Formula 4

A seventh degree formula for three dimensions using 27 integrand evaluations can be obtained in the form

$$J \approx K = A_0 f(0, 0, 0) + A_1 \sum_P f(x_1, 0, 0) + A_2 \sum_P f(x_2, x_2, 0) + A_3 \sum_P f(x_3, x_3, x_3).$$

There are two possible choices for A_0, A_1, A_2, A_3, x_1, x_2 and x_3, thus:

$A_0 =$	$0 \cdot 1184868$ or	$0 \cdot 1821729$
$A_1 =$	$0 \cdot 0053074$	$0 \cdot 0466670$
$A_2 =$	$0 \cdot 0629095$	$0 \cdot 0049431$
$A_3 =$	$0 \cdot 0118472$	$0 \cdot 0598136$
$x_1 =$	$1 \cdot 2795819$	$0 \cdot 8484180$
$x_2 =$	$0 \cdot 7000973$	$1 \cdot 1064129$
$x_3 =$	$0 \cdot 8550443$	$0 \cdot 6528165$

These formulae have the undesirable feature that one of the abscissae lies outside the range of integration.

Estimate of the maximum error in best polynomial approximations

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By using Chebyshev's equioscillation theorem and the well-known error formula for the interpolating polynomial, inequalities are derived for the minimax error in polynomial approximation. These results are extended to piecewise polynomial approximations.

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1. Approximations over a single interval

Suppose a function $f(x)$ is defined on $[a, b]$ and that $f^{(n+1)}(x)$ exists and is continuous on that interval. Let $p_n(x)$ be the polynomial of degree not greater than n which is the best approximating polynomial for $f(x)$ on $[a, b]$ in the Chebyshev sense. By the equioscillation theorem (see, for example, Davis (1963)), $\exists$ at least $n + 2$ points on which

$$\max_{a \leqslant x \leqslant b} | f(x) - p_n(x) |$$

is attained, with the error $f(x) - p_n(x)$ alternating in sign over those points. Hence, by continuity, $\exists$ at least $n + 1$ distinct points, say $x_0, x_1, \ldots, x_n$, on $[a, b]$, where $f(x) - p_n(x) = 0$, and so we may write the usual estimate for the error in the interpolating polynomial,

$$f(x) - p_n(x) = \frac{(x - x_0) \ldots (x - x_n)}{(n+1)!} f^{(n+1)}(\xi_x), \quad (1)$$

where ξ_x is some function of x.

Now let $x_0^*, x_1^*, \ldots, x_n^*$ be the zeros of $T_{n+1}((2x - b - a)/(b - a))$, where $T_{n+1}(x)$ is the Chebyshev polynomial $\cos((n + 1) \cos^{-1} x)$. If we let $q_n(x)$ be the interpolating polynomial for $f(x)$ constructed at $x_0^*, x_1^*, \ldots, x_n^*$, we will have

$$f(x) - q_n(x) = \frac{(x - x_0^*) \ldots (x - x_n^*)}{(n+1)!} f^{(n+1)}(\eta_x), \quad (2)$$

η_x being some function of x. Therefore

$$\max_{a \leqslant x \leqslant b} |f(x) - q_n(x)| \leqslant \frac{1}{(n+1)!} \max_{a \leqslant x \leqslant b} |(x - x_0^*) \ldots (x - x_n^*)| \cdot \max_{a \leqslant x \leqslant b} |f^{(n+1)}(x)|. \quad (3)$$

Putting $y = (2x - b - a)/(b - a)$ and

$$y_r^* = (2x_r^* - b - a)/(b - a) \text{ for } r = 0, 1, \ldots, n,$$

$$\max_{a \leqslant x \leqslant b} |(x - x_0^*) \ldots (x - x_n^*)| = \left(\frac{b-a}{2}\right)^{n+1} \cdot \max_{-1 \leqslant y \leqslant 1} |(y - y_0^*) \ldots (y - y_n^*)|. \quad (4)$$

Thus the inequality (3) gives

$$\max_{a \leqslant x \leqslant b} |f(x) - q_n(x)| \leqslant \frac{(b-a)^{n+1}}{2^{2n+1}(n+1)!} \cdot \max_{a \leqslant x \leqslant b} | f^{(n+1)}(x)|. \quad (5)$$

From the definition of $p_n(x)$, it follows that

$$\max_{a \leqslant x \leqslant b} |f(x) - p_n(x)| \leqslant \frac{(b-a)^{n+1}}{2^{2n+1}(n+1)!} \cdot \max_{a \leqslant x \leqslant b} |f^{(n+1)}(x)|. \quad (6)$$

Also, from (1),

$$\max_{a \leqslant x \leqslant b} | f(x) - p_n(x)| \geqslant \frac{1}{(n+1)!} \max_{a \leqslant x \leqslant b} |(x - x_0) \ldots (x - x_n)| \cdot \min_{a \leqslant x \leqslant b} |f^{(n+1)}(x)| \quad (7)$$

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and from the minimax property of the Chebyshev polynomials we have from (7) that

$$\max_{a \leqslant x \leqslant b} |f(x) - p_n(x)| \geqslant \frac{(b-a)^{n+1}}{2^{2n+1}(n+1)!} \cdot \min_{a \leqslant x \leqslant b} |f^{(n+1)}(x)|. \quad (8)$$

Combining the results (6) and (8), by the continuity of $f^{(n+1)}(x)$ ∃ some $\xi \epsilon [a, b]$ such that

$$\max_{a \leqslant x \leqslant b} |f(x) - p_n(x)| = \frac{2}{(n+1)!} \cdot \left(\frac{b-a}{4}\right)^{n+1} \cdot |f^{(n+1)}(\xi)|. \quad (9)$$

2. Piecewise approximations

Let us now approximate to $f(x)$ by splitting $[a, b]$ into k sub-intervals and using a polynomial approximation of degree at most n on each sub-interval. Let us choose the points of sub-division and the k approximating polynomials so as to minimize the maximum error. It is easily seen that the maximum error, say $E(n, k)$, will be attained at least once on each sub-interval.

Let $I_1, I_2, \ldots, I_k$ be the sub-intervals and let $x_1, x_2, \ldots x_{k-1}$ be the sub-dividing points. Then from (9) we may write, for $r = 1, 2, \ldots, k$,

$$E(n, k) = \frac{2}{(n+1)!}\left(\frac{x_r - x_{r-1}}{4}\right)^{n+1} \cdot |f^{(n+1)}(\xi_r)|, \quad (10)$$

where $\xi_r \epsilon I_r$ and $x_0 = a$, $x_k = b$. Thus

$$\left(\frac{(n+1)!\,E(n,k)}{2}\right)^{\frac{1}{n+1}} = \tfrac{1}{4}(x_r - x_{r-1}) \cdot \left|f^{(n+1)}(\xi_r)\right|^{\frac{1}{n+1}} \quad (11)$$

and

$$k\left(\frac{(n+1)!\,E(n,k)}{2}\right)^{\frac{1}{n+1}} = \tfrac{1}{4}\sum_{r=1}^{k}(x_r - x_{r-1}) \left|f^{(n+1)}(\xi_r)\right|^{\frac{1}{n+1}}. \quad (12)$$

As $k \to \infty$, the largest sub-interval $x_r - x_{r-1} \to 0$ and we may replace the right side of (12) by the Riemann integral, giving

$$\lim_{k \to \infty} k^{n+1}E(n, k) = \frac{2}{(n+1)!} \left\{\tfrac{1}{4}\int_a^b \left|f^{(n+1)}(x)\right|^{\frac{1}{n+1}} dx\right\}^{n+1}. \quad (13)$$

The special case of (13) with $n = 1$ is given by Ream (1961).

Returning to (10), at least one sub-interval I_r must have length not greater than $(b - a)/k$, so that

$$E(n, k) \leqslant \frac{2}{(n+1)!} \cdot \left(\frac{b-a}{4k}\right)^{n+1} \cdot \max_{a \leqslant x \leqslant b} |f^{(n+1)}(x)|. \quad (14)$$

Similarly at least one sub-interval must have length not smaller than $(b - a)/k$, giving

$$E(n, k) \geqslant \frac{2}{(n+1)!} \cdot \left(\frac{b-a}{4k}\right)^{n+1} \cdot \min_{a \leqslant x \leqslant b} |f^{(n+1)}(x)|. \quad (15)$$

From (14) and (15), by continuity of $f^{(n+1)}(x)$, ∃ some $\xi \epsilon [a, b]$ such that

$$E(n, k) = \frac{2}{(n+1)!} \cdot \left(\frac{b-a}{4k}\right)^{n+1} \cdot |f^{(n+1)}(\xi)|. \quad (16)$$

References

Davis, P. J. (1963). *Interpolation and Approximation*, Blaisdell, New York.

Ream, N. (1961). Note on "Approximation of Curves by Line Segments", *Mathematics of Computation*, Vol. 15, pp. 418–419.