Generation of time delays on analogue computers

By J. L. Riley and P. H. Walker*

An analogue computer method of achieving a constant transport delay over a particular frequency range is justified. A circuit is given for representing a Padé approximation to the delay. Three different sets of values for the Padé coefficients are proposed and compared with respect to their accuracy over the frequency range. Two of the sets are considerably more accurate than the third.

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1. Introduction

Delays arise in the investigation of many physical systems. The simplest example occurs in the study of the control of a steel-strip rolling mill where the thickness of the sheet is required to remain constant at a particular value. The thickness is measured at some time after it has been determined by the rolls, and no information about the measurement is available to reposition the rolls until a finite time after the sheet leaves them.

There are several methods available for achieving this delay in practice. The first involves the use of a tape recorder, where the input variable is recorded and the output variable is played back after the elapse of the delay time. The value of the delay time depends on the tape speed and the spacing between the roller and replay heads. For the achievement of satisfactory results, very sophisticated and expensive equipment is required and this usually prohibits its use.

A second method uses a rotating drum with a number of capacitors at its circumference. The drum acts as a sequential switch whereby the input voltage variable is applied to each capacitor in turn and the capacitors are discharged after the required delay time. This time depends solely on the speed with which the drum rotates. Over a hundred capacitors are normally required for this technique and these must be of high quality. This factor, together with its general unsatisfactory performance, make its use undesirable.

A modification of this method replaces the capacitors by a magnetic core store (Electronic Associates Ltd. (1964)). The sampled input data is passed through an analogue to digital converter and stored in a digital memory unit. A "stepped" approximation to the data is obtained by reading out each stored value after T_D seconds. The true output variable is recovered by reconverting into analogue form. This requires a large number of components and is more suitable for hybrid computing systems.

A fourth alternative enables the delay to be simulated on an analogue computer. This has the advantage that the same type of components can be used to generate the delay as are used in the simulation of the rest of the system.

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2. Basic theory

A time delay is characterised by

$$f_0(t) = f_i(t - T_D)$$

where f_0 and f_i are the output and input variables and T_D is the delay.

Taking the Laplace transform,

$$F(s) = \frac{F_0(s)}{F_i(s)} = \exp(-T_D s). \qquad (1)$$

The frequency characteristic is given by replacing s by $j\omega$

$$\therefore\ F(j\omega) = \exp(-j\omega T_D) = \cos\omega T_D - j\sin\omega T_D$$

$$\therefore\ |F(j\omega)| = 1: \quad \therefore \angle F(j\omega) = -\omega T_D.$$

Thus the delay unit requires a level frequency response and a phase angle proportional to frequency for all frequencies of interest, if the delay is to be constant.

$$\therefore\ T_D = \frac{d}{d\omega}\left[\angle F(j\omega)\right].$$

Since no electric network has the exact response of the ideal transfer function, the analogue computer method of simulating the delay employs a linear computer circuit in which this transfer function is approximated by a ratio of polynomials (often referred to as Padé polynomials).

The transfer function can be written

$$\frac{F_0}{F_i} = \frac{1 - a_1 s + a_2 s^2 - a_3 s^3 + a_4 s^4.}{1 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4} \qquad (2)$$

The accuracy of the expression depends on the number of terms taken and the values of the coefficients. In a fourth order approximation in which terms of up to the fourth degree are included, the phase angle remains proportional to frequency to within 1° up to a maximum value of

$$T_D = 7 \cdot 50 \text{ rads.}$$

For convenience a delay of 50 msec is chosen; a value representative of the size of delay often encountered. A frequency range extending up to 20 hz, therefore, gives a required maximum value of

$$T_D = 6 \cdot 28 \text{ rads.}$$

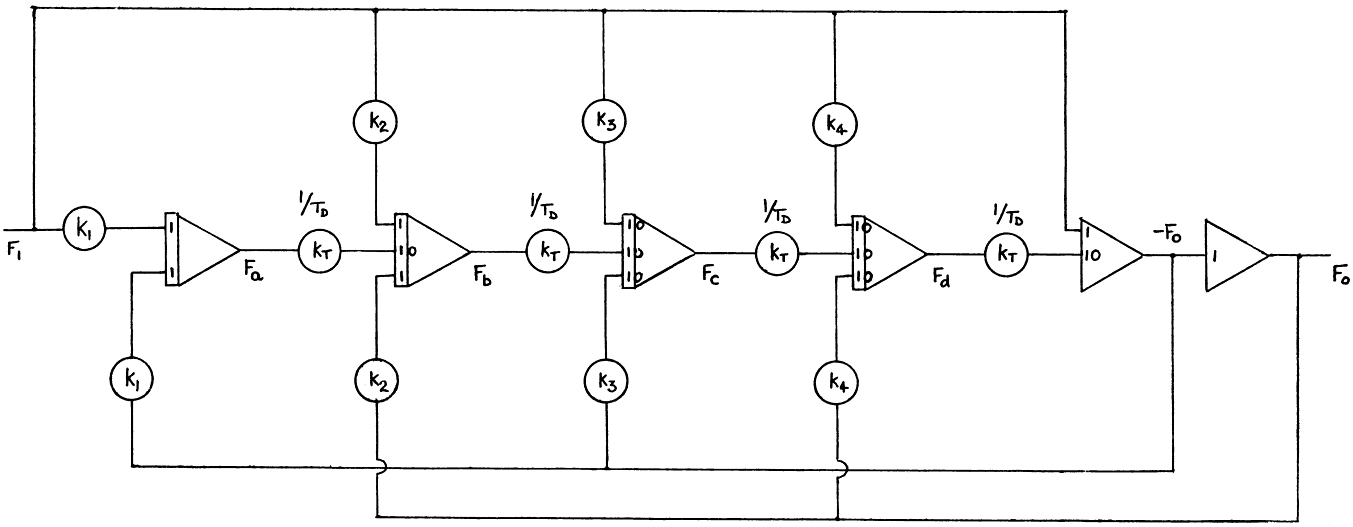

Fig. 1. Computer circuit of transport delay

A fourth order Padé approximation is, therefore, a reasonable choice for the generation of this time delay.

The circuit used to reproduce this approximation is shown in **Fig. 1** and confirmation of its validity is given in the Appendix.

3. Calculation of coefficients

The magnitudes of the coefficients seem to be a point of disagreement among several authors (Rogers and Connolly (1960), Cunningham (1954), Morrill (1954), Gilbert (1964)).

Three sets of solutions are considered and shown in **Table 1.** The derivation of two of them is given and the other is included for comparison purposes.

Table 1

	a_1	a_2	a_3	a_4
Set 1	$\frac{1}{2} T_D$	$\frac{1}{9\cdot 34} T_D{}^2$	$\frac{1}{84} T_D{}^3$	$\frac{1}{1680} T_D{}^4$
Set 2	$\frac{1}{2} T_D$	$\frac{1}{9\cdot 1} T_D{}^2$	$\frac{1}{78\cdot 6} T_D{}^3$	$\frac{1}{1420} T_D{}^4$
Set 3	$\frac{1}{2} T_D$	$\frac{1}{8\cdot 933} T_D{}^2$	$\frac{1}{79\cdot 12} T_D{}^3$	$\frac{1}{1072} T_D{}^4$

Set 1 can be obtained in two ways:

(*a*) Expand the exponential term as an infinite series and equate it to the ratio of the Padé polynomials

$$\frac{F_0}{F_i} = \exp(-sT_D) = \frac{1 - a_1 s + a_2 s^2 - a_3 s^3 + a_4 s^4}{1 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4}$$

$$= 1 - sT_D + \frac{s^2 T_D^2}{2!} - \frac{s^3 T_D^3}{3!} + \frac{s^4 T_D^4}{4!} - .$$

Substituting the values of the a coefficients in terms of T (see Appendix) and multiplying out, we have

$$1 - \frac{k_2}{10k_1} sT + \frac{k_3}{10k_1} s^2T^2 - \frac{k_4}{100k_1} s^3T^3 + \frac{1}{10{,}000k_1} s^4T^4$$

$$= 1 + sT\left(\frac{k_2}{10k_1} - 1\right) + s^2T^2\left(\frac{k_3}{10k_1} - \frac{k_2}{10k_1} + \frac{1}{2}\right)$$

$$+ s^3T^3\left(\frac{k_4}{100k_1} - \frac{k_3}{10k_1} + \frac{k_2}{20k_1} - \frac{1}{6}\right)$$

$$+ s^4T^4\left(\frac{1}{10{,}000k_1} - \frac{k_4}{100k_1} + \frac{k_3}{20k_1} - \frac{k_2}{60k_1} + \frac{1}{24}\right)$$

$$+ \dots$$

By equating terms of the same degree, an infinite number of simultaneous equations is produced.

Equate sT terms,

$$1 - \frac{k_2}{10k_1} = \frac{k_2}{10k_1} \qquad k_2 = 5k_1.$$

Equate s^2T^2 terms,

$$\frac{k_3}{10k_1} - \frac{k_2}{10k_1} + \frac{1}{2} = \frac{k_3}{10k_1} \quad k_2 = 5k_1.$$

Equate s^3T^3 terms,

$$\frac{k_4}{100k_1} - \frac{k_3}{10k_1} + \frac{k_2}{20k_1} - \frac{1}{6} = -\frac{k_4}{100k_1}$$

$$6k_4 - 30k_3 + 15k_2 - 50k_1 = 0. \qquad (3)$$

Equate s^4T^4 terms,

$$-\frac{k_4}{100k_1} + \frac{k_3}{20k_1} - \frac{k_2}{60k_1} + \frac{1}{24} = 0$$

$$-6k_4 + 30k_3 - 10k_2 + 25k_1 = 0. \qquad (4)$$

By adding (3) and (4) we have $k_2 = 5k_1$.

Thus the first four equations are not independent and yield only one result, which can be re-stated as

$$a_1 = \tfrac{1}{2}T_D.$$

Equations from the 5th, 6th and 7th powers give an independent set which can be solved for a_2, a_3 and a_4.

We have:

5th: $-6 + 300k_4 - 1000k_3 + 250k_2 - 500k_1 = 0$

6th: $36 - 1200k_4 + 3000k_3 - 600k_2 + 1000k_1 = 0$

7th: $-42 + 1050k_4 - 2100k_3 + 350k_2 - 500k_1 = 0.$

These can be solved to give:

$$k_1 = 0\cdot168,\ k_2 = 0\cdot840,\quad k_3 = 0\cdot180,\ k_4 = 0\cdot200$$

$$a_1 = \tfrac{1}{2}T_D \quad a_2 = \frac{1}{9\cdot34}T_D^2 \quad a_3 = \frac{1}{84}T_D^3 \quad a_4 = \frac{1}{1680}T_D^4$$

(*b*) The second way is shown by Morrill (1954) and Perron (1950). The Padé approximation is stated as:

$$F_{\mu,\nu}(X) = 1 + \frac{\mu X}{(\mu+\nu)1!} + \frac{\mu(\mu-1)X^2}{(\mu+\nu)(\mu+\nu-1)2!} + \frac{\mu(\mu-1)(\mu-2)X^3}{(\mu+\nu)(\mu+\nu-1)(\mu+\nu-2)3!} + \frac{\mu(\mu-1)(\mu-2)(\mu-3)X^4}{(\mu+\nu)(\mu+\nu-1)(\mu+\nu-2)(\mu+\nu-3)4!}$$

$$G_{\mu,\nu}(X) = 1 - \frac{\mu X}{(\mu+\nu)1!} + \frac{\mu(\mu-1)X^2}{(\mu+\nu)(\mu+\nu-1)2!} - \frac{\mu(\mu-1)(\mu-2)X^3}{(\mu+\nu)(\mu+\nu-1)(\mu+\nu-2)3!} + \frac{\mu(\mu-1)(\mu-2)(\mu-3)X^4}{(\mu+\nu)(\mu+\nu-1)(\mu+\nu-2)(\mu+\nu-3)4!}$$

and

$$\operatorname*{Lt}_{(\mu+\nu)\to\infty} \frac{F_{\mu,\nu}(X)}{G_{\mu,\nu}(X)} = e^X,$$

Putting $X = -sT_D$, $\mu = \nu = 4$ we have a fourth-order Padé approximation to the delay

$$\exp(-sT_D) = \frac{1 - \tfrac{1}{2}T_D s + \frac{1}{9\cdot34}s^2T_D^2 - \ldots}{1 + \tfrac{1}{2}T_D s + \ldots}.$$

The set 2 coefficients are quoted by Gilbert (1964), but no proof of their derivation is given.

A third set of coefficients is derived by expressing the fourth-order Padé approximation in a product form (Rogers and Connolly (1960)):

$$\exp(-sT_D) = \frac{(1 - 2\zeta_1\tau_1 s + \tau_1^2 s^2)(1 - 2\zeta_2\tau_2 s + \tau_2^2 s^2)}{(1 + 2\zeta_1\tau_1 s + \tau_1^2 s^2)(1 + 2\zeta_2\tau_2 s + \tau_2^2 s^2)}$$

where $T_D = 4(\zeta_1\tau_1 + \zeta_2\tau_2)$.

The delay can thus be considered to be made up of two second-order delay elements with time constants of $4\zeta_1\tau_1$ and $4\zeta_2\tau_2$ respectively. By proper choice of ζ and τ for each element, the accuracy of the representation can be improved for larger values of ωT_D. The amplitude response is level for all frequencies and the ζ and τ parameters can be adjusted to extend the required linear-phase relationship as far as is required.

It is found that for any term in the product above, a value of ζ less than $\tfrac{1}{2}\sqrt{3}$ gives a phase angle which for increasing frequency differs from the ideal $-\omega T_D$ by an error that is initially positive. A value of ζ greater than $\tfrac{1}{2}\sqrt{3}$ gives a phase angle which has an error that is initially negative.

By selecting a suitable value of ζ_2 designed to cancel the initial phase error produced in the first term, the required maximum phase shift can be achieved.

The values of such parameters are

$$\zeta_1 = \tfrac{1}{2}\sqrt{3}, \quad \zeta_2 = 0\cdot4, \quad \frac{\tau_1}{\tau_2} = 1\cdot68.$$

Inserting these values yields the set 3 coefficients.

4. Practical investigations

The Padé coefficients are used to obtain the settings of the coefficient potentiometers in the delay circuit. The delay circuit is investigated for each set of coefficients by measuring the phase shift of a sinusoidal input signal over the frequency range 0–20 hz with the time delay set to give 50 msec. Because the value of the delay appears in the setting of a potentiometer, the facility of a variable time delay is available.

In the steady state $s = j\omega$ and the time delay can be expressed:

$$\frac{F_0}{F_1} = \frac{(1 - a_2\omega^2 + a_4\omega^4) - j(a_1\omega - a_3\omega^3)}{(1 - a_2\omega^2 + a_4\omega^4) + j(a_1\omega - a_3\omega^3)} = \frac{A - jB}{A + jB}.$$

The phase of the time delay can now be expressed:

$$\angle F(j\omega) = \tan^{-1}\frac{B}{A} - \tan^{-1}\frac{A}{B} = 2\tan^{-1}\frac{B}{A}$$

Inserting the Set 1 coefficients give

$$A = 1 - \frac{1}{9\cdot34}T^2\omega^2 + \frac{1}{1680}T^4\omega^4$$

$$B = \frac{1}{2}T\omega - \frac{1}{84}T^3\omega^3.$$

For $T = 50$ msec,

$$A = 1 - 0\cdot000268\omega^2 + 0\cdot0000000037\omega^4$$

$$B = 0\cdot025\omega - 0\cdot00000149\omega^3.$$

At 20 hz,

$$A = -2\cdot30, \quad B = 0\cdot19$$

$$\therefore \quad \phi = F(j\omega) = 2\tan^{-1}(-0\cdot083) = 351°$$

i.e. a 9° phase error from the ideal 360°.

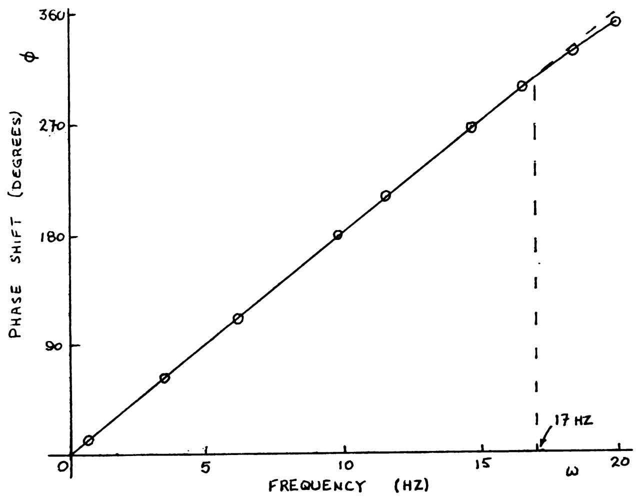

Fig. 2.—Phase shift of time delay with Set 1 coefficients

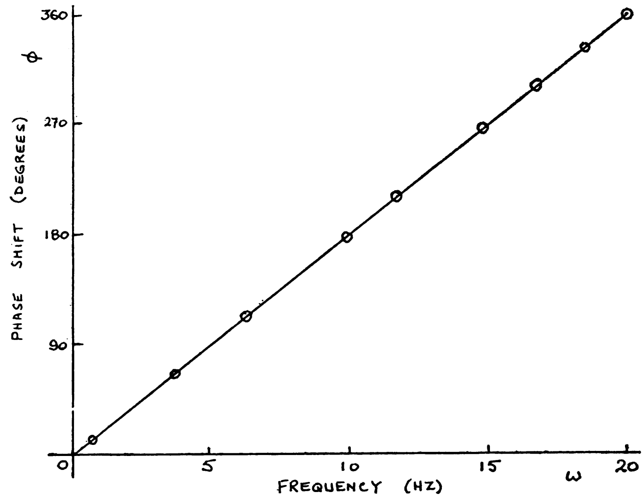

Fig. 3. Phase shift of time delay with Sets 2 and 3 coefficients

The phase error for Sets 2 and 3 coefficients can be obtained in a similar manner.

At 20 hz, Set 2 coefficients are

$$A = -1 \cdot 347, B = -0 \cdot 01$$

$$\therefore \quad \phi = 2 \tan^{-1}(0 \cdot 0074) = 360° \, 48'$$

i.e. less than 1° phase error.

At 20 hz, Set 3 coefficients are

$$A = -1 \cdot 97, \quad B = 0 \cdot 01$$

$$\therefore \quad \phi = 2 \tan^{-1}(-0 \cdot 005) = 359° \, 24'$$

i.e. less than 1° phase error.

These calculations are confirmed in practice and the resulting frequency responses of the delay for the three sets of coefficients are given in **Figs. 2** and **3.** Set 1 is found to give a phase shift which is linear only up to 17 hz. The other two sets have linear phase shifts right up to 20 hz.

To obtain a delay of 50 msec two ten-input operational amplifiers must be included in series with each coefficient potentiometer, k_T, in the delay circuit. With these potentiometers set at 0·2, a delay of exactly 50 msec is expected, but it is found that the delay is only 46·3 msec. Adjusting k_T to 0·197 restores the delay time to 50 msec. This disagreement can only be attributed to the difficulties created by the large number of operational amplifiers in series.

5. Conclusions

The computer circuit provides an accurate simulation of a time delay which can be made constant over a particular frequency range within the limits of the approximation. In the case of the fourth-order approximation, the maximum value of the product of the frequency range and the magnitude of the delay is 7·5 rads. This means that a small delay can be produced over a large frequency range or conversely a larger delay over a smaller frequency range. This product can be increased if necessary by taking higher order approximations.

The circuit is arranged in such a manner that the amount of delay can be varied.

The Set 1 coefficients do not give the largest delay possible over a given frequency range, whereas the other two sets come tolerably close. There is, in fact, little difference between the effects of the Sets 2 and 3 but the author's preference is for Set 3 for which the derivation is known.

Appendix

Referring to Fig. 1 and writing down the Laplacian form of the indicated voltages, we have:

$$F_a = -(k_1F_1 - k_1F_0)1/s$$

$$F_b = -(k_2F_1 + k_2F_0 + 10F_a/T_D)1/s$$

$$F_c = -(10k_3F_1 - 10k_3F_0 + 10F_b/T_D)1/s$$

$$F_d = -(10k_4F_1 + 10k_4F_0 + 10F_c/T_D)1/s$$

$$F_0 = +F_1 + 10F_d/T_D$$

$$\therefore \; F_b = -(k_2F_1 + k_2F_0)1/s + 10(k_1F_1 - k_1F_0)/s^2T_D$$

$$\therefore \; F_c = -(10k_3F_1 - 10k_3F_0)1/s + 10(k_2F_1 + k_2F_0)/s^2T_D - 100(k_1F_1 - k_1F_0)/s^3T_D^2$$

$$\therefore \; F_d = -(10k_4F_1 + 10k_4F_0)1/s + 10\,(10k_3F_1 - 10k_3F_0)/s^2T_D - 100(k_2F_1 + k_2F_0)/s^3T_D^2 + 1000\,(k_1F_1 - k_1F_0)/s^4T_D^3$$

$$\therefore F_0 = F_1 - 10(10k_4F_1 + 10k_4F_0)/sT_D$$
$$+100(10k_3F_1 - 10k_3F_0)/s^2T_D^2$$
$$-1000(k_2F_1 + k_2F_0)/s^3T_D^3$$
$$+10{,}000(k_1F_1 - k_1F_0)/s^4T_D^4.$$

Multiply throughout by $T_D^4s^4/10{,}000k_1$,

$$\therefore s^4T_D^4F_0/10{,}000k_1 = s^4T_D^4F_1/10{,}000k_1$$
$$- s^3T_D^3k_4(F_1 + F_0)/100k_1$$
$$+ s^2T_D^2k_3(F_1 - F_0)/10k_1$$
$$- sT_Dk_2(F_1 + F_0)/10k_1 + (F_1 - F_0).$$

Let

$$a_1 = k_2T_D/10k_1,\ a_2 = k_3T_D^2/10k_1,\ a_3 = k_4T_D^3/100k_1$$
$$a_4 = T_D^4 10{,}000k_1.$$

Re-arranging, we have

$$\therefore \frac{F_0}{F_1} = \frac{1 - a_1s + a_2s^2 - a_3s^3 + a_4s^4}{1 + a_1s + a_2s^2 + a_3s^3 + a_4s^4}$$

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Book Review

Mathematical Methods for Digital Computers, Volume 2, edited by A. RALSTON and H. S. WILF, 1967; 287 pages. (New York, Chichester, Sydney: *John Wiley and Sons Inc. 112s.*)

Volume 1 of *Mathematical Methods for Digital Computers* appeared in 1960 and contained twenty-six chapters, each giving a computer-oriented description of various processes or applications of numerical analysis. Volume 2 presents entirely new material but has the same format as Volume 1. Thus the formulation and mathematical description of a problem is followed by a concise summary of the computational procedure, a detailed schematic flow chart and a box-by-box description of each step. A complete FORTRAN program is also listed where space permits, and a small sample problem is given to illustrate typical behaviour of the process under description. A count of the number of arithmetic operations involved provides an estimate of the running time.

The present volume contains thirteen chapters grouped into six parts. Parts I and II each contain a single chapter, the first giving an introduction to the FORTRAN and ALGOL programming languages and the second describing applications of the quotient-difference algorithm. Parts III, IV and V have the titles 'Numerical Linear Algebra', 'Numerical Quadrature and Related Topics', and 'Numerical Solution of Equations', respectively, and contain three chapters each. The solution of ill-conditioned linear equations, the Givens-Householder method for symmetric matrices and the LU and QR algorithms for non-symmetric matrices are discussed in Part III, whilst Part IV includes Romberg quadrature, approximate multiple integration and the use of spline functions for interpolation and quadrature. Part V deals with general iterative methods for solving transcendental equations, the use of the 'resultant' procedure for the numerical solution of polynomial equations, and the application of alternating-direction methods to the solution of heat-conduction problems—partial differential equations are not encountered elsewhere in the book, incidentally. Part VI contains two chapters on random number generation and rational Chebyshev approximation.

The reduction in the number of chapters, as compared to Volume 1, has allowed considerably more space to be devoted in this volume to mathematical discussion and development. This is a particularly valuable feature of the book—combined with the extensive list of references, the chapters provide an excellent introduction to the topics discussed, besides giving a detailed presentation of particular methods. A minor criticism of Chapter 8 is the suggested use of successive overrelaxation to solve a set of linear equations with a triple-diagonal matrix of coefficients: a direct elimination method is surely preferable, and is indeed given in Chapter 11. The reader is also advised that incorrect results are quoted for the sample problem in Chapter 6 to illustrate the use of Romberg quadrature. The given FORTRAN program has, however, been rerun and produces correct answers.

The Editors have obviously been at pains to secure contributions from active research workers in the particular fields covered, and the result is a book which can be recommended for its expertise to numerical analysts and programmers alike. Inevitably some topics, such as the numerical solution of integral equations and the use of Chebyshev series for the solution or ordinary differential equations, are not discussed in either volume. Perhaps some future Volume 3 will cater for growing interest in these subjects.

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