A numerical approach to biharmonic problems*

By Donald Greenspan†

A new numerical method is described for biharmonic problems. The essence of the technique is to combine variational and difference methods in approximating a functional for which the biharmonic equation is an Euler equation. Typical problems and computer results are described.

1. Introduction

The solution of biharmonic problems has long been of importance in the study of elasticity (see, e.g., references [4], [7], [14]) and the advent of the high speed digital computer has been conducive to the development of new, and renewed interest in old, numerical techniques for such problems (see [2], [3], [6]–[8], [11]–[13], [15], [16]). We shall explore here a new numerical method which is distinctly different from those already mentioned.

2. The problems

The two fundamental problems associated with the biharmonic equation can be described as follows.

Let R be a simply connected, bounded region whose boundary is S. The positive normal direction, when it exists, at a point of S will be the *outward* direction. Let $\phi_1(x, y)$, $\phi_2(x, y)$ and $\phi_3(x, y)$ be defined on S. Then,

Problem 1. Find a function $u(x, y)$ which is defined and continuous on $R \cup S$, which satisfies on R the biharmonic equation

$$\Delta\Delta u \equiv \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0 \tag{2.1}$$

and which satisfies on S the boundary conditions

$$u \equiv \phi_1 \tag{2.2}$$

$$\frac{\partial u}{\partial n} \equiv \phi_2. \tag{2.3}$$

Problem 2. Find a function $u(x, y)$ which is defined and continuous on $R \cup S$, which satisfies (2.1) on R, and which satisfies on S the boundary conditions (2.2) and

$$\frac{\partial^2 u}{\partial n^2} \equiv \phi_3. \tag{2.3'}$$

Although for Problems 1 and 2, above, existence and uniqueness theorems are available under various restrictive assumptions on ϕ_1, ϕ_2, ϕ_3 and S, (see, for example, Miranda (1955) and Schroder (1943)) there exists, in general, no constructive method for producing a solution. For such problems, then, attention will be directed here toward developing a high speed, digital computer method for approximating a solution. Emphasis will be placed on Problem 1 since, as indicated in Section 4, for the important case when R is rectangular all the numerical difficulties present in Problem 2 are also present in Problem 1.

3. Numerical preliminaries

For h_x, h_y positive constants, and for fixed $(\bar{x}, \bar{y}) \in (R \cup S)$, the set of points $(\bar{x} + ph_x, \bar{y} + qh_y)$; $p = 0, \pm 1, \pm 2, \ldots$; $q = 0, \pm 1, \pm 2, \ldots$, is called a *set of lattice points*. The horizontal and vertical lines through a set of lattice points will be called *a set of lattice lines*. Now, define R_h to be those lattice points which are also points of R and define S_h to be those points of S which are also points of at least one lattice line. With regard to $R \cup S$, then, R_h is said to be a set of interior lattice points while S_h is said to be a set of boundary lattice points. Throughout it will be assumed that R_h contains m points, numbered $1, 2, \ldots, m$, and that S_h contains n points, numbered $m + 1$, $m + 2, \ldots$, $m + n$. Further, if a point $(x, y) \in (R_h \cup S_h)$ has been numbered r, it will be convenient in practice to use the subscript notation $u(x, y) = u_r$.

Next, it will be useful to describe certain finite difference approximations for second order derivatives. For this purpose, let the points (x, y), $(x + h_x, y)$, $(x - h_x, y)$ be denoted 0, 1, 2, respectively, as in **Fig. 1**(*a*). Then we shall utilize the well-known (see, e.g., Milne (1949), difference approximation

$$u_{xx}|_0 \sim \frac{u_1 - 2u_0 + u_2.}{(h_x)^2} \tag{3.1}$$

If (x, y), $(x, y + h_y)$, $(x, y - h_y)$ are denoted by 0, 1, 2, respectively, as in **Fig. 1**(*b*), a formula for $u_{yy}|_0$ completely analogous to (3.1) is valid. Next let (x, y), $(x + h_x, y)$ be denoted 0, 1, respectively, as in **Fig. 1**(*c*). Then, assuming a valid Taylor expansion, one has approximately

$$u_1 \sim u_0 + h_x u_x|_0 + \frac{h_x^2}{2} u_{xx}|_0,$$

so that

$$u_{xx}|_0 \sim \frac{2(u_1 - u_0 - h_x u_x|_0)}{(h_x)^2}. \tag{3.2}$$

* Sponsored by the Mathematics Research Center, United States Army, Madison, Wisconsin, under Contract No.: DA-31-124-ARO-D-462.

† *The University of Wisconsin, Madison, Wisconsin, U.S.A.*

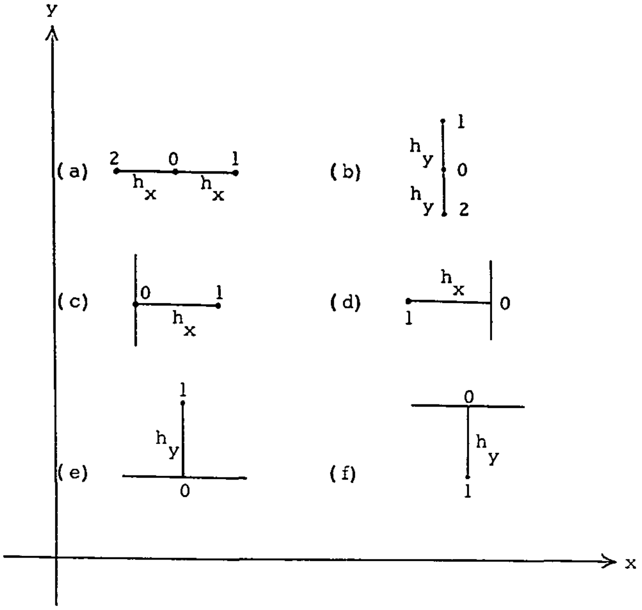

Fig. 1

Similarly, if (x, y), $(x - h_x, y)$ are denoted 0, 1, respectively, as in **Fig. 1**(*d*), then one has

$$u_{xx}|_0 \sim \frac{2(u_1 - u_0 + h_x u_x|_0)}{(h_x)^2}. \tag{3.3}$$

If (x, y), $(x, y + h_y)$ are denoted 0, 1, respectively, as in **Fig. 1**(*e*), then a formula for $u_{yy}|_0$ completely analogous to (3.2) is valid, while if (x, y), $(x, y - h_y)$ are denoted 0, 1, respectively, as in **Fig. 1**(*f*), then a formula for $u_{yy}|_0$ completely analogous to (3.3) is valid.

4. The numerical method

Instead of approaching the problems of Section 2 directly, we shall consider (2.1) to be an Euler equation (see Courant and Hilbert, 1962) and adapt the approximation method of Greenspan (1966) for minimizing the associated functional:

$$\frac{1}{2}\iint_{R \cup S} (u_{xx}^2 + 2u_{xx}u_{yy} + u_{yy}^2)dA. \tag{4.1}$$

For fixed $h_x > 0$ and $h_y > 0$, construct and label R_h and S_h as described in Section 3. Suppose the lattice lines subdivide R into k^* subregions $R_1, R_2, \ldots, R_{k^*}$. If any one of these subregions has a polygonal boundary, then further subdivide it into triangular regions by the insertion of diagonals. Assume then that there results finally a subdivision of R into $k(\geqslant k^*)$ subregions $R_1, R_2, \ldots, R_k$, the respective boundaries of which are $S_1, S_2, \ldots, S_k$, and the respective areas of which are $A_1, A_2, \ldots, A_k$. To each R_i; $i = 1, 2, \ldots, k$, associate if possible on S_i a point of $R_h \cup S_h$ at which u_{xx} and u_{yy} are known, or at which one can approximate u_{xx} and u_{yy} by difference approximations $u_{xx,\,i}$ and $u_{yy,\,i}$, respectively, each of which utilizes only function values at points of R_h and/or function values plus first order partial derivatives at points of S_h. Then, approximate functional (4.1) by the function

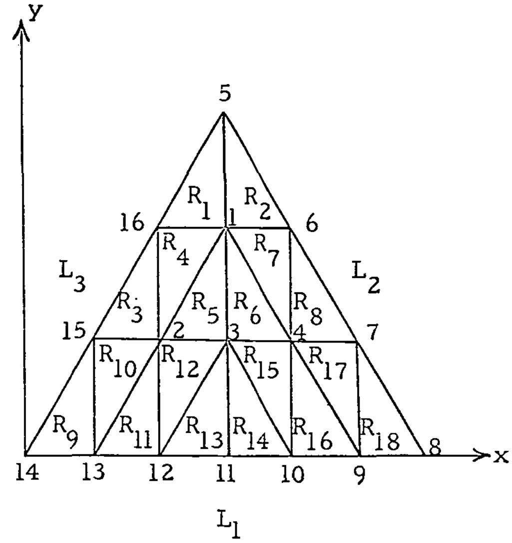

Fig. 2

$$J_k = \frac{1}{2}\sum_{i=1}^{k} \{A_i[(u_{xx,\,i})^2 + 2u_{xx,\,i}u_{yy,\,i} + (u_{yy,\,i})^2]\}. \tag{4.2}$$

Since J_k is a function only of $u_1, u_2, \ldots, u_m$, we attempt to find an extremal of J_k by considering the linear algebraic system

$$\frac{\partial J_k}{\partial u_r} = 0,\ r = 1, 2, \ldots, m. \tag{4.3}$$

Let the solution of system (4.3) constitute an approximation of $u(x, y)$ at the points numbered $1, 2, \ldots, m$ of R_h.

Illustrative example

Let S be the triangle with vertices (0, 0), (1, 0), $(\frac{1}{2}, 1)$. (Consult **Fig. 2** in connection with the present discussion.) Let R be the interior of S and let

$$\phi_1 = x^3 - 2y^2 \text{ on } S. \tag{4.4}$$

Denote by L_1, L_2, L_3 the sides of S which join (0, 0) to (1, 0), (1, 0) to $(\frac{1}{2}, 1)$, $(\frac{1}{2}, 1)$ to (0, 0), respectively. Let

$$\phi_2 \equiv \begin{cases} 0, & \text{on } L_1 \\ \dfrac{6}{\sqrt{5}}x^2 - \dfrac{4}{\sqrt{5}}y, & \text{on } L_2 \\ -\dfrac{6}{\sqrt{5}}x^2 - \dfrac{4}{\sqrt{5}}y, & \text{on } L_3. \end{cases} \tag{4.5}$$

In order to approximate the solution of the biharmonic problem then defined by (2.1)–(2.3), set $(\bar{x}, \bar{y}) = (0, 0)$, $h_x = \frac{1}{6}$, $h_y = \frac{1}{3}$. Construct the lattice lines and triangulate those rectangular subregions which result, as shown in Fig. 2, so that R is divided into the eighteen subregions $R_1, R_2, \ldots, R_{18}$. Finally, to each R_i associate the vertex of the right angle of the triangular boundary of R_i.

Now, from (4.4) one knows $u_5, u_6, \ldots, u_{16}$ exactly. In order to calculate u_x and u_y at various points of S_h, we shall utilize both (4.4) and (4.5) in the following fashion. In general, let S be given parametrically in terms of parameter s (arc length) from some fixed point. Also, let s increase as one traverses S in the counter clock-wise direction. Then by means of (2.2) one can calculate at each point the tangential derivative $\frac{\partial u}{\partial t}$, provided it exists. Then (see **Fig. 3**), one has from the definition of the directional derivative that

$$\begin{cases} \dfrac{\partial u}{\partial n} = u_x \cos \alpha + u_y \sin \alpha \\ \dfrac{\partial u}{\partial t} = u_x \cos \left(\alpha + \dfrac{\pi}{2}\right) + u_y \sin \left(\alpha + \dfrac{\pi}{2}\right). \end{cases} \quad (4.6)$$

Hence, from (4.6),

$$\begin{cases} u_x = u_n \cos \alpha - u_t \sin \alpha \\ u_y = u_n \sin \alpha + u_t \cos \alpha. \end{cases} \quad (4.7)$$

Thus, one can show readily with regard to (4.4), (4.5) and Fig. 2 that

$$\begin{aligned} u_x|_6 &= \tfrac{4}{3}, & u_y|_9 &= 0, & u_x|_{15} &= \tfrac{1}{12}, \\ u_y|_6 &= -\tfrac{8}{3}, & u_y|_{10} &= 0, & u_y|_{15} &= -\tfrac{4}{3}, \\ u_x|_7 &= \tfrac{25}{12}, & u_y|_{11} &= 0, & u_x|_{16} &= \tfrac{1}{3}, \\ u_y|_7 &= -\tfrac{4}{3}, & u_y|_{12} &= 0, & u_y|_{16} &= -\tfrac{8}{3}, \\ & & u_y|_{13} &= 0. \end{aligned} \quad (4.8)$$

Next, for convenience, set

$$J_{18}^{(i)} = A_i[(u_{xx,\,i})^2 + 2(u_{xx,\,i})(u_{yy,\,i}) + (u_{yy,\,i})^2]; \quad i = 1, 2, \ldots, 18.$$

Thus

$$J_{18} = \tfrac{1}{2} \sum_{i=1}^{18} J_{18}^{(i)},$$

and we proceed to determine

$$J_{18}^{(1)}, J_{18}^{(2)}, \ldots, J_{18}^{(18)}.$$

With regard to $J_{18}^{(1)}$, the area A_1 of R_1 is $\frac{1}{36}$. Moreover, since the point associated with R_1 is the vertex of the right angle of S_1, that is, the point numbered 1 in Fig. 2, then

$$u_{xx}|_1 \sim \frac{u_6 - 2u_1 + u_{16}}{1/36} = \frac{-\frac{16}{27} - 2u_1 - \frac{23}{27}}{1/36} = \frac{-2u_1 - \frac{13}{9}}{1/36}$$

$$u_{yy}|_1 \sim \frac{u_5 - 2u_1 + u_3}{1/9} = \frac{-\frac{15}{8} - 2u_1 + u_3}{1/9}.$$

Thus

$$J_{18}^{(1)} = \tfrac{1}{36}\,[1296(2u_1 + \tfrac{13}{9})^2 + 648(-2u_1 - \tfrac{13}{9}) \times \times (-2u_1 + u_3 - \tfrac{15}{8}) + 81(2u_1 - u_3 + \tfrac{15}{8})^2]. \quad (4.9)$$

On the other hand, when considering a subregion like R_4,

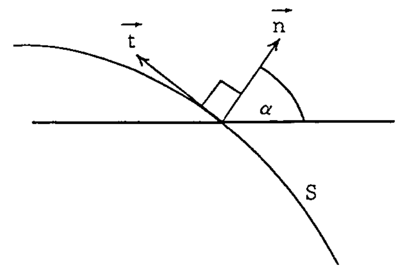

Fig. 3

one has

$$u_{xx}|_{16} \sim \frac{2(u_1 - u_{16} - \frac{1}{6}u_x|_{16})}{1/36}$$

$$u_{yy}|_{16} \sim \frac{2(u_2 - u_{16} + \frac{1}{3}u_y|_{16})}{1/9},$$

so that

$$J_{18}^{(4)} = \tfrac{1}{36}\,[5184(u_1 + \tfrac{43}{54})^2 + 2592(u_1 + \tfrac{43}{54})(u_2 - \tfrac{1}{27}) + 324(u_2 - \tfrac{1}{27})^2]. \quad (4.10)$$

Similarly, all the other $J_{18}^{(i)}$ can be readily constructed.

It follows then that (4.3) is equivalent to the system

$$\begin{cases} 117u_1 + 6u_2 - 20u_3 + 6u_4 = -88{\cdot}875 \\ 6u_1 + 125u_2 - 80u_3 + 16u_4 = -20{\cdot}148 \\ -20u_1 - 80u_2 + 134u_3 - 80u_4 = 11{\cdot}083 \\ 6u_1 + 16u_2 - 80u_3 + 125u_4 = 9{\cdot}875 \end{cases} \quad (4.11)$$

The coefficient matrix of (4.11) is symmetric and positive definite and the solution of the system is approximately

$$u_1 = -0{\cdot}776 \quad u_2 = -0{\cdot}179 \quad u_3 = -0{\cdot}081 \quad u_4 = 0{\cdot}082. \quad (4.27)$$

However, the analytical solution of the given boundary value problem is known in this case to be $u = x^3 - 2y^2$. At the points numbered 1, 2, 3, 4 then the exact solution is

$$u_1 = -\tfrac{55}{72},\ u_2 = -\tfrac{5}{27},\ u_3 = -\tfrac{7}{72},\ u_4 = \tfrac{2}{27},$$

with which approximate solution (4.27) compares most favourably.

Note finally that the method described in this section applies even more easily to the prototype problems of the second kind (see, e.g. [2]) described in Section 2. For suppose S is a rectangle and R is the interior of S. Without loss of generality, assume that the sides of S are parallel to the x and y axes. Then condition (2.3′) is equivalent to prescribing u_{xx} on the vertical sides of S and u_{yy} on the horizontal sides of S. Moreover, from (2.2) one can calculate explicitly u_{xx} on the horizontal sides of S and u_{yy} on the vertical sides of S. Thus, u_{xx} and u_{yy} are known at each point of S, and hence of S_h, and there is no need to approximate these at points of S_h. The resulting calculation, and hence the numerical method itself, thereby simplify considerably.

5. Further examples

A number of examples in which S was selected to be square, rectangular, triangular and trapezoidal have been run on the CDC 1604 and the CDC 3600 at the University of Wisconsin. The following typical ones are presented to illustrate the ease with which the method of Section 4 can be applied.

Example 1. All the details were the same as in the illustrative example of Section 4 except for the choices $h_x = 0\cdot025$, $h_y = 0\cdot050$. The resulting set of 361 linear algebraic equations was solved by successive over-relaxation with zero initial vector and $\omega = 1\cdot85$, determined by the method of Carré (1961). The running time on the CDC 1604 was 29 minutes and the number of iterations was 370. The maximum error was $0\cdot0097$ and it occurred at the point $x = 0\cdot4750$, $y = 0\cdot4000$.

Example 2. Let S be the square with vertices (0, 0), (1, 0), (1, 1), (0, 1) and let R be the interior of S. Define

$$\phi_1 = x^3 - 2y^2, \text{ on } S. \tag{5.1}$$

Let the sides of the square joining (0, 0) to (1, 0), (1, 0) to (1, 1), (1, 1) to (0, 1), (0, 1) to (0, 0) be denoted, respectively, by L_1, L_2, L_3, L_4. Define

$$\phi_2 = \begin{cases} 4y, & \text{on } L_1 \\ 3x^2, & \text{on } L_2 \\ -4y, & \text{on } L_3 \\ -3x^2, & \text{on } L_4. \end{cases} \tag{5.2}$$

In order to approximate the solution of the biharmonic problem then defined by (2.1)–(2.3), (5.1) and (5.2), we set $(\bar{x}, \bar{y}) = (0, 0)$ and $h_x = h_y = 0\cdot05$. R was triangulated in a fashion analogous to that described in the illustrative example of Section 4 and to each subregion R_i was associated the vertex of the right angle of the boundary S_i of R_i. The system (4.3) of 361 linear algebraic equations which resulted in this example was solved by successive over-relaxation with zero initial vector and $\omega = 1\cdot7$. The running time on the CDC 3600 was 24 minutes and the number of iterations was 1101. The maximum error was $0\cdot00035$ and it occurred at the point $x = 0\cdot65$, $y = 0\cdot50$. The exact solution is $u = x^3 - 2y^2$.

With regard to the latter example, it is also of interest to note that when the grid was selected with $h_x = h_y = 0\cdot01$, then the convergence of successive over-relaxation with each of $\omega = 0\cdot5, 0\cdot6, 0\cdot7, \ldots, 1\cdot9$ was so slow that no significant results were obtained, even though the CDC 3600 was allowed to compute for periods up to three hours.

6. Concluding remarks

For the method of this paper, convergence proofs and treatment of curved boundaries for large classes of problems have been given only recently [5]. No error estimates are as yet available, since the method of proof in [5] is that of the classical calculus of variations.

Note also that the method of Section 4 can be modified easily so as to incorporate higher order formulas for numerical integration and differentiation. Recent calculations indicate that such modifications often result in increased accuracy when the given data is sufficiently smooth.

References

1. Carré, B. A. (1961). "The determination of the optimum accelerating factor for successive over-relaxation", *The Computer Journal*, Vol. 4, pp. 73–78.
2. Conte, S. D., and Dames, R. T. (1960). "On an alternating direction method for solving the plate problem with mixed boundary conditions", *Jour. Assoc. Comp. Mach.*, Vol. 7, pp. 264–273.
3. Courant, R., Friedrichs, K., and Lewy, H. (1928). "Über die partielle Differentialgleichungen der mathematischen Physik", *Math. Ann.*, Vol. 100, pp. 32–74.
4. Courant, R., and Hilbert, D. (1962). *Methods of Mathematical Physics*, Vol. I, Interscience, New York, p. 192.
5. Greenspan, D. (1966). "On approximating extremals of functionals. Part II", Tech. Rpt. 689, Math. Res. Ctr., Univ. Wisconsin. (To appear in the *Int. Journ. of Eng. Sci.*)
6. Griffin, D. S., and Varga, R. S. (1963). "Numerical solution of plane elasticity problems", *Jour. SIAM*, Vol. 11, pp. 1046–1062.
7. Kantorovich, L. V., and Krylov, V. I. (1958). *Approximate Methods of Higher Analysis*, Noordhoff, Groningen, pp. 61–66.
8. Mangasarian, O. L. (1963). "Numerical solution of the first biharmonic problem by linear programming", *Int. Jour. Eng. Sci.*, Vol. 1, pp. 231–240.
9. Milne, W. E. (1949). *Numerical Calculus*, Princeton Univ. Press, Princeton, p. 142.
10. Miranda, C. (1955). *Equazioni alle derivate parziali di tipo ellitico*. Springer, Berlin, pp. 178–184.
11. Omarov, E. O. (1963). "An approximate solution to a partial differential equation of the type $u_{xxxx} + au_{xxyy} + bu_{yyyy} = f(x, y)$ by the method of lines in a trapezoidal region", *Izv. Akad. Nauk Uz SSSR Ser. Fiz.-Mat. Nauk*, No. 4, pp. 33–38.
12. Parter, S. V. (1959). "On 'two-line' iterative methods for the Laplace and biharmonic difference equations", *Num. Math.*, Vol. 1, pp. 240–252.
13. Prikhodko, E. M. (1964). "Application of the method of summary representations to the numerical solution of biharmonic problems", *Dopovidi Akad. Nauk Ukr. R.S.R.*, No. 7, p. 856.
14. Schroder, K. (1943). "Zur Theorie der Randwertaufgaben der Differentialgleichungen $\Delta\Delta U = 0$", *Math. Zeit.*, Vol. 48, pp. 553–675.
15. Waschakidze, D. (1941). "On the numerical solution of the biharmonic equation", *Tbliss. Mat. Inst., Trudy*, Vol. 9, pp. 61–73.
16. Windsor, E. (1957). "Iterative solutions of biharmonic differential equations", Master's thesis, New York University.