Some techniques for rational interpolation

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Simplified forms of Stoer's rational interpolation algorithms are presented as special cases of a generalization of the Neville–Aitken method. These algorithms offer convenient means for effecting rational interpolation in a given set of data points, either numerically or analytically.

1. Introduction

The problem of finding a rational function which assumes given function values at prescribed positions of the independent variable may be approached from several viewpoints. One can, of course, assume an explicit form for the function

$$R(x) = \frac{\sum_{r=0}^{p} a_r x^r}{\sum_{r=0}^{q} b_r x^r} \quad (1)$$

and attempt to determine the coefficients $\{a_r;\ 0 \leqslant r \leqslant p\}$ and $\{b_r;\ 0 \leqslant r \leqslant q\}$ from the $p + q + 1$ linear equations which result from insisting that

$$R(x_s) = f_s;\ 1 \leqslant s \leqslant p + q + 1, \quad (2)$$

where the (distinct) interpolating points $\{(x_s, f_s);\ 1 \leqslant s \leqslant p + q + 1\}$ are given. Notice that, although expression (1) contains $p + q + 2$ coefficients this number can always be reduced by one by cancellation. Thence, given that the interpolating function exists, uniqueness follows from the fact that, if the linear, homogeneous equations

$$f_s \sum_{r=0}^{q} b_r x_s^r = \sum_{r=0}^{p} a_r x_s^r;\ 1 \leqslant s \leqslant p + q + 1 \quad (3)$$

obtained from (2) have a solution in which the coefficients $\{a_r\}$ and $\{b_r\}$ are not all zero, then this solution is unique except for an arbitrary, non-zero, constant multiplying factor. Furthermore, this construction of $R(x)$ is entirely independent of the order of the given positions $\{x_s;\ 1 \leqslant s \leqslant p + q + 1\}$. However, the amount of numerical work involved in solving equations (3) for the relative magnitudes of the coefficients makes this approach less attractive than others.

The classical technique for constructing $R(x)$, in the special case when $p = q$, or $q + 1$, is due to Thiele (1909). This special rational interpolating function $T(x)$ is expressed in the form of a terminating continued fraction

$$T(x) = a_0 + \cfrac{x - x_1}{a_1 + \cfrac{x - x_2}{a_2 + \cfrac{\cdots}{\cdots \cfrac{x - x_{p+q}}{a_{p+q}}}}} \quad (4)$$

where the coefficients $\{a_r;\ 0 \leqslant r \leqslant p + q\}$ are determined by constructing a table of inverted differences, or reciprocal differences, from the co-ordinates of the given interpolation points.

Wynn (1960) and Stoer (1961) have given tabular methods for the purpose of rational interpolation, Stoer's algorithms being somewhat simpler than those of Wynn. A further, slight simplification is achieved by casting Stoer's algorithms in the forms to be discussed. Moreover, a conceptual advantage is obtained since the forms presented here arise naturally as special cases of a generalization of the Neville–Aitken method which is described in another paper (Larkin, 1967).

2. Definitions and nomenclature

Consider a set of points $\{(x_j, f_j); j = 1, 2 \ldots\}$. For any $j \geqslant 1$ the f_j may be thought of as defined in terms of some originating function $f(x)$ by the relation

$$f_j = f(x_j). \quad (5)$$

We assume that $x_j \neq x_k$ unless $j = k$. The quantities $\{x_j; j = 1, 2 \ldots\}$ and $\{f_j; j = 1, 2 \ldots\}$ may be real or complex and their order in the implied sequence is quite arbitrary. Our object will be to construct an array of functions $\{f_{jk}(x);\ j = 1, 2, \ldots;\ k = 1, 2, \ldots\}$ each one having the property that

$$f_{jk}(x_r) = f_r;\ j \leqslant r \leqslant j + k \quad (6)$$

except in certain special circumstances.

For ease of presentation it is convenient to arrange these functions in a table of triangular form, as shown in **Table 1.**

For any $j \geqslant 1$, $k \geqslant 1$ we shall refer to the set of points $\{x_r;\ j \leqslant r \leqslant j + k\}$ as the *domain* of $f_{jk}(x)$, and we shall write

$$D_{jk} = \bigcup_{r=j}^{j+k} \{x_r\}. \quad (7)$$

Moreover, we define the *domain of interpolation*, D^I_{jk}, of $f_{jk}(x)$ as the set of points $x_s \in D_{jk}$ such that

$$f_{jk}(x_s) = f_s. \quad (8)$$

Thus, if equation (6) is satisfied we can write

$$D^I_{jk} = D_{jk} \quad (9)$$

and in this case we shall say that the function $f_{jk}(x)$

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Table 1

A table of interpolating functions

x_1	f_1								
		f_{11}							
x_2	f_2		f_{12}						
		f_{21}		f_{13}					
x_3	f_3		f_{22}		f_{14}				
		f_{31}		f_{23}		.			
x_4	f_4		f_{32}	.	.		.		
		f_{41}	.	.	.			.	
x_5	f_5	.	.	.	.				.
.	.	.	.	.	.				
.	.	.	.						
.	.	.							
.	.	.							
.	.								

possesses *Property I.* By extension, we see that it is reasonable to define

$$f_{j0} = f_j \quad ; \quad j = 1, 2, \ldots \tag{10}$$

and to say that the point x_j constitutes the domain of interpolation of f_{j0}, so that

$$D^I_{j0} = D_{j0} \quad ; \quad j = 1, 2, \ldots \tag{11}$$

Fig. 1 shows how a function $f_{jk}(x)$ stands, in a table of the form of Table 1, in relation to its domain. Clearly, if $f_{jk}(x)$ does not possess Property I

$$D^I_{jk} \subset D_{jk}. \tag{12}$$

3. The triangle and rhombus rules

For the remainder of this paper we shall restrict ourselves to consideration of the case where the $\{f_{jk}\}$ are rational functions of x. In order to construct these functions we shall make use of the two "triangle rules"

$$f_{jk} = \frac{(x - x_j)f_{j+1,k-1} + (x_{j+k} - x)f_{j,k-1}}{x_{j+k} - x_j} \tag{13}$$

and

$$f_{jk} = \frac{x_{j+k} - x_j}{\dfrac{x - x_j}{f_{j+1,k-1}} + \dfrac{x_{j+k} - x}{f_{j,k-1}}}, \tag{14}$$

and the "rhombus rule"

$$f_{jk} = f_{j+1,k-2} + \frac{x_{j+k} - x_j}{\dfrac{x - x_j}{f_{j+1,k-1} - f_{j+1,k-2}} + \dfrac{x_{j+k} - x}{f_{j,k-1} - f_{j+1,k-2}}}. \tag{15}$$

Equation (13) is, of course, the Neville–Aitken formula, which, when the starting conditions

$$f_{r0} = f_r \quad ; \quad j \leqslant r \leqslant j + k \tag{16}$$

are used, leads to interpolating functions

$$\{f_{rs}; j \leqslant r \leqslant j + k - s, 1 \leqslant s \leqslant k\}$$

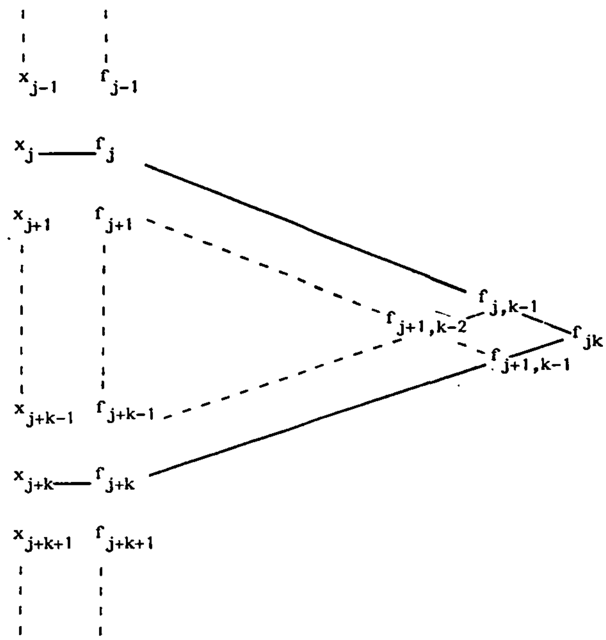

Fig. 1. The domain of f_{jk}

such that f_{rs} is a polynomial of degree less than or equal to s. Also, it is easily shown by induction that recurrence formula (14), starting from conditions (16), leads to functions of the form

$$f_{rs} = \left\{ \sum_{t=0}^{s} a_t x^t \right\}^{-1} \quad ; \quad j \leqslant r \leqslant j + k - s, 1 \leqslant s \leqslant k, \tag{17}$$

where the coefficients $\{a_t; 0 \leqslant t \leqslant s\}$ are constants. Moreover, provided that none of the given function values $\{f_r; j \leqslant r \leqslant j + k\}$ is equal to zero, all of the functions constructed from them will possess Property I.

Before recurrence formula (15) can be applied, for the purpose of generating the functions in the kth column of Table 1, the two previous columns must be available. It is shown in the following section that if the first few columns of functions in Table 1 are constructed either exclusively by recurrence (13), or exclusively by recurrence (14), and the succeeding columns exclusively by recurrence (15), then all the functions in the table will possess Property I, except in certain special circumstances. Naturally, if there does not exist a rational function, with prescribed degrees of numerator and denominator, which interpolates a certain given set of points, we cannot expect an algorithm to produce one.

Table 2a illustrates the use of formula (15), after one initial application of the Neville–Aitken rule, in constructing a table of rational interpolating functions. **Table 2b** illustrates the use of the same algorithm for numerical interpolation at the point $x = 3{\cdot}5$. The values $\{(x_r, f_r); 1 \leqslant r \leqslant 6\}$ are also discussed by Hildebrand (1956) as an example in rational interpolation.

Notice that the function $f_{13}(x)$ is peculiar in that it does not possess Property I. The possibility of loss of Property I, and the necessity of taking this into con-

Table 2a

Example of a table of rational interpolating functions

j	x_j	f_j	$k = 1$	2	3	4	5
1	0	2					
			$\frac{4-x}{2}$				
2	1	3/2		$\frac{14-5x}{7-x}$			
			$\frac{22-7x}{10}$		$\frac{(2-x)(4-x)}{2(2-x)}$		
3	2	4/5		$\frac{10-x}{2+4x}$		$\frac{x+2}{x^2+1}$	
			$\frac{14-3x}{10}$		$\frac{x^2-10x+48}{2(7x+6)}$		$\frac{x+2}{x^2+1}$
4	3	1/2		$\frac{22-x}{13x-1}$		$\frac{x+2}{x^2+1}$	
			$\frac{32-5x}{34}$		$\frac{x^2-14x+136}{2(33x+4)}$		
5	4	6/17		$\frac{40-x}{28x-10}$			
			$\frac{304-37x}{442}$				
6	5	7/26					

Table 2b

Numerical rational interpolation at the point $x = 3{\cdot}5$

j	x_j	f_j	$k = 1$	2	3	4	5
1	0	2					
			0·25				
2	1	3/2		−1·0			
			−0·25		0·25		
3	2	4/5		0·406250		0·415094	
			0·35		0·413934		0·415094
4	3	1/2		0·415730		0·415094	
			0·426471		0·415272		
5	4	6/17		0·414773			
			0·394796				
6	5	7/26					

sideration, accounts for much of the complexity in the arguments of the following section. By consideration of equation (15) in the limits as $\{x \to x_r; j \leqslant r \leqslant j+k\}$ it is easily seen that f_{jk} may lose Property I where $x = x_r$ if any of the special relations

$$f_{j,\,k-1}(x_{j+k}) = f_{j+1,\,k-2}(x_{j+k})$$

or $$f_{j+1,\,k-1}(x_j) = f_{j+1,\,k-2}(x_j)$$

or

$$\frac{x_r - x_j}{f'_{j+1,k-1}(x_r) - f'_{j+1,\,k-2}(x_r)} + \frac{x_{j+k} - x_r}{f'_{j,\,k-1}(x_r) - f'_{j+1,\,k-2}(x_r)} = 0$$

holds.

4. The algorithms and their resultant functions

The two algorithms for constructing tables of the form of Table 1 which we shall consider are as follows:

Algorithm A_i:

Use the Neville–Aitken recurrence (13) *to construct the first i columns of Table* 1, *i.e. up to and including the column of ith degree polynomials. Then use recurrence* (15) *to construct all succeeding columns of the table.*

Algorithm B_i:

Use recurrence (14) *to construct the first i columns of Table* 1, *i.e. up to and including the column of ith degree inverse polynomials. Then use recurrence* (15) *to construct all succeeding columns of the table.*

It is obvious from consideration of the triangle and rhombus rules that all the functions $\{f_{jk}(x)\}$ are rational in x. Let us then write

$$f_{jk}(x) = \frac{P^*_{jk}(x)}{Q^*_{jk}(x)} = \frac{P_{jk}(x)}{Q_{jk}(x)}\,;\quad \begin{matrix} j = 1, 2, \ldots \\ k = 0, 1, 2, \ldots \end{matrix} \tag{18}$$

where $P^*_{jk}(x)$ and $Q^*_{jk}(x)$ are polynomials in x having no non-constant common factor. $P_{jk}(x)$ and $Q_{jk}(x)$ are also polynomials, constructed from $P^*_{jk}(x)$ and $Q^*_{jk}(x)$ by the following process:

(i) If $f_{jk}(x)$ possesses Property I

$$\left.\begin{aligned} P_{jk}(x) &= P^*_{jk}(x) \\ Q_{jk}(x) &= Q^*_{jk}(x) \end{aligned}\right\}. \tag{19}$$

(ii) If $f_{jk}(x)$ does not possess Property I

$$\left.\begin{aligned} P_{jk}(x) &= P^*_{jk}(x)\,.\,E_{jk}(x) \\ Q_{jk}(x) &= Q^*_{jk}(x)\,.\,E_{jk}(x) \end{aligned}\right\} \tag{20}$$

where
$$E_{jk}(x) = \prod_{x_r \in D_{jk} - D_{jk}^{(i)}} (x - x_r). \tag{21}$$

At first sight the introduction of the quantities $\{P_{jk}, Q_{jk}\}$, by the above construction, seems rather arbitrary. However, it turns out that they are actually more fundamental to the theory than are the $\{P^*_{jk}, Q^*_{jk}\}$, as is shown by the following theorems and corollaries.

There seems to be no obvious reason why the degrees of the polynomials $P_{jk}(x)$ and $Q_{jk}(x)$ should not increase very rapidly with k. However, it turns out that these degrees are the smallest possible, consistent with allowing $f_{jk}(x)$ to possess Property I in the general case—an assertion which is expressed more precisely in the propositions which follow. Let us introduce the notation $\deg\{P\}$ to indicate the degree of the polynomial $P(x)$. Now, using the above definitions of the polynomials $\{P^*_{jk}, Q^*_{jk}, P_{jk}, Q_{jk}\}$, we have:

Theorem 1:

If Table 1 *is constructed by the use of algorithm A_i, the kth column consists of rational functions satisfying the conditions*

$$\left.\begin{aligned} \deg\{P_{jk}\} &\leqslant k \\ \deg\{Q_{jk}\} &= 0 \end{aligned}\right\};\ j \geqslant 1, 0 \leqslant k \leqslant i \tag{22}$$

and

$$\left.\begin{aligned} \deg\{P_{jk}\} &\leqslant \left[\frac{k+i}{2}\right] \\ \deg\{Q_{jk}\} &\leqslant \left[\frac{k-i+1}{2}\right] \end{aligned}\right\};\ j \geqslant 1, k \geqslant i+1. \tag{23}$$

The expression $[y]$ indicates "largest integer not greater than y".

Theorem 2:

If none of the given values $\{f_r;\ r = 1, 2, 3, \ldots\}$ is zero, and if Table 1 *is constructed by the use of algorithm B_i, the kth column consists of rational functions satisfying the conditions*

$$\left.\begin{aligned} \deg\{P_{jk}\} &= 0 \\ \deg\{Q_{jk}\} &\leqslant k \end{aligned}\right\};\ j \geqslant 1, 0 \leqslant k \leqslant i \tag{24}$$

and

$$\left.\begin{aligned} \deg\{P_{jk}\} &\leqslant \left[\frac{k-i+1}{2}\right] \\ \deg\{Q_{jk}\} &\leqslant \left[\frac{k+i}{2}\right] \end{aligned}\right\};\ j \geqslant 1, k \geqslant i+1. \tag{25}$$

Notice that the restriction in Theorem 2, that the given function values be non-zero, is simply analogous to the implied restriction in Theorem 1 that they be finite. In fact, the functions generated by applying algorithm B_i to the given points $\{(x_r, f_r);\ r \geqslant 1\}$ are the reciprocals of those generated by applying algorithm A_i to the points $\{(x_r, 1/f_r);\ r \geqslant 1\}$.

We now proceed with the proof of Theorem 1. The proof of Theorem 2 will not be given, since it trivially parallels that of Theorem 1.

Proof of Theorem 1:

Equations (22) simply state a well known property of the Neville–Aitken algorithm, so our task reduces to proving the truth of equations (23). This will be done by induction, after noting that for k equal to i and $i - 1$ equations (23) are indeed satisfied.

For $j \geqslant 1$ and $k \geqslant i + 1$, we define quantities Z_{jk}, R_{jk}, S_{jk} and T_{jk} by the relations

$$Z_{jk}(x) = \prod_{r=j+1}^{j+k-1} (x - x_r) \tag{26}$$

$$R_{jk}\cdot Z_{jk} = P_{jk}\cdot Q_{j+1,\,k-2} - P_{j+1,\,k-2}\cdot Q_{jk} \tag{27}$$

$$S_{jk}\cdot Z_{jk} = P_{j,k-1}\cdot Q_{j+1,k-2} - P_{j+k,k-2}\cdot Q_{j,k-1} \tag{28}$$

$$T_{jk}\cdot Z_{jk} = P_{j+1,k-1}\cdot Q_{j+1,\,k-2} - P_{j+1,k-2}\cdot Q_{j+1,k-1}. \tag{29}$$

By construction of the polynomials $\{P_{rs}(x)\}$ and $\{Q_{rs}(x)\}$, we see that the right hand sides of equations (27), (28) and (29) all vanish whenever

$$\text{such that}\quad \left.\begin{aligned} x &= x_t \\ j+1 &\leqslant t \leqslant j+k-1 \end{aligned}\right\}, \tag{30}$$

so that Z_{jk} must be a proper divisor of each of these right hand sides, except possibly when one of them vanishes identically. The case when one of the right hand sides of equations (27), (28) and (29) vanishes identically will be considered separately, so in the meantime we can assume

$$\left.\begin{aligned} R_{jk} &\not\equiv 0 \\ S_{jk} &\not\equiv 0 \\ T_{jk} &\not\equiv 0 \end{aligned}\right\}. \tag{31}$$

It is clear now that R_{jk}, S_{jk} and T_{jk} must all be polynomials; in the following Lemma we go on to show that they must be polynomials of degree zero.

Lemma: *R_{jk}, S_{jk} and T_{jk} are all constants.*

Suppose the polynomials $\{P_{rs}(x),\ Q_{rs}(x);\ r \geqslant 1,\ k-2 \leqslant s \leqslant k-1\}$ all satisfy conditions (23). From equation (28) we then have

$$\begin{aligned} \deg\{S_{jk}.Z_{jk}\} &\leqslant \operatorname{Max}\{\deg\{P_{j,k-1}\} + \deg\{Q_{j+1,k-2}\};\ \deg\{P_{j+1,k-2}\} + \deg\{Q_{j,k-1}\}\} \\ &\leqslant \operatorname{Max}\left\{\left[\frac{k+i-1}{2}\right] + \left[\frac{k-i-1}{2}\right];\ \left[\frac{k+i-2}{2}\right] + \left[\frac{k-i}{2}\right]\right\} \end{aligned}$$

i.e.

$$\deg\{S_{jk}.Z_{jk}\} \leqslant k-1. \tag{32}$$

Similarly, from equation (29), we obtain

$$\deg\{T_{jk}.Z_{jk}\} \leqslant k-1. \tag{33}$$

However, by construction

$$\deg\{Z_{jk}\} = k-1, \tag{34}$$

which enables us to deduce that strict equality holds in equations (32) and (33), and that S_{jk} and T_{jk} must be constants, as required.

Now notice that equation (15) may be written in the form

$$\frac{x_{j+k}-x_j}{\dfrac{P_{jk}}{Q_{jk}} - \dfrac{P_{j+1,k-2}}{Q_{j+1,k-2}}} = \frac{x-x_j}{\dfrac{P_{j+1,k-1}}{Q_{j+1,k-1}} - \dfrac{P_{j+1,k-2}}{Q_{j+1,k-2}}} + \frac{x_{j+k}-x}{\dfrac{P_{j,k-1}}{Q_{j,k-1}} - \dfrac{P_{j+1,k-2}}{Q_{j+1,k-2}}},$$

and that a factor $\dfrac{Q_{j+1,k-2}}{Z_{jk}}$ may be cancelled throughout, leaving

$$(x_{j+k}-x_j).\frac{Q_{jk}}{R_{jk}} = (x-x_j).\frac{Q_{j+1,k-1}}{T_{jk}} + (x_{j+k}-x).\frac{Q_{j,k-1}}{S_{jk}}. \tag{35}$$

We next multiply through equation (27) by $\dfrac{(x_{j+k}-x_j)}{R_{jk}}$, through equation (28) by $\dfrac{x_{j+k}-x}{S_{jk}}$ and through equation (29) by $\dfrac{x-x_j}{T_{jk}}$, and then combine the results linearly with equation (35) to yield

$$(x_{j+k}-x_j)\frac{P_{jk}}{R_{jk}} = (x-x_j).\frac{P_{j+1,k-1}}{T_{jk}} + (x_{j+k}-x).\frac{P_{j,k-1}}{S_{jk}}. \tag{36}$$

Notice that, if the quantities R_{jk}, S_{jk} and T_{jk} were known, equations (35) and (36) would provide separate, linear recurrence formulae for the $\{P_{jk}\}$ and $\{Q_{jk}\}$.

Now, by construction, the only non-constant factors common to $P_{jk}(x)$ and $Q_{jk}(x)$ are those occurring in $E_{jk}(x)$, defined in equation (21). But, since neither of the right hand sides of equations (35) and (36) contains a singularity in the finite part of the complex plane, R_{jk} must be a divisor of both P_{jk} and Q_{jk}, and so it may only consist of a product of single factors of the form $(x-x_s)$, where

$$x_s \epsilon D_{jk} - D^I_{jk},$$

i.e. where

$$\left.\begin{aligned} & P_{jk}(x_s) = 0 = Q_{jk}(x_s) \\ \text{and}\quad & \frac{P^*_{jk}(x_s)}{Q^*_{jk}(x_s)} \neq f_s \end{aligned}\right\}. \tag{37}$$

However, from equations (35) and (36), we have

$$\frac{P_{jk}}{Q_{jk}} = \frac{P^*_{jk}}{Q^*_{jk}} = \frac{(x-x_j).\dfrac{P_{j+1,k-1}}{T_{jk}} + (x_{j+k}-x).\dfrac{P_{j,k-1}}{S_{jk}}}{(x-x_j).\dfrac{Q_{j+1,k-1}}{T_{jk}} + (x_{j+k}-x)\dfrac{Q_{j,k-1}}{S_{jk}}}, \tag{38}$$

and separate consideration of the three possibilities

$$s = j$$
$$j < s < j+k$$
$$s = j+k$$

leads to the conclusion that $\dfrac{P^*_{jk}(x_s)}{Q^*_{jk}(x_s)}$ can only fail to equal f_s when both numerator and denominator of the right hand side of equation (38) possess a factor $(x-x_s)$. But, if that is so, equations (35) and (36) indicate that $\dfrac{P_{jk}}{R_{jk}}$ and $\dfrac{Q_{jk}}{R_{jk}}$ both vanish at x_s, implying that P_{jk} and Q_{jk} both possess a factor $(x-x_s)^2$, which is absurd since the construction of P_{jk} and Q_{jk} ensures that a common factor of the form $(x-x_s)$ can only occur singly. This argument applies separately to all the points

$$x_s \epsilon D_{jk} - D^I_{jk},$$

thus enabling us to conclude that R_{jk} must be a constant, as required.

To recapitulate, the Lemma shows that, under the assumption of the induction hypothesis, the three quantities R_{jk}, S_{jk} and T_{jk}, appearing in equations (27), (28) and (29), are all constants.

Before proceeding to determine the bounds on the degrees of P_{jk} and Q_{jk} notice that, from equations (28) and (29), we can write

$$Q_{j+1,k-2}\cdot\left(\frac{P_{j+1,k-1}}{T_{jk}}-\frac{P_{j,k-1}}{S_{jk}}\right)=P_{j+1,k-2}\cdot\left(\frac{Q_{j+1,k-1}}{T_{jk}}-\frac{Q_{j,k-1}}{S_{jk}}\right). \quad (39)$$

Also, from equation (28) or (29), we have

$$k-1\leqslant \mathrm{Max}\left\{\left[\frac{k+i-1}{2}\right]+\deg\{Q_{j+1,k-2}\};\ \deg\{P_{j+1,k-2}\}+\left[\frac{k-i}{2}\right]\right\}, \quad (40)$$

and when $k-i$ is even, $k+i$ is also even, so equation (40) becomes

$$k-1\leqslant \mathrm{Max}\left\{\frac{k+i-2}{2}+\frac{k-i-2}{2};\ \deg\{P_{j+1,k-2}\}+\frac{k-i}{2}\right\}$$

$$\therefore \deg\{P_{j+1,k-2}\}=\frac{k+i}{2}-1. \quad (41)$$

Hence, we can deduce from equation (39) that

$$\frac{k-i-2}{2}+\frac{k+i-2}{2}\geqslant\frac{k+i}{2}-1+\deg\left\{\frac{Q_{j+1,k-1}}{T_{jk}}-\frac{Q_{j,k-1}}{S_{jk}}\right\}$$

$$\therefore \deg\left\{\frac{Q_{j+1,k-1}}{T_{jk}}-\frac{Q_{j,k-1}}{S_{jk}}\right\}\leqslant\frac{k-i-2}{2}. \quad (42)$$

Similarly, when $k-i$ is odd, $k+i$ is also odd, so equation (40) becomes

$$k-1\leqslant \mathrm{Max}\left\{\frac{k+i-1}{2}+\deg\{Q_{j+1,k-2}\};\ \frac{k+i-3}{2}+\frac{k-i-1}{2}\right\}$$

$$\therefore \deg\{Q_{j+1,k-2}\}=\frac{k-i-1}{2}. \quad (43)$$

Hence, we can also deduce from equation (39) that

$$\frac{k-i-1}{2}+\deg\left\{\frac{P_{j+1,k-1}}{T_{jk}}-\frac{P_{j,k-1}}{S_{jk}}\right\}\leqslant\frac{k+i-3}{2}+\frac{k-i-1}{2}$$

$$\therefore \deg\left\{\frac{P_{j+1,k-1}}{T_{jk}}-\frac{P_{j,k-1}}{S_{jk}}\right\}\leqslant\frac{k+i-3}{2}. \quad (44)$$

Now, from equation (42) and the induction hypothesis, we can write

$$\deg\left\{\frac{Q_{j+1,k-1}}{T_{jk}}-\frac{Q_{j,k-1}}{S_{jk}}\right\}\leqslant\left[\frac{k-i-1}{2}\right], \quad (45)$$

and from equation (44) and the induction hypothesis, we can write

$$\deg\left\{\frac{P_{j+1,k-1}}{T_{jk}}-\frac{P_{j,k-1}}{S_{jk}}\right\}\leqslant\left[\frac{k+i-2}{2}\right]. \quad (46)$$

Hence, using equations (35) and (45), and the induction hypothesis, we find

$$\deg\{Q_{jk}\}\leqslant \mathrm{Max}\left\{\left[\frac{k-i-1}{2}\right]+1;\ \left[\frac{k-i}{2}\right]\right\}$$

i.e. $$\deg\{Q_{jk}\}\leqslant\left[\frac{k-i+1}{2}\right], \quad (47)$$

as required. Also, using equations (36) and (46), and the induction hypothesis, we obtain

$$\deg\{P_{jk}\}\leqslant \mathrm{Max}\left\{\left[\frac{k+i-2}{2}\right]+1;\ \left[\frac{k+i-1}{2}\right]\right\}$$

i.e. $$\deg\{P_{jk}\}\leqslant\left[\frac{k+i}{2}\right], \quad (48)$$

as required.

In the foregoing reasoning it was assumed that none of the quantities R_{jk}, S_{jk} and T_{jk} vanished identically. For completeness we now give separate consideration to that possibility. Suppose, for example, that

$$P_{j,k-1}\cdot Q_{j+1,k-2}-P_{j+1,k-2}\cdot Q_{j,k-1}\equiv 0$$

i.e. $$f_{j,k-1}\equiv f_{j+1,k-2}, \quad (49)$$

then, from equation (15) we see that

$$f_{jk}\equiv f_{j+1,k-2} \quad (50)$$

thus satisfying equations (23) automatically. The same conclusion follows if we suppose that either

$$P_{j+1,k-1}\cdot Q_{j+1,k-2}-P_{j+1,k-2}\cdot Q_{j+1,k-1}\equiv 0$$

or $$P_{jk}\cdot Q_{j+1,k-2}-P_{j+1,k-2}\cdot Q_{jk}\equiv 0,$$

which finally confirms the truth of equations (23), under the assumption of the induction hypothesis. However, we need only recall that equations (23) are certainly satisfied for k equal to i and $i-1$ to see that the induction is complete.

Corollary 1:

For all $j\geqslant 1, k\geqslant i+1$, except when $f_{jk}\equiv f_{j,k-1}\equiv f_{j+1,k-1}$,

$$\deg\{P_{jk}\}=\frac{k+i}{2},\quad \textit{whenever } k+i \textit{ is even}, \quad (51)$$

and

$$\deg\{Q_{jk}\}=\frac{k-i+1}{2},\quad \textit{whenever } k+i \text{ is odd}. \quad (52)$$

To prove this, consider

$$X_{jk} = P_{jk} \cdot Q_{j,k-1} - P_{j,k-1} \cdot Q_{jk}, \tag{53}$$

which is a polynomial with zeros at the k points $\{x_s; j \leqslant s \leqslant j + k - 1\}$. Thus, unless $X_{jk} \equiv 0$, implying $f_{jk} \equiv f_{j,k-1} \equiv f_{j+1,k-1}$ from equation (15), we have

$$\deg \{X_{jk}\} \geqslant k. \tag{54}$$

Hence

$$k \leqslant \text{Max} \{\deg \{P_{jk}\} + \deg \{Q_{j,k-1}\}; \deg \{P_{j,k-1}\} + \deg \{Q_{jk}\}\}. \tag{55}$$

Thus, when $k + i$ is even and $k \geqslant i + 1$,

$$k \leqslant \text{Max} \{\deg \{P_{jk}\} + \deg \{Q_{j,k-1}\}; k - 1\}$$

$$\therefore \quad \deg \{P_{jk}\} = \frac{k+i}{2} \text{ and } \deg \{Q_{j,k-1}\} = \frac{k-i}{2}. \tag{56}$$

Similarly, when $k + i$ is odd and $k \geqslant i + 1$,

$$k \leqslant \text{Max} \{k - 1; \deg \{P_{j,k-1}\} + \deg \{Q_{jk}\}\}$$

$$\therefore \quad \deg \{P_{j,k-1}\} = \frac{k+i-1}{2} \text{ and } \deg \{Q_{jk}\} = \frac{k-i+1}{2}, \tag{57}$$

which completes the proof of the corollary.

Remembering the relations between the quantities $\{P^*_{jk}, Q^*_{jk}, E_{jk}, P_{jk}, Q_{jk}\}$, expressed in equations (18), (19), (20) and (21) we see that f_{jk} can only fail to possess Property I by virtue of the cancellation of a factor E_{jk} from P_{jk} and Q_{jk}. Thus, we can be sure that if

$$\deg \{P^*_{jk}\} = \frac{k+i}{2} \qquad \text{when } k + i \text{ is even}$$

and

$$\deg \{Q^*_{jk}\} + \frac{k-i+1}{2} \text{ when } k + i \text{ is odd}$$

then f_{jk} must possess Property I, regardless of how many times Property I was lost by functions in previous columns of the table. By definition, of course, if a non-constant factor E_{jk} is common to P_{jk} and Q_{jk} then f_{jk} loses Property I at the zeros of E_{jk}.

Corollary 2:

The polynomials $\{P_{jk}(x), Q_{jk}(x); j \geqslant 1, k \geqslant i + 1\}$ *may be constructed from the recurrence relations*

$$\left.\begin{aligned} P_{jk} &= \alpha_{jk} . (x - x_j) . P_{j+1,k-1} \\ &\quad + \beta_{jk} . (x_{j+k} - x) . P_{j,k-1} \\ Q_{jk} &= \alpha_{jk} . (x - x_j) . Q_{j+1,k-1} \\ &\quad + \beta_{jk} . (x_{j+k} - x) . Q_{j,k-1} \end{aligned}\right\} \tag{58}$$

where the constant weighting factors α_{jk} *and* β_{jk} *are chosen, not both zero, so that when* $k + i$ *is even the coefficient of* $x^{(k-i)/2+1}$ *in* Q_{jk} *vanishes and when* $k + i$ *is odd the coefficient of* $x^{(k+i+1)/2}$ *in* P_{jk} *vanishes.*

This observation, which forms the starting point for Stoer's development, follows from the results of the previous Lemma and Theorem, and from equations (35) and (36).

Corollary 3:

If $f_{jk}(x)$ *possesses Property I it is unique and independent of the order of the given points* $\{x_r; j \leqslant r \leqslant j + k\}$.

This follows from considerations discussed in the introduction.

As mentioned earlier, the proof of Theorem 2 follows closely along the lines of the proof of Theorem 1. Corollary 3 to Theorem 1 also applies to Theorem 2, as do the following two corollaries which are analagous to Corollaries 1 and 2.

Corollary 4:

If the functions $\{f_{jk}\}$ *in Table* 1 *are constructed by means of Algorithm* B_i, *then for all* $j \geqslant 1, k \geqslant i + 1$, *except when* $f_{jk} \equiv f_{j,k-1} \equiv f_{j+1,k-1}$,

$$\deg \{P_{jk}\} = \frac{k-i+1}{2}, \text{ whenever } k + i \text{ is odd} \tag{59}$$

and

$$\deg \{Q_{jk}\} = \frac{k+i}{2}, \qquad \text{whenever } k + i \text{ is even.} \tag{60}$$

Corollary 5:

If the functions $\{f_{jk}\}$ *in Table* 1 *are constructed by means of Algorithm* B_i *the polynomials* $\{P_{jk}(x), Q_{jk}(x); j \geqslant 1, k \geqslant i + 1\}$ *satisfy the recurrence relations*

$$\left.\begin{aligned} P_{jk} &= \alpha_{jk} . (x - x_j) . P_{j+1,k-1} \\ &\quad + \beta_{jk} . (x_{j+k} - x) . P_{j,k-1} \\ Q_{jk} &= \alpha_{jk} . (x - x_j) . Q_{j+1,k-1} \\ &\quad + \beta_{jk} . (x_{j+k} - x) . Q_{j,k-1} \end{aligned}\right\}, \tag{61}$$

where the constant weighting factors α_{jk} *and* β_{jk} *are chosen, not both zero, so that when* $k + i$ *is even the coefficient of* $x^{(k-i)/2+1}$ *in* P_{jk} *vanishes, and when* $k + i$ *is odd the coefficient of* $x^{(k+i+1)/2}$ *in* Q_{jk} *vanishes.*

5. Further remarks on the algorithms

The successive advances in the degrees of numerator and denominator of the function $\{f_{jk}\}$, as k increases, are shown schematically in **Fig. 2.** The three paths starting from the square (0, 0) illustrate columnar progressions of the three algorithms A_1, A_3 and B_4. Notice also that all the $\{f_{jk}\}$ which are not either polynomials or inverse polynomials can, in general, be constructed by two separate algorithms; for example, functions with numerators of degree 3, 4, 5, etc., and corresponding denominators of degree 1, 2, 3 . . . etc., may be constructed both by Algorithm A_2 and Algorithm A_3. Furthermore, it is clear that, by consulting the diagram in Fig. 2, we can choose algorithms specifically for the purpose of interpolating given points by a rational function with prescribed degrees for its numerator and denominator. **Table 3** illustrates the use of Algorithm A_3 in constructing an interpolant with numerator of degree 3 and denominator of degree 1.

Algorithm A_1 indicated by the solid line in Fig. 2, is of particular interest since it completes the analogy between the classical Newton–Neville–Aitken techniques for

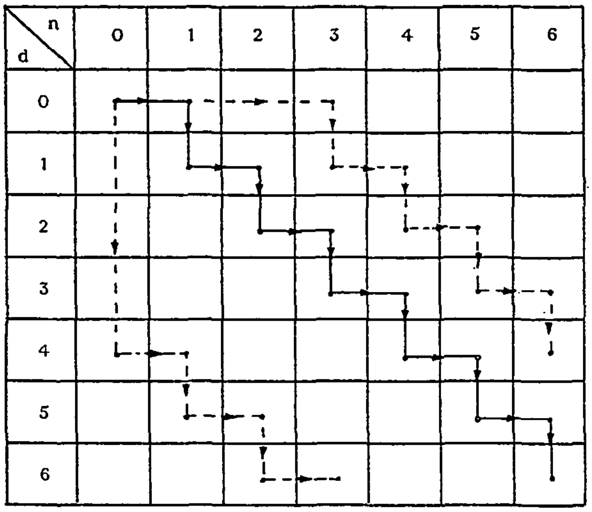

Fig. 2.—Representation of successive advances in the degree of numerator and denominator or rational interpolating functions with successive increments in *k*. *n* and *d* indicate permitted maximum degrees of numerator and denominator respectively

polynomial interpolation and the Thiele continued fraction method for the special type of rational interpolation mentioned in the introduction. It is well known that the rationalized form of $T(x)$, in equation (4) satisfies the same restrictions upon the degrees of its numerator and denominator as does $f_{1,p+q+1}$, constructed from the points $\{(x_s, f_s);\ 1 \leqslant s \leqslant p + q + 1\}$ by means of Algorithm A_1. Hence, if $f_{1,p+q+1}$ possesses Property I it is unique and therefore identical with $T(x)$.

If we express the Newton interpolating polynomial in the form

$$N(x) = b_0 + (x - x_1)\{b_1 + (x - x_2)\{b_3 + \dots (x - x_{p+q})b_{p+q}\}\}, \quad (62)$$

where the constants $\{b_s;\ 0 \leqslant s \leqslant p + q\}$ are constructed by means of a table of divided differences, the correspondence between the two forms of interpolation is easily seen from **Table 4.**

It is of interest, also, to consider the limiting form of the table generated by Algorithm A_1 (or, equally, by Algorithm B_1) as the interpolation point x moves to

Table 3

Construction of a rational interpolation function of prescribed form

j	x_j	f_j	$k = 1$	2	3	4
1	-2	1				
			$x + 3$			
2	-1	2		$-\dfrac{3x^2 + 7x}{2}$		
			$-2x$		$\dfrac{5x^3 + 6x^2 - 11x}{6}$	
3	0	0		$x^2 - x$		$\dfrac{4x^3 + 3x^2 - 7x}{3(3x + 4)}$
			0		$-\dfrac{x^3 - 6x^2 + 5x}{6}$	
4	1	0		$\dfrac{x^2 - x}{2}$		
			$x - 1$			
5	2	1				

Table 4

Illustration of analogy between Newton and Thiele interpolations

Interpolating function	N(x)	T(x)
Method of constructing coefficients	Table of divided differences	Table of inverted, or reciprocal, differences
Method of direct construction	Neville–Aitken algorithm	Algorithm A_1

Table 5

Scheme of calculation for rational extrapolation to infinity

x_1	f_1						
		e_{11}					
x_2	f_2		e_{12}				
		e_{21}		e_{13}			
x_3	f_3		e_{22}		e_{14}		
		e_{31}		e_{23}	. .		
x_4	f_4		e_{32}	.	.	.	
		e_{41}	.	.	.		.
x_5	f_5	.	.		.		
.	.	.	.		.		
.	.	.					
.	.						
.	.						
.							

infinity. From Theorem 1 we know that, in general, $f_{jk}(x)$ will have a simple pole at infinity when k is odd and a finite value when k is even. Accordingly, let us construct a table, of the same form as Table 1, by listing extrapolated values of the quantities $\{f_{jk}; j \geqslant 1, k \text{ even}\}$ in the even-k columns. However, in the odd-k columns we shall list reciprocals of the residues, at the assumed simple pole at infinity, of the functions $\{f_{jk}; j \geqslant 1, k \text{ odd}\}$. **Table 5** illustrates the scheme.

It is easily verified that the numbers $\{e_{jk}; j \geqslant 1, k \geqslant 1\}$ may be constructed, both for odd and even k, by the single recurrence relation

$$e_{jk} = e_{j+1,k-2} + \frac{x_{j+k} - x_j}{e_{j+1,k-1} - e_{j,k-1}}, \tag{63}$$

with starting conditions

$$\left.\begin{aligned} e_{j0} &= f_j \\ e_{j,-1} &= 0 \end{aligned}\right\}. \tag{64}$$

We then consult the "highest-j" members of the even-k columns of Table 5 for estimates of the value of

$$\lim_{x\to\infty} f(x).$$

In the special case when

$$x_j = j\,;\; j = 1, 2, 3 \ldots \tag{65}$$

equation (63) reduces to

$$e_{jk} = e_{j+1,k-2} + \frac{k}{e_{j+1,k-1} - e_{j,k-1}}, \tag{66}$$

a formula which is given by Wynn (1958).

Notice that, whereas the direct application of Algorithms $\{A_i\}$ and $\{B_i\}$ provides efficient means for interpolating numerically at a single, specified position x, Corollaries 2 and 5 give us convenient methods for evaluating the coefficients in the polynomials $\{P_{jk}(x)\}$ and $\{Q_{jk}(x)\}$.

Let us write

$$\left.\begin{aligned} P_{jk}(x) &= \sum_r p_{jkr}.x^r \\ Q_{jk}(x) &= \sum_r q_{jkr}.x^r \end{aligned}\right\}. \tag{67}$$

Then, from equations (58), recurrence relations for the coefficients $\{p_{jkr}\}$ and $\{q_{jkr}\}$ may be written in the form

$$\left.\begin{aligned} p_{jkr} = \alpha_{jk}.(p_{j+1,k-1,r-1} - x_j.p_{j+1,k-1,r}) \\ + \beta_{jk}.(x_{j+k}.p_{j,k-1,r} - p_{j,k-1,r-1}) \\ q_{jkr} = \alpha_{jk}.(q_{j+1,k-1,r-1} - x_j.q_{j+1,k-1,r}) \\ + \beta_{jk}.(x_{j+k}.q_{j,k-1,r} - q_{j,k-1,r-1}) \end{aligned}\right\}, \tag{68}$$

with starting conditions

$$p_{jk,-1} = q_{jk,1} = 0\,;\; j \geqslant 1, k \geqslant 0 \tag{69}$$

$$\left.\begin{aligned} p_{j00} &= f_j \\ q_{j00} &= 1 \end{aligned}\right\}; j \geqslant 1, \tag{70}$$

and for Algorithm A_i

$$\left.\begin{aligned} \alpha_{jk} &= \frac{q_{j,k-1,0}}{q_{j,k-1,0} + q_{j+1,k-1,0}} \\ \beta_{jk} &= \frac{q_{j+1,k-1,0}}{q_{j,k-1,0} + q_{j+1,k-1,0}} \end{aligned}\right\};\; \begin{aligned} j &\geqslant 1 \\ 1 \leqslant k &\leqslant i, \end{aligned} \tag{71}$$

$$\left.\begin{aligned} \alpha_{jk} &= \frac{p_{j,k-1,(k+i-1)/2}}{p_{j,k-1,(k+i-1)/2} + p_{j+1,k-1,(k+i-1)/2}} \\ \beta_{jk} &= \frac{p_{j+1,k-1,(k+i-1)/2}}{p_{j,k-1,(k+i-1)/2} + p_{j+1,k-1,(k+i-1)/2}} \end{aligned}\right\}\; \begin{aligned} &j \geqslant 1, \\ &;k \geqslant i+1 \\ &k+i \text{ odd}. \end{aligned} \tag{72}$$

$$\left.\begin{aligned} \alpha_{jk} &= \frac{q_{j,k-1,(k-i)/2}}{q_{j,k-1,(k-i)/2} + q_{j+1,k-1,(k-i)/2}} \\ \beta_{jk} &= \frac{q_{j+1,k-1,(k-i)/2}}{q_{j,k-1,(k-i)/2} + q_{j+1,k-1,(k-i)/2}} \end{aligned}\right\}\; \begin{aligned} &j \geqslant 1 \\ &; k \geqslant i+1 \\ &k + i \text{ even}. \end{aligned} \tag{73}$$

Formulae (71), (72) and (73) are simply precise statements of the obvious rules for choosing the $\{\alpha_{jk}\}$ and $\{\beta_{jk}\}$ in order to suppress increments in the degrees of numerators or denominators of the $\{f_{jk}\}$ at appropriate stages in the construction of Table 1. Similar formulae apply when constructing the interpolating functions generated by Algorithm B_i.

Table 3 illustrated the construction of a rational function having not more than one pole. Like Table 2, it may be regarded as having been developed, either directly from Algorithm A_3, or by application of rules (71), (72) and (73). From the latter viewpoint we can regard f_{14}, for example, as constructed from

$$f_{14}(x) = \frac{5.(x+2).(x^3 - 6x^2 + 5x) + (2-x).(5x^3 + 6x^2 - 11x)}{6\{-5(x+2) + (2-x)\}}$$

i.e.

$$f_{14} = \frac{4x^3 + 3x^2 - 7x}{3(3x+4)}. \tag{74}$$

6. Acknowledgement

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Book Review

The Memory System of the Brain, by J. Z. Young, 1967, 128 pages. (London: *Oxford University Press*, 28s.)

In any computer, whether in flesh or metal, there must be two main classes of mechanism, clocks and stores. This book is a collection of lectures delivered by Professor Young in California in 1964 about some properties of the latter, the storage mechanisms, but "the Brain" is not of man but of Octopus. As in other works in this domain the definite article is still misleading, and authors may envy their Russian colleagues whose language enforces ambiguity of reference.

Lacking a systematic functional taxonomy, neurophysiologists must be content with whatever information they can glean from any creature whose behaviour can be observed without too much disturbance or expense. Young and his colleagues have exploited the cephalopod with incomparable ingenuity and patience for many years and have been able to identify specific structure-function relationships more confidently than would be possible in a more advanced animal.

The natural history of Octopus is particularly fascinating and this is surveyed in more detail in Young's longer work, *A Model of the Brain*, but in these lectures too there is enough detail for the reader to be able to identify not only with the author but with the subject-animal. As Young admits, the first step in cryptography is often to guess what the messages are likely to be about, and here this means looking at the submarine world with the large and liquid eye of an Octopus. One has eight sensitive and powerful tentacles, excellent eyesight (with colour-vision) and a voracious appetite particularly for small crabs. One lives in a stone-built cottage in a charming seascape but difficult of access. One can crawl or boost one's power by jet-propulsion. When in trouble one can withdraw to one's home behind a smoke screen. The sea being, as we all know, cruel, one must approach a doubtful quarry cautiously, ready to retreat at once if it bites, but prepared to attack more quickly if it is harmless and edible. One can learn from such experiences in a few trials, remember the lesson for several weeks and change one's mind if first impressions turn out wrong. One has two memory stores in distinct locations each with specialized storage elements. One is thus almost ideally fitted to be the prey of the universal predator—the human scientist. The only slight protection against extensive interference is that, living in salt water, one is unlikely to have electrodes implanted in one's brain as happens to most terrestrial animals. Surgical mutilation, however, cannot be avoided, and it is mainly by this means that one contributes to human knowledge.

What hypotheses have been constructed from observations on this obliging beast? The basic proposition is "that learning consists in the limitation of choice between alternatives". The establishment of the models of the alternatives in the memory of an animal is "like the printing of a book in that it involves selecting appropriate items from a pre-established alphabet". But since each species of animal has a limited repertoire of responses, its stock of symbols must also be limited. "Brains are not general-purpose computers but specialized analogues." Whether this last assertion is universally true and useful is still a matter for discussion and experiment. The human brain seems to have such a vast capacity and to be so nearly independent of hereditary constraints that any statement about the class of computer to which it belongs would have to be qualified by definition of the conditions in which it is to be used or studied.

No analogy or metaphor can conceal the overwhelming magnitude of the human brain. That there are 10,000,000,000 neurons is bad, but not incommensurable with the imaginable scale of the artificial molecular circuitry of the future. If these were living flip-chips we could hope to penetrate their logic. But on some of these cells there are at least 10,000 contacts and in some of these contacts there may be thousands of sub-microscopic chemical vesicles.

It is because of this intractable complexity that Octopus brain is so reassuring—combined of course with the supposition, based more on hope than conviction, that there may be principles common to cephalopod, computer and ourselves. Assuming that learning in all cases involves a modification of memory resulting from a choice between two specific alternatives—a binary decision, how is this choice or decision implemented? In the simplest models one can start either with a system in which all channels start open and those unused or ineffective are progressively blocked, or with one in which no throughput is possible until associations have been classified and accepted, when the appropriate channels are opened for these specific contingencies. It is the first system that is suggested by Young's studies of Octopus, that is, learning by blocking un-needed channels. From the biologic standpoint this hypothesis has the advantage that the raw material for conditional inhibition is present at an elementary level of evolution in the negative feedback pathways that are so prominent in primitive reflex action. The complementary positive feedback circuits exist, but they would leave the alternative channels open, which would seem extravagant.

(*Continued on p. 189*)