The acceleration of the Peaceman–Rachford method by Chebyshev polynomials

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1. Introduction

Considerable interest centres at the present time on the application of alternating direction implicit (ADI) procedures to the numerical solution of elliptic systems of equations (see for example the excellent book of Wachspress (1966)). Under model problem conditions such methods have very rapid convergence rates. It is the purpose of this note to show how such convergence rates may be improved on by building into the ADI method an acceleration procedure based on the use of Chebyshev polynomials.

In Section 2, the Chebyshev semi-iterative procedure is summarized; a fuller account may be found in Varga (1962). In Section 3, the ADI process is defined. Section 4 contains the new process applied to an ADI procedure with a constant acceleration parameter, whilst in Section 5 this new process is generalized to the important case when the ADI procedure has a cycle of acceleration parameters. An example is given in Section 6 where this process is applied to solving Laplace's equation by two well-known methods.

In Section 7 a more efficient version of the procedure outlined in Section 4 is derived. This procedure is of general value.

2. Chebyshev semi-iteration

We shall be interested in solving the linear matrix equation

$$Ax = b \tag{2.1}$$

where the matrix A is symmetric and positive definite. Equation (2.1) may be solved by an iterative procedure of the form

$$x^{(m+1)} = Mx^{(m)} + g \tag{2.2}$$

where the matrix M is a convergent symmetric matrix (i.e. $\rho(M) < 1$ where $\rho(M)$ is the spectral radius of the matrix M). An acceleration process can be applied to (2.2) in general, but we shall restrict ourselves to a consideration of (2.2) when the eigenvalue spectrum of the iteration matrix M satisfies

$$-\rho \leqslant \lambda_m \leqslant \rho \tag{2.3}$$

where $|M - \lambda_m I| = 0$, and $\rho \equiv \rho(M)$. Then, the Chebyshev acceleration process takes the form

$$x^{(m+1)} = w_m[Mx^{(m)} + g] + (1 - w_m)x^{(m-1)} \tag{2.4}$$

where the acceleration parameters w_i are defined by

$$\left.\begin{aligned} w_0 = 1 \quad w_1 &= \left(1 - \frac{\rho^2}{2}\right)^{-1} \\ w_{i+1} &= \left(1 - \frac{\rho^2}{4} w_i\right)^{-1} \qquad i = 1, 2, 3, \ldots \end{aligned}\right\} \tag{2.5}$$

so that $1 \leqslant w_i \leqslant 2$. A fuller account of the derivation of this process may be found in Varga (1962). Moreover the spectral radius of the process (2.4) may be shown to be

$$||\tilde{p}_m(M)|| = (w - 1)^{m/2}\left\{\frac{2}{1 + (w - 1)^m}\right\} \tag{2.6}$$

where w is the limit of the sequence defined by (2.5), that is

$$w = 2[1 + \sqrt{(1 - \rho^2)}]^{-1}$$

3. The ADI procedure

Let us now define our ADI procedure for the numerical solution of (2.1). We shall impose model problem conditions, that is, assume that there exists a decomposition of the matrix A in the form

$$A = H + V \tag{3.1}$$

where H and V are symmetric and commute. The Peaceman–Rachford ADI iterative process for solving (2.1) is given by

$$[r_m I + H]x^{(m+\frac{1}{2})} = [r_m I - V]x^{(m)} + b \tag{3.2}$$

$$[r_m I + V]x^{(m+1)} = [r_m I - H]x^{(m+\frac{1}{2})} + b \tag{3.3}$$

where r_m is an acceleration parameter, and I is the unit $n \times n$ matrix. Elimination of $x^{(m+\frac{1}{2})}$ from (3.2), (3.3) gives

$$\begin{aligned}[r_m I + H][r_m I + V]x^{(m+1)} \\ = [r_m I - H][r_m I - V]x^{(m)} + 2r_m b\end{aligned} \tag{3.4}$$

which is equivalent to the iterative process

$$\begin{aligned}x^{(m+1)} = [r_m I + V]^{-1}[r_m I + H]^{-1} \\ \{[r_m I - H][r_m I - V]x^{(m)} + 2r_m b\}.\end{aligned} \tag{3.5}$$

An alternative splitting to (3.2), (3.3) of the Peaceman–Rachford procedure is given by

$$[r_m I + H]x^{(m+\frac{1}{2})} = [r_m I - V]x^{(m)} + b \tag{3.2}$$

$$[r_m I + V]x^{(m+1)} = 2r_m x^{(m+\frac{1}{2})} - [r_m I - V]x^{(m)} \tag{3.6}$$

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and it will in fact be this form of the splitting which we shall employ later.

For the ADI process (3.5) to converge we require that the spectral radius of T_m (denoted by $\rho(T_m)$) be less than one in modulus, where

$$T_m = [r_m I + V]^{-1}[r_m I + H]^{-1}[r_m I - H][r_m I - V]. \tag{3.7}$$

The eigenvalues of the matrix T_m are of the form

$$\left[\frac{r_m - \lambda_i}{r_m + \lambda_i}\right]\left[\frac{r_m - \gamma_j}{r_m + \gamma_j}\right]$$

where

$$Hz = \lambda_i z, \quad Vz = \gamma_j z \quad \text{since } HV = VH.$$

If the parameter r_m is allowed to vary from one application of (3.5) to the next, we must then ensure that $\rho\left(\prod_{m=1}^{K} T_m\right) < 1$ where K is the number of parameters r_m. The eigenvalues of the matrix $\prod_{m=1}^{K} T_m$ are then of the form

$$\mu = \prod_{m=1}^{K}\left(\frac{r_m - \lambda_i}{r_m + \lambda_i}\right)\left(\frac{r_m - \gamma_j}{r_m + \gamma_j}\right).$$

The sequence of parameters $\{r_m\}$ are chosen to minimize the spectral radius $\rho_K = \rho\left(\prod_{m=1}^{K} T_m\right)$. This leads to a Chebyshev minimization problem and it follows that

$$-1 < -\rho_K \leqslant \mu \leqslant \rho_K < 1 \tag{3.8}$$

We shall not state here as to how the minimization is effected and the sequence of parameters obtained but refer to the exact solutions given in Wachspress (1966). The cycle of the parameters $\{r_m\}$ is repeated until convergence is achieved. We have now defined our ADI process.

4. Acceleration of stationary ADI process

In this section we will introduce a new process which combines the procedure in Section 2 with the ADI method in Section 3, restricted to the case of a single ADI acceleration parameter ($K = 1$).

For the ADI method (3.5) the Chebyshev acceleration process (2.4) takes the form

$$x^{(m+1)} = w_m[T_m x^{(m)} + 2[r_m I + V]^{-1}[r_m I + H]^{-1}b] + (1 - w_m)x^{(m-1)} \tag{4.1}$$

where the w_m are calculated from (2.5) with $\rho = \rho(T_m)$. Equation (4.1) can now be written in a form analogous to (3.4),

$$\begin{aligned}[r_m I + H][r_m I + V]x^{(m+1)} &= w_m\{[r_m I - H][r_m I - V]x^{(m)} + 2b\} \\ &\quad + (1 - w_m)[r_m I + H][r_m I + V]x^{(m-1)}\end{aligned} \tag{4.2}$$

of which the splitting analogous to (3.2) and (3.6) is given by

$$[r_m I + H]x^{(m+\frac{1}{2})} = [r_m I - V]x^{(m)} + b \tag{4.3}$$

$$[r_m I + V]x^{(m+1)} = w_m[2r_m x^{(m+\frac{1}{2})} - [r_m I - V]x^{(m)}] + (1 - w_m)[r_m I + V]x^{(m-1)}. \tag{4.4}$$

The equation (4.4) is now a three-level scheme but the additional term in $x^{(m-1)}$ affords no extra computation, only storage, since it would have been calculated during the application of the process on the interval $x^{(m-2)}$ to $x^{(m-1)}$. It would appear, therefore, that the process (4.3), (4.4) would not be appreciably more time consuming than the process (3.2), (3.6). Moreover, as we shall see in the next section, this new process produces more rapid convergence.

5. Acceleration of non-stationary ADI process

We shall now generalize the results of the previous section to the case of K-variable parameters in the ADI scheme. We will also show that the resulting process converges more rapidly than the Peaceman–Rachford procedure.

A general step in the $(m + 1)$th cycle of the ADI scheme takes the form

$$x^{(mK+i)} = T_i x^{(mK+i-1)} + g_i \quad (i = 1, \ldots K) \tag{5.1}$$

where

$$g_i = 2r_i[r_i I + V]^{-1}[r_i I + H]^{-1}b. \tag{5.2}$$

If we consider the result of applying (5.1) over a complete cycle we have

$$x^{((m+1)K)} = \prod_{i=1}^{K} T_i x^{(mK)} + g_K^*, \tag{5.3}$$

where

$$\prod_{i=1}^{K} T_i = T_K T_{K-1} \ldots T_1$$

and

$$g_K^* = g_K + T_K g_{K-1} + T_K T_{K-1} g_{K-2} + \ldots + T_K \ldots T_2 g_1. \tag{5.4}$$

The application of our semi-iterative procedure to (5.3) leads to

$$x^{((m+1)K)} = w_m\left[\prod_{i=1}^{K} T_i x^{(mK)} + g_K^*\right] + (1 - w_m)x^{((m-1)K)}. \tag{5.5}$$

Making use of (5.4) we can re-write (5.5) in the form

$$\begin{aligned} x^{((m+1)K)} &= w_m\left[T_K\left\{\prod_{i=1}^{K-1} T_i x^{(mK)}\right\} + T_K g_{K-1}^* + g_K\right] + (1 - w_m)x^{((m-1)K)} \\ &= w_m\left[T_K\left\{\prod_{i=1}^{K-1} T_i x^{(mK)} + g_{K-1}^*\right\} + g_K\right] + (1 - w_m)x^{((m-1)K)} \\ &= w_m[T_K x^{((m+1)K-1)} + g_K] + (1 - w_m)x^{((m-1)K)} \end{aligned} \tag{5.6}$$

since

$$x^{((m+1)K-1)} = \prod_{i=1}^{K-1} T_i x^{(mK)} + g_{K-1}^*.$$

Notice that our acceleration is not applied to solutions over consecutive intervals but to solutions over consecutive *cycles*. Therefore our accelerated K-step procedure takes the form

$$x^{(mK+i)} = T_i x^{(mK+i-1)} + g_i \quad i = 1, \ldots K-1$$
$$x^{((m+1)K)} = w_m[T_K x^{((m+1)K-1)} + g_K] + (1 - w_m)x^{((m-1)K)}$$

and this leads to the splitting

$$\left.\begin{aligned}[r_i I + H]x^{(mK+i-\frac{1}{2})} &= [r_i I - V]x^{(mK+i-1)} + b \\ [r_i I + V]x^{(mK+i)} &= 2r_i x^{(mK+i-\frac{1}{2})} - [r_i I - V]x^{(mK+i-1)}\end{aligned}\right\} \quad (5.7)$$

for $i = 1, \ldots K-1$ and

$$\left.\begin{aligned}[r_K I + H]x^{((m+1)K-\frac{1}{2})} &= [r_K I - V]x^{((m+1)K-1)} + b \\ [r_K I + V]x^{((m+1)K)} &= w_m[2r_K x^{((m+1)K-\frac{1}{2})} - [r_K I - V]x^{((m+1)K-1)}] + (1 - w_m)[r_K I + V]x^{((m-1)K)}.\end{aligned}\right\} \quad (5.8)$$

Notice that the equations (5.7) and the first of (5.8) are exactly the same as occur in the non-accelerated case of the Peaceman–Rachford method with K ADI parameters. Therefore the only modification of this process occurs in the very last equation of the splitting. This new process requires the storage of the solutions from the final equation of the two cycles prior to the one being currently computed. However, since these values are required only once in a cycle they can easily be stored in a backing store and therefore need not prove a drawback to the method.

Once again the sequence $\{w_m\}$ is calculated from the equations (2.5) with $\rho = \rho\left(\prod_{i=1}^{K} T_i\right)$.

Having now defined the process let us consider the convergence rate. Let $M = \prod_{i=1}^{K} T_i$, and denote the spectral radius of the (symmetric) matrix M by $\rho(M) = \rho$. Then from Varga (1962), the convergence factor for the Chebyshev semi-iterative method after m iterations is

$$||\tilde{p}_m(M)|| = (w-1)^{m/2}\left\{\frac{2}{1 + (w-1)^m}\right\} \quad (5.9)$$

where

$$w = \frac{2}{1 + \sqrt{(1 - \rho^2)}}. \quad (5.10)$$

Also the convergence factor of the Peaceman–Rachford method after m cycles of K parameters is given by

$$||M^m|| = \rho^m = \left(\frac{2}{w}\right)^m (w-1)^{m/2} \quad (5.11)$$

so that

$$\frac{||\tilde{p}_m(M)||}{||M^m||} = \frac{w^m}{2^{m-1}[1 + (w-1)^m]}$$

where $1 \leqslant w \leqslant 2$. Consider the function

$$f(w) = w^m - 2^{m-1} - 2^{m-1}(w-1)^m$$

in the range $1 \leqslant w \leqslant 2$. It is easy to show that this function has one turning point, a maximum, at $w = 2$ and that $f(2) = 0$. It follows that $f(w) < 0$ for $1 \leqslant w < 2$ and $m > 1$. Therefore

$$\frac{||\tilde{p}_m(M)||}{||M^m||} < 1 \quad \text{for} \quad m > 1$$

and the Chebyshev acceleration process converges more rapidly than does the ordinary Peaceman–Rachford method. The sequence $\{w_i\}$ is strictly decreasing [3] and tends to a limit w which is usually much closer to 1 than to 2. Therefore in order to obtain a measure of the ratio of the convergence rates we let $w = 1 + \epsilon$ where ϵ is small. It is then easily shown for $m \geqslant 2$, to first order terms

$$\frac{||\tilde{p}_m(M)||}{||M^m||} \doteq \frac{1 + m(w-1)}{2^{m-1}}$$

which gives a measure of the increase in convergence rate. $\left(\text{This in fact requires } \rho \doteq \frac{2\sqrt{\epsilon}}{1+\epsilon}.\right)$

6. Numerical examples

We will now consider as an example, the solution of Laplace's equation by the two well-known ADI methods of Peaceman and Rachford (1955) and Mitchell and Fairweather (1964). Although the latter method does not fall into the classification of Section 3, it is of a similar form to the Peaceman–Rachford method. However, we shall not dwell on such distinctions.

In finite difference notation the above methods take the respective forms

$$\left.\begin{aligned}\left(1 - \frac{r}{2}\delta_y^2\right)u^{(m+\frac{1}{2})} &= \left(1 + \frac{r}{2}\delta^2\right)u^{(m)} \\ \left(1 - \frac{r}{2}\delta_x^2\right)u^{(m+1)} &= 2u^{(m+\frac{1}{2})} - \left(1 + \frac{r}{2}\delta_x^2\right)u^{(m)}\end{aligned}\right\} \quad (6.1)$$

and

$$\left.\begin{aligned}(1 - \tfrac{1}{2}(r - \tfrac{1}{6})\delta_y^2)u^{*(m+1)} &= (1 + \tfrac{1}{2}(r + \tfrac{1}{6})\delta_x^2)u^{(m)} \\ (1 - \tfrac{1}{2}(r - \tfrac{1}{6})\delta_x^2)u^{(m+1)} &= \frac{2r}{r - \frac{1}{6}}u^{*(m+1)} - \frac{(r + \frac{1}{6})}{(r - \frac{1}{6})}(1 + \tfrac{1}{2}(r + \tfrac{1}{6})\delta_x^2)u^{(m)}\end{aligned}\right\} \quad (6.2)$$

where δ^2 is the central difference operator. ADI parameters were chosen for the model problem with $h = \frac{1}{10}$ for the above methods (both constant parameter and a set of (four) variable parameters). These methods

we then used to solve Laplace's equation under Dirichlet boundary conditions on two regions

(*a*) a rectangle
(*b*) an L-shaped region.

Comparative runs were undertaken for the above two ADI methods with Chebyshev acceleration built into the process.

The number of cycles of iterations of the ADI parameters required for convergence are quoted in Table 1 for problems (*a*) and (*b*) for the methods of Peaceman–Rachford (PR) and Mitchell and Fairweather (MF) respectively.

Table 1

	(*a*)		(*b*)	
	PR	MF	PR	MF
Constant parameter	44	47	32	42
Constant parameter with acceleration	36	37	23	34
Variable parameters	1	1	5	6
Variable parameters with acceleration	1	1	4	5

It can be seen that if the ADI process has but one acceleration parameter the convergence may be accelerated by about 20% for both problems. The convergence of the ADI procedures with a variable set of parameters is much more rapid and consequently the Chebyshev acceleration does not have such a pronounced effect. However, it does reduce the number of cycles required. It would be hoped that in more general regions the convergence of the ADI method with a variable set of parameters might be more rapid if the acceleration process is incorporated. The latter process can be defined in a slightly modified form even if the eigenvalue distribution is not as in (3.8).

In the next section we derive an alternative version of the procedure of Section 5. This uses the acceleration process usually known as Richardson's method. The new technique can be used to advantage whenever a Chebyshev acceleration strategy is being employed.

7. An alternative formulation using Richardson's method

This is constructed from (2.2) by the incorporation of an (acceleration) parameter α_{m+1} in the manner

$$x^{(m+1)} = x^{(m)} - \alpha_{m+1}[(I - M)x^{(m)} - g]. \quad (7.1)$$

If we assume the real eigenvalues of the matrix M are given by λ_m where

$$-1 < a \leqslant \lambda_m \leqslant b < 1 \quad (7.2)$$

then the eigenvalues of the matrix $(I - M)$, denoted by μ, are given by

$$0 < \alpha \leqslant \mu \leqslant \beta$$

where

$$\alpha = 1 - b, \quad \beta = 1 - a.$$

Then from the paper of Young (1954) it follows that the optimum choice of the $\{\alpha_m\}$ $(m = 1, \ldots N)$ where a cycle length N has been chosen, are given by

$$\alpha_m^N = 2[(\beta + \alpha) - (\beta - \alpha)t_m^{(N)}\}^{-1} \; m = 1, \ldots N \quad (7.3)$$

where

$$t_m^{(N)} = \cos[(2m - 1)\pi/2N], \quad m = 1, \ldots N$$

are the zeroes of the Nth Chebyshev polynomial. With this choice of $\{\alpha_m^N\}$ the factor of convergence for the cycle is then given by

$$\left[T_N\left(\frac{\beta + \alpha}{\beta - \alpha}\right)\right]^{-1} = \left[T_N\left(\frac{2 - b - a}{b - a}\right)\right]^{-1} < 1. \quad (7.4)$$

This process requires a fixed length of cycle N. The larger the value of N the more rapid the convergence. However, the disadvantage of this method lies in the fact that for large N some of the parameters α_m^N may become quite large and the process (7.1) then becomes very susceptible to instabilities through growth of round-off error. The strategy usually employed is that of choosing a fixed cycle length N, ordering the parameters in a certain way (see Young (1954)) and using this cycle of parameters repeatedly. After S such cycles the error is reduced by the factor

$$\left[T_N\left(\frac{\beta + \alpha}{\beta - \alpha}\right)\right]^{-S}. \quad (7.5)$$

If the parameter cycle had been of length NS then the error would have been reduced by the factor

$$\left[T_{NS}\left(\frac{\beta + \alpha}{\beta - \alpha}\right)\right]^{-1} \quad (7.6)$$

and such a process would converge more rapidly. However, as we have observed, the use of NS parameters may lead to instabilities.

An alternative formulation of the above Chebyshev acceleration process is given by

$$x^{(m+1)} = w_{m+1}\left\{\frac{2M - (b + a)I}{2 - (b + a)}x^{(m)} + \frac{2}{2 - (b + a)}g - x^{(m-1)}\right\} + x^{(m-1)} \quad (7.7)$$

where

$$w_{m+1} = \frac{2\xi T_m(\xi)}{T_{m+1}(\xi)} \text{ with } \xi = \frac{2 - (b + a)}{b - a} \quad (7.8)$$

and

$$w_1 = 1.$$

This process is a simple extension of the process considered above. Once again for a sequence of N ws the

factor of convergence is given by (7.4). However, for this process we do not have to calculate the sequence $\{w_m\}$ before we start. They can be calculated recursively from the relations

$$\left.\begin{aligned} w_1 &= 1 \quad w_2 = \left(1 - \frac{1}{2\xi^2}\right)^{-1} \\ w_{m+1} &= \left(1 - \frac{w_m}{4\xi^2}\right)^{-1} \quad m = 2, 3, \ldots \end{aligned}\right\} \tag{7.9}$$

and so the process carries on automatically.

Therefore it would appear that the semi-iterative process (7.7) is greatly superior to Richardson's method (7.1). Unfortunately if the matrix M involves the inverses of matrices (in other words, any method different from Jacobi iteration) then the process (7.7) may require considerable extra storage and computing time since it is a three-level method. In contrast method (7.1) is a two-level scheme, no different from the unaccelerated process ($\{\alpha_m\} = \{1\}$).

The actual distinction between the two methods can be seen to be the following:

(*a*) Richardson's method uses the property that the zeroes of the Chebyshev polynomial are real and lie in a certain interval.

(*b*) Chebyshev semi-iteration uses the property that there exists a three-term recurrence relation for Chebyshev polynomials.

We will now show that a suitable process can be constructed which enjoys most of the advantages of both processes, and which also endeavours to minimize any disadvantage.

Our process is effectively to apply the procedure (7.7) to the method (7.1) with a "stable" range of $\{\alpha_m^N\}$. The property of Chebyshev polynomials which allows us to do this is the relation

$$T_r(T_s(x)) = T_{rs}(x). \tag{7.10}$$

We now state our modified semi-iterative method as follows.

We first of all define the kth cycle of the Richardson process in the form

$$\begin{aligned} \tilde{x}^{((k-1)N+i+1)} &= \tilde{x}^{((k-1)N+i)} \\ &\quad - \alpha_{i+1}^N[(I - M)\tilde{x}^{((k-1)N+i)} - g] \end{aligned} \tag{7.11}$$

for $i = 0, \ldots, N-1$ where $\{\alpha_i^N\}$ are given by equation (7.3) and

$$\tilde{x}^{((k-1)N)} = x^{((k-1)N)}.$$

The vector $x^{(kN)}$ is then obtained from the vectors $\tilde{x}^{(kN)}$, $x^{((k-2)N)}$ by the process

$$x^{(kN)} = w_k\tilde{x}^{(kN)} + (1 - w_k)\tilde{x}^{((k-2)N)} \tag{7.12}$$

where w_k is obtained from (7.9) with

$$\xi = \left[T_N\left(\frac{2 - b - a}{b - a}\right)\right]. \tag{7.13}$$

If $Q_i = I - \alpha_{i+1}(I - M)$ then it is easily seen that the eigenvalues λ_Q of $\prod_{i=1}^{N} Q_i$ are given by

$$-\left[T_N\left(\frac{2 - b - a}{b - a}\right)\right]^{-1} \leqslant \lambda_Q \leqslant \left[T_N\left(\frac{2 - b - a}{b - a}\right)\right]^{-1}$$

where b, a are given by (7.2). It follows that after S cycles of the procedure (7.11) and (7.12) the error is reduced by a factor

$$\left[T_S\left(T_N\left(\frac{2 - b - a}{b - a}\right)\right)\right]^{-1} = \left[T_{SN}\left(\frac{2 - b - a}{b - a}\right)\right]^{-1}.$$

The above process can be regarded as applying Chebyshev semi-iteration to Richardson's method. The method (7.11) and (7.12) has the following advantages:

(*a*) It has a factor of convergence of the form (7.6) rather than (7.5).

(*b*) It only employs a *stable* cycle of Richardson parameters. The choice of N is at our disposal and is chosen to ensure the stability of Richardson's method.

(*c*) It is largely a two-step method since the three-step Chebyshev process is applied only to the final iterates in each cycle.

(*d*) It does not necessarily require extra storage for the end-of-cycle iterates. Such vectors may be stored in a fast-access backing store rather than in core, since they are required infrequently.

The above process will be of use when our iteration method corresponds to that of symmetric successive over-relaxation or to an alternating direction method with a single ADI parameter. In the case of an ADI method with a sequence of ADI parameters it is more efficient to use the procedure outlined in Section 5.

References

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