$$+ \frac{125}{5832} D(\sqrt{3/5}) + O(h^6). \qquad (5.3)$$

All three formulae use twenty-seven points per sub-interval. With $h = \frac{1}{2}$, (5.3) gives the approximation $0 \cdot 725\ 524\ 912$ with error $0 \cdot 000\ 034\ 104$, while the upper and lower-sign formulae give errors of $-0 \cdot 000\ 011\ 343$ and $0 \cdot 000\ 007\ 835$, respectively. For $h = \frac{1}{4}$, when each formula uses 216 points, (5.3) gives an error of $0 \cdot 000\ 000\ 067$ and the two formulae (4.11) give errors of $0 \cdot 000\ 000\ 002$ and $0 \cdot 000\ 000\ 003$, respectively.

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Gaussian numerical integration of a function depending on a parameter

By H. Tompa*

It is shown that the advantages claimed for Romberg integration for the computation of definite integrals as functions of a parameter can also be obtained by using Gaussian integration, and that the latter is usually more economical in terms of computer time than the former.

In a recent note Rabinowitz (1966) drew attention to the advantages of using Romberg's technique (Romberg, 1955, and references given by Rabinowitz) for the integration of a set of functions depending on a parameter α if the integrand is of the form $h(x, \alpha) = f(x).g(x, \alpha)$. The advantage of Romberg's technique lies in the fact that the value of the integrand is required at fixed values x_j of the abscissa and so $f(x_j)$ can be stored after it has been first computed and need not be recomputed for each value of α; Rabinowitz gives the outline of a very ingenious FORTRAN program for putting this idea into effect.

Gauss' n-point formula for numerical integration has the advantage of giving higher precision for a given number of points at which the integrand is computed, but has the disadvantage of using different values of the abscissa for different values of n; at first sight this disadvantage seems to be overwhelming when the function to be integrated depends on a parameter. It is, however, possible to use an algorithm with a series of n-point Gaussian formulae with a fixed set of increasing values of n, so that the part $f(x)$ of $h(x, \alpha)$ which does not depend on α need only be computed once, but now at the abscissae corresponding to all the n-point formulae used; the abscissae and weight factors are available for all values of n up to 64 (Gawlick, 1958) and for some higher values (Davis and Rabinowitz, 1958).

It is the purpose of this note to show that the total number of points at which the integrand has to be computed is usually smaller in this scheme than in Romberg integration or in adaptive Simpson integration quoted by Rabinowitz.

It seems reasonable to take a set of values of n forming a geometric series, $n_j = n_1 q^{j-1}$, and we assume that the required precision of the integral be reached with the k-point formula. Then the integral has to be evaluated $(i + 1)$ times to confirm the precision, where n_i is the first integer of the set of n which equals or exceeds k. The integrand is computed N times where

$$N = \sum_{1}^{i+1} n_1 q^{j-1} = n_1 \frac{q^{i+1} - 1}{q - 1}$$

and if we put $n_1 q^{i-1}$ equal to k, which is approximately so, we have

$$N = (kq^2 - n_1)/(q - 1).$$

The value of N is a minimum for given k and n_1 if $q = 1 + (1 - n_1/k)^{1/2}$, which is just below 2 since k will usually be much larger than n_1.

The integrals from 0 to 1 of the seven functions which Rabinowitz has evaluated by Romberg integration and by adaptive Simpson integration to a precision of 10^{-3} and of 10^{-6} have been evaluated using a set of n-point Gaussian formulae with three series of values of n. In the first two series the values of n form a geometric series with $q = 2$, in series A with $n_1 = 2 (2, 4, 8, 16, \ldots)$, in series B with $n_1 = 3 (3, 6, 12, 24, \ldots)$; the third series, C, consists of the values of n of both series A and B in increasing order so that the ratio of successive values is alternatively 3/2 and 4/3 $(2, 3, 4, 6, 8, \ldots)$. **Table 1** gives the total number of points at which the integrand has been computed until two successive values of the integral agreed with a relative error of less than 10^{-3} or 10^{-6}, respectively; the corresponding figures for

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Table 1

Number of points at which the integrand has been computed for Romberg (R), adaptive Simpson (AS) and three schemes of Gaussian integration (A, B, C) in order to obtain the integral from 0 to 1 of the seven functions given in col. 1 with relative errors below 10^{-3} and 10^{-6}, respectively. The minimum number of points used is 9 for R, 19 for AS, 6 for A, 9 for B, 5 for C. Figures in parentheses are extrapolated values.

	R		AS		A		B		C	
Function $f(x)$	10^{-3}	10^{-6}	10^{-3}	10^{-6}	10^{-3}	10^{-6}	10^{-3}	10^{-6}	10^{-3}	10^{-6}
$x^{1/2}$	65	4097	55	199	30	(254)	21	189	23	219
$x^{3/2}$	17	129	19	91	14	62	9	45	9	51
$\frac{1}{1+x}$	9	33	19	55	14	30	9	21	9	23
$\frac{1}{1+x^4}$	17	65	19	67	14	30	9	21	9	23
$\frac{1}{1+\exp(x)}$	9	17	19	19	6	14	9	9	5	9
$\frac{x}{\exp(x)-1}$	9	17	19	19	6	14	9	9	5	9
$\frac{2}{2+\sin 10\pi x}$	65	257	163	883	(254)	(510)	189	(381)	219	(443)

Romberg integration and adaptive Simpson integration, as given by Rabinowitz, are also shown.

It is evident that this algorithm of Gaussian integration is in no way inferior to Romberg or adaptive Simpson integration, and that there is not much to choose between sets A, B, and C, with a slight preference for the last. It is to be expected that some adaptive scheme in which the integral is suitably subdivided will bring about an improvement for the last function because of its periodicity and for the first function because of the singularity of the derivative at the lower limit of integration. This has been confirmed in the latter case; if the integral 0 to 1 is evaluated as the sum of the integrals 0 to 0·14 and 0·14 to 1, the values given in the table for sets A, B, C are reduced to 20, 18, 14, respectively, for a relative accuracy of 10^{-3} and to 156, 114, 130, respectively, for a relative accuracy of 10^{-6}.

There would be no difficulty in applying Rabinowitz' scheme for computing $f(x)$ only once at any of the abscissae used in this scheme so that all the advantages claimed for Romberg integration in the evaluation of a definite integral as a function of a parameter are also obtained by the present scheme.

Note added in proof: At the time of writing, the formula with $n = 96$ was the highest-order Gaussian formula available to the author, and the precision of higher-order formulae was estimated by extrapolation; since then formulae with larger values of n have become available (Stroud and Secrest, 1966).

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