phase simplex method (H. p. 149) is capable of solving the discrete T problem even when the rank of A is less than p . In this case some of the artificial variables will persist in the optimal basis (H. p. 121). This is a definite advantage not possessed by the Stiefel exchange. However, Stiefel (1960) argues the advantage of his algorithm, and it is certainly more economical in storage than the simplex algorithm. However, our second test of Section 3 would appear to be necessary for its satisfactory implementation.

Certainly it is possible in the continuous case to provide an algorithm which combines the merits of both approaches. It is our intention to describe such an algorithm together with some applications in a further paper. A program for implementing the simplex algorithm to solve the discrete T problem has been given by Barrowdale and Young (1966).

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