Note on two methods of solving ordinary linear differential equations

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Two assumptions are formulated, based on recent results concerning two methods of approximating to the solutions of ordinary linear differential equations. They are shown to be false by means of counter examples.

1. An important class of methods for finding global solutions to ordinary linear differential equations involves assuming a trial solution containing free parameters, and determining these by some strategy.

Attention has recently focussed on two methods of this kind:

(*a*) methods equating coefficients of the independent variable,
(*b*) collocation methods, where the residual is made equal to zero at certain values of the independent variable.

In this note we consider two assumptions based on recent results regarding these methods, and show them to be false by means of counter examples.

2. We consider first the most common method of type (*a*), viz. the Lanczos τ-method (Lanczos 1957). Here we obtain polynomial approximations to the solution of the linear differential equation

$$L(y) = f(z), \tag{2.1}$$

by solving the system

$$L(y) = R_n(z, \tau_1, \tau_2, \ldots \tau_p) + f(z), \tag{2.2}$$

subject to the imposed boundary conditions. The $\tau_j, j = 1, 2, \ldots, p$, are free parameters, and R_n is chosen so that equation (2.2) is satisfied by a polynomial of degree n. For problems of the form

$$(A + Bz)y^{(1)} + Cy = 0, \; y(0) = K, \tag{2.3}$$

A, B, C, K constant,

a possible form for R_n is $\tau T_n(z)$, where $T_n(z)$ is the Chebyshev polynomial of degree n for the given range. Some results of Rivlin† would seem to indicate that this choice of R_n determines the polynomial approximation of degree n which gives the smallest residual in the Chebyshev sense (subsequently referred to as the minimax solution). This gives the basis for our first postulate:

Assumption 1

For equations of type (2.3), the Lanczos τ-method, as detailed above, determines the polynomial approximation of degree n which is the minimax solution.

Methods of type (*b*) normally differ in the strategy adopted to determine the collocation points, and Kizner (1966) has compared various choices. He concludes that for "well behaved" solutions, the smallest maximum error in the solution occurs when the points are taken to be the zeros of the derivative of $T_{n+1}^{**}(z)$, the Chebyshev polynomial stretched so that it is zero at the end points of the range of solution.

Assumption 2

There exists a choice of collocation points which will give a good computational strategy for all equations with well-behaved solutions.

3. Consider the following example of an equation of type (2.3), which is to be solved by the Lanczos τ-method.

$$(1 + 9z)y^{(1)} - 17y = 0, \quad 0 \leqslant z \leqslant 1, \tag{3.1}$$

$$y(0) = 1.$$

We seek a polynomial solution of degree 2,

$$\sum_{i=0}^{2} a_i z^i,$$

where a_i, $i = 0, 1, 2$ are unknowns.

Substituting into (3.1) we obtain

$$\sum_{i=0}^{2} a_i \psi_i(z) = \tau T_2(z), \tag{3.2}$$

where $\psi_i(z) = (9i - 17)z^i + iz^{i-1}, \quad i = 0, 1, 2.$

From the boundary condition,

$$a_0 = 1,$$

and so (3.2) reduces to

$$\sum_{i=0}^{1} a_{i+1}\phi_i(z) = \tau T_2(z) + 17, \tag{3.3}$$

where $\phi_i(z) = \psi_{i+1}(z), \quad i = 0, 1.$

The solution of (3.3) gives the residual

$$r_1(z) = 8 \cdot 5 T_2(z) = 68z^2 - 68z + 8 \cdot 5$$

with maximum value $8 \cdot 5$. This is shown in **Fig. 1**. The

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† Lecture at Stanford University, February 1967.

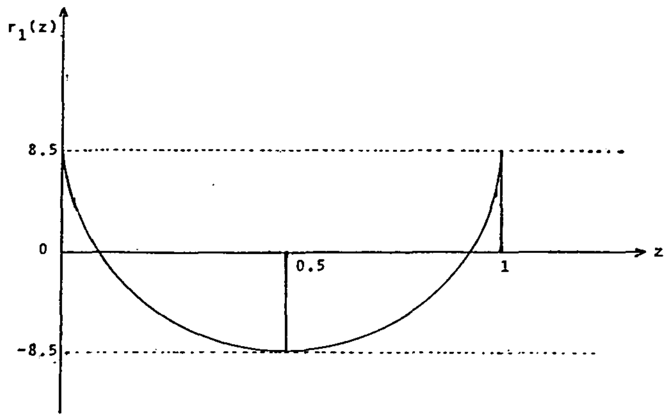

Fig. 1. Residual for Lanczos τ-method

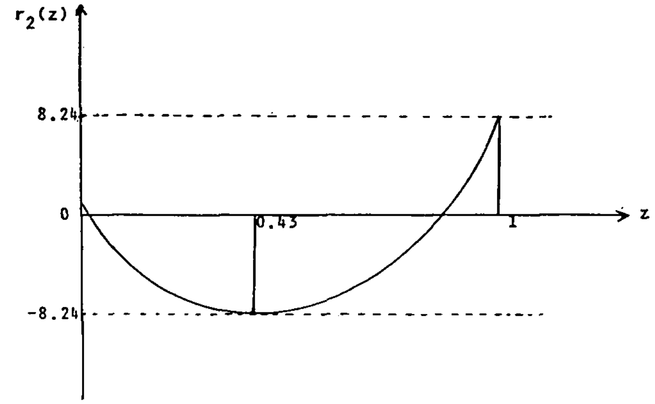

Fig. 2. Residual for minimax method

minimax approximation, however, has residual

$$r_2(z) = 50{\cdot}48z^2 - 43{\cdot}27z + 1{\cdot}03$$

with maximum value $8{\cdot}24$. This is shown in **Fig. 2.** We note here that there are just 2 points in the range when the minimax residual attains its maximum (in modulus) value.

The approximation problem in the form (3.3) is produced from (3.1) by

(i) the mapping by the differential operator,
(ii) the constraint imposed by the boundary condition.

The application of (i) results in equation (3.2), where the functions $\psi_i(z)$, $i = 0, 1, \ldots, n = 2$ form a Chebyshev set, i.e. no linear combination has more than n zeros. It is well known that in this case the minimax approximation has the residual of maximum modulus occurring at $(n + 1)$ points, and alternating in sign at those points (e.g. Rice, 1964).

However, we have to consider the effect of applying the boundary condition. In our example, the function $\psi_0(z)$ is effectively deleted, leaving just 2 (polynomial) approximating functions. The minimax approximation, however, may still have the maximum residual occurring at 3 points, and alternating in sign at these points. If, and only if, this is the case, it can be shown that the residual is characterized as a multiple of the Chebyshev polynomial of degree 2, and the Lanczos τ-method will give the minimax solution.

It seems clear, then, that no general theory on the equivalence of the two solutions can be formulated which does not take the boundary condition explicitly into consideration.

4. In order to contradict the second assumption, we consider the solution, by a collocation method, of

$$Ay^{(1)} + (z + B)y = 0, \quad 0 \leqslant z \leqslant 1, \qquad (4.1)$$
$$y(0) = K,$$

where A, B, K are constants.

The exact solution of this equation is

$$y = K \exp\left(-(z^2 + 2Bz)/2A\right).$$

Taking as solution the polynomial $\sum_{i=0}^{2} a_i z^i$, we require 2 collocation points z_1 and z_2 at which

$$a_0(z + B) + a_1(A + Bz + z^2) + a_2(2Az + Bz^2 + z^3) = 0.$$

We also have the constraint

$$a_0 = K.$$

Now if

$$A = z_1 z_2,$$
$$B = -(z_1 + z_2),$$

the coefficient of a_1 vanishes for both points z_1 and z_2, and no solution can be obtained.

Thus, by suitably adjusting the coefficients in equation (4.1), any prescribed set of collocation points can be shown to be worst possible for some differential equation with a well behaved solution.

References

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