A note on the estimation of the optimum successive overrelaxation parameter for Laplace's equation

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The optimum parameter for any given region is estimated by finding an equivalent rectangle intuitively. Experimental evidence is produced in support of the method for two dimensions and for three dimensions with axial symmetry.

1. Introduction

The SOR method for solving the set of linear simultaneous equations $Ax = k$ is given by the iterative formula

$$x^{(m+1)} = (I - wL)^{-1}\{(1 - w)I + wV\}x^{(m)} + w(I - wL)^{-1}D^{-1}k,$$

where $A = D - E - F$, $L = D^{-1}E$ and $V = D^{-1}F$. E and F are, respectively, upper and lower triangular matrices with zero leading diagonals and D is a diagonal matrix. w is the overrelaxation parameter. If $w = 1$ we have the Gauss–Seidel method

$$x^{(m+1)} = (I - L)^{-1}Vx^{(m)} + (I - L)^{-1}D^{-1}k.$$

Young (1954) shows that for a certain class of matrices, A, with what he terms property (A), the optimum parameter, w_0, is given by $w_0 = 2/[1 + \sqrt{(1 - \lambda)}]$ where λ is the maximum of the moduli of the latent roots of the matrix $(I - L)^{-1}V$ associated with the Gauss–Seidel method. For the general case, the problem of obtaining the optimum parameter without a considerable computing effort has not been solved but, for the Dirichlet problem in a rectangular region using the finite difference analogue of Laplace's equation, Frankel (1950) proves that $\lambda = \left(\cos\frac{\pi}{p} + \cos\frac{\pi}{q}\right)^2/4$ where p and q are the dimensions of the rectangle in mesh lengths.

For regions other than rectangles, Young (1955) recommends using w_0 for a square of the same area. This was based on experiments with the following five regions which are subsets of the square of side 20:

A: the complete square.
B: the square with a 6 by 6 hole in the centre.
C: the square with a 5 by 5 square removed from each corner.
D: the square with a ten by ten square removed from one corner.
E: one of the two triangular regions obtained by cutting the square along a diagonal.

2. The equivalent rectangle

The estimation of w_0 by using the value for a square of the same area can give a very poor result. For region B, Young (1955) shows that nearly twice as many iterations are required using this method as are required with w_0 to reduce the residuals to 2^{-32}.

The following method of choosing an equivalent rectangle gives much better estimates of w_0. The breadth is the diameter of the largest circle which can be drawn inside the boundary of the region: the circle must not cut a boundary line nor contain a hole. The length is then found by dividing the area of the region by the breadth. For each of the five regions studied by Young (1955) the values of w found by this method are either better or as good as those calculated using the square of the same area. The values are also very close to the optimum and err, if at all, on the high side which, according to Forsythe and Wasow (1960), is preferable. **Table 1** gives the results.

Table 1

Comparison of overrelaxation parameters for five regions

	A	B	C	D	E
w_0	1·73	1·55	1·68	1·65	1·61
w_{ER}	1·73	1·57	1·68	1·66	1·63

w_0: the optimum parameter.
w_{ER}: the equivalent rectangle parameter.

3. Extension to a three-dimensional region with axial symmetry

Laplace's equation for a three-dimensional region with axial symmetry is

$$\frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} = 0.$$

The corresponding difference equations for a square mesh are very similar to those for two dimensions, except when r is small. An experiment was conducted to ascertain whether the equivalent retangle method for a corresponding two-dimensional region would give a satisfactory estimate of w_0 for the three-dimensional region. The region considered was that between two concentric spheres. It is generated by the rotation, about a diameter, of two concentric semicircles. The

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boundary values were unity on the inner semicircle and zero on the outer semicircle. A net having mesh length of one-eighth the radius of the inner sphere was used. Interpolation was required at all nodes adjacent to the boundaries.

Various values of w were tried and the optimum value determined. The results are given in **Table 2**. The equivalent rectangle method gave the optimum value and the square of the same area gave a value which required 30% more iterations. The results confirm that an overestimate is better than an underestimate but they also show that overestimating by 0·2 involves twice as many iterations as are required with w_0.

Table 2

Comparison of overrelaxation parameters for the region between concentric spheres

w	$N(w)$	w	$N(w)$
1·50	39	1·60	34
1·51	38	1·61	34
1·52	37	1·62	35
1·53	36	1·63	36
1·54	34	1·64	37
1·55	33	1·65	37
1·56	32	1·66	38
1·5634 (w_{ER})	32	1·67	40
1·57	33	1·68	40
1·58	33	1·695 (w_A)	42
1·59	33	1·77	60

w: overrelaxation parameter.
$N(w)$: the number of iterations to reduce the residuals to less than 0·00001.
w_A: the parameter for a square of the same area.
w_{ER}: the equivalent rectangle parameter.

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Book Review

Introduction to Computational Linguistics, by David G. Hays, 1967; 231 pages. (London: Macdonald & Co., 70s.)

The declared purpose of this book is to serve as an "introductory university text for courses in computer applications to linguistics". The author moves fairly swiftly from the elements of computing to discuss in detail linguistic techniques such as dictionary lookup, parsing strategies, and the handling of concordances. Of wider interest are the chapters on Documentation, covering topics such as KWIC indexing, and on Automatic Translation, though the latter chapter rapidly reaches a level of detailed technical discussion which demands assimilation of the preceding material. For this reason, it is doubtful whether the book will be useful to "computer specialists and managers", as the blurb hopefully suggests, except where they have a very specialist interest in linguistics.

A considerable number of the specimen programs in this book are presented in a symbolic machine language, the rest being written in ALGOL. It is arguable to what extent the linguistic specialist should need to know low-level machine languages at all; perhaps a case might be made out that greater comprehension of the "mysteries" of the computer might thereby be gained, but one suspects that the author has been influenced by the lack of a widely available high-level language with adequate character-handling and list-processing facilities. Perhaps PL/1, whose specifications cover both these areas in some depth, will fill this gap.

One slightly unfortunate feature of the book is that the author has omitted to tie by number references in the text to the sources quoted at the end of each chapter. Until one realizes this, sentences such as the following seem to have an unnecessarily "in-group" ring: "Eugene D. Pendergraft has proposed a different way of elaborating phrase-structure grammar".

The technical chapters of the book cover the material in good detail, and topics are developed very clearly despite the highly involved nature of some of them. Each chapter is accompanied by a number of exercises which often encourage the student to apply the material covered by means of writing algorithms or whole programs of his own. In summary, the book is aimed at the linguistic specialist approaching the computer, rather than the computer specialist approaching linguistics.

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