# An offset vector iteration method for solving two-point boundary-value problems 

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#### Abstract

An offset vector iteration technique is proposed for solving two-point boundary-value problems. In this paper the properties of the method are explored. Application to parameter selection is first considered and convergence properties are described; comparison is made with other numerical methods. The two-point boundary-value problem is shown to be equivalent to the parameter selection problem. The method generally has a lower convergence rate than second order techniques; however, in many applications each iteration requires relatively few computational operations. Therefore it is competitive with higher order numerical procedures in applications that require few iterations to obtain an acceptably accurate solution. A modification to the offset vector method is suggested which takes advantage of the finite difference information generated at each iteration.


(First received September 1967 and in revised form February 1968)

## 1. Introduction

The use of offset vectors to develop iterative techniques for solving two-point boundary-value problems is a numerical procedure that has been proposed and investigated for use in near-earth (Godal, 1961), (Price and Boylan, 1964) and interplanetary guidance applications (Battin, 1964a), (Slater, 1966). The advantage of the method, when it can be applied, is that each iteration is often computationally simple to mechanise, relative to other techniques. In fact, there is evidence that it converges sufficiently rapidly in some cases to permit its use in real-time airborne guidance systems (Price et al., 1964). This study was motivated by the desire to utilise an offset vector method for solving certain two-point boundary-value problems that represent necessary conditions for optimal trajectories. An example of such an application is presented in a recent paper (Price, 1967).
The concept of the offset vector method is easily understood and motivated through a simple, familiar example. Consider the problem of hitting a target with a projectile fired from a gun that is stationary with respect to the target. Let the direction of the gun barrel on the $j$ th shot be designated by a unit vector, $i_{j}, j=1,2, \ldots$, expressed in an appropriate coordinate system. On the first shot, $j=1, i_{1}$ is some function,

$$
i_{1}=i_{1}\left(r_{T}\right)
$$

of the target's position, $r_{T}$. Suppose the first shot misses the target by a miss-vector, $\Delta r_{1}$, such that an impact point, $r_{1}$, is defined by

$$
r_{1}=r_{T}+\Delta r_{1}
$$

Using whatever quantitative knowledge of the miss he has, the gunner attempts to make an intelligent choice of the pointing direction on the next shot. If it happens
that $i_{2}$ is expressed in the functional form (however crude)

$$
i_{2}=i_{2}\left(r_{T}-\Delta r_{1}\right)
$$

where $\left(r_{T}-\Delta r_{1}\right)$ is a 'dummy' target position, we say that an offset vector iteration technique is being used. By analogy, on the $k$ th iteration
$i_{k+1}=i_{k+1}\left(r_{T}-\Delta r_{1}-\Delta r_{2}-\ldots-\Delta r_{k}\right) ; k=1,2 \ldots$
The philosophy is that on each iteration the aiming point is changed by the negative of the miss-vector. It is shown in this paper that such an approach is applicable to solving two-point boundary-value problems; in fact the above example can be formulated as such a problem.

Offset vector methods are ad hoc in nature because no general quantitative prescription is given for implementing the iterations. In the projectile example, the functional form of $i_{k+1}()$ depends upon the sophistication of the fire control system. This point is emphasised in the subsequent discussion. However, it appears that the convergence properties of the technique can be described, to some extent, without reference to any special application, and comparisons can be made with other numerical procedures. That is the primary purpose of this paper.

In the next three Sections the concept of offset vectors for solving parameter selection problems is more precisely defined, convergence properties are described, and a simple example is presented. In Sections 5 and 6 it is shown that the two-point boundary-value problem reduces to that of parameter selection and results of utilising the method in a typical physical application are given. In Section 7 a modification to the offset vector method is suggested which takes advantage of the finitedifference information generated at each iteration. This provides a means for making a transition from the offset vector method to a finite-difference version of the

[^0]Newton-Raphson technique in situations where many iterations are required.

## 2. An offset vector method for solving the parameter selection problem

Parameter selection or equation solving is simply the task of finding a value of an $n$-dimensional vector $x=x_{\infty}$ which satisfies the vector equation

$$
\begin{equation*}
g(x)=0 \tag{1}
\end{equation*}
$$

In parameter optimisation problems, equations of this form are necessary conditions that a function $\phi(x)$ have a stationary point. We assume that $g(x)$ also has dimension $n$ and that at least one solution of eqn. (1) exists.

Numerical techniques for solving eqn. (1) depend upon having an initial guess $x_{0}$ that is 'near' the desired solution $x_{\infty}$ and improving that guess by iteratively generating a sequence $\left\{x_{0}, x_{1}, \ldots\right\}$ which converges to $x_{\infty}$. Criteria for convergence of the sequence are usually given in terms of sufficient conditions satisfied by $g(x)$ in a region about $x_{\infty}$ containing $x_{0}$.

The most important property of any particular numerical method is the total time required to achieve a sufficiently accurate solution for $x_{\infty}$. This is dependent upon two factors-the number $m$ of iterations required to obtain a value $x_{m}$ that is sufficiently close to $x_{\infty}$, and the computational complexity of each iteration. One often observes that these factors are inversely related; that is, the simpler each iteration is to perform, the more iterations required to obtain a desired level of accuracy in the solution. This characteristic is evidence of the fact that the amount of progress made in each iteration toward $x_{\infty}$, i.e. the convergence rate, depends upon the amount of information used about $g(x)$ in deriving the recursion expressions.

Because the total time required for convergence is often dependent upon inversely related factors, it is difficult to state a priori in any particular application which of the various numerical methods is most advantageous from a computational point of view. However, if any initial guess $x_{0}$ is quite close to $x_{\infty}$, relatively simple iteration techniques may accomplish the required degree of accuracy with no more, or few more, iterations than more elaborate methods. This rationale provides the motivation for describing an offset vector iteration technique which is potentially simple to implement and is based upon the idea of having a reasonably accurate initial guess $x_{0}$; in fact, the structure of the method is defined by the manner in which $x_{0}$ is chosen.

Suppose one can find an $n$-dimensional vector function $\hat{g}(x)$ that approximates $g(x)$ such that the solution $x \equiv x_{0}$ of

$$
\begin{equation*}
\hat{g}(x)=0 \tag{2}
\end{equation*}
$$

is relatively easily determined.* For example, $g(x)$ and

[^1]

Fig. 1. Graphical development of the first two iterations of the offset vector method applied to a scalar function $g(x)$
$\hat{g}(x)$ may be of the form

$$
\left.\begin{array}{l}
g(x)=g+G x+\epsilon f(x)=0  \tag{3}\\
\hat{g}(x)=g+G x
\end{array}\right\}
$$

with $\epsilon$ a constant scalar, $g$ a constant vector, $G$ a nonsingular matrix and $f(x)$ some nonlinear function of $x$. If the term $\epsilon f(x)$ is small relative to $g(x)$ for $x$ near $x_{\infty}$, the solution $x_{0}=-G^{-1} g$, is near $x_{\infty}$. Let us write the solution to eqn. (2) as

$$
\begin{equation*}
x_{0}=\hat{g}^{-1}(0) \equiv h(0) \tag{4}
\end{equation*}
$$

where $\hat{g}^{-1}()$ represents the required inversion of $\hat{g}()$, and the argument 0 refers to the value of the right-hand side of eqn. (2). The situation is illustrated graphically in Fig. 1a for $n=1$.

Having $x_{0}$, we can evaluate

$$
\begin{equation*}
g\left(x_{0}\right) \equiv g_{0} \tag{5}
\end{equation*}
$$

noting the eqn. (1) is in general not satisfied, that is, $g_{0} \neq 0$. Based upon this observation an improvement to $x_{0}$ can be determined by the following reasoning. Suppose $\hat{g}(x)$ differs from $g(x)$ by only a constant vector $f_{0}$, that is,

$$
\begin{equation*}
\hat{g}(x)=g(x)+f_{0} ; \text { for all } x \tag{6}
\end{equation*}
$$

Then

$$
g\left(x_{0}\right)=-f_{0}=g_{0}
$$

If this be true, the solution to eqn. (1) is also the solution to

$$
\hat{g}(x)-f_{0}=0
$$

or

$$
\begin{equation*}
\hat{g}(x)=-g_{0} \tag{7}
\end{equation*}
$$

Thus we offset the approximating function by the negative of the error determined in eqn. (5) and calculate
$x_{1}$ from eqn. (7), using the notation of eqn. (4).

$$
\begin{equation*}
x_{1}=h\left(-g_{0}\right) . \tag{8}
\end{equation*}
$$

This sequence of operations is illustrated in Fig. 1b. The quantity $-g_{0}$ is analogous to $-\Delta r_{1}$ in the projectile problem of the previous section.

In general, $x_{1}$ does not satisfy eqn. (6) either, as evidenced by

$$
g\left(x_{1}\right) \equiv g_{1} \neq 0
$$

Accordingly, replace eqn. (6) by the conjecture

$$
\begin{equation*}
\hat{g}(x)=g(x)-g_{0}+f_{1} \tag{9}
\end{equation*}
$$

which leads to

$$
\begin{align*}
& g\left(x_{1}\right)=-f_{1}=g_{1} \\
& \hat{g}(x)=-g_{0}-g_{1} \tag{10}
\end{align*}
$$

resulting in

$$
\begin{equation*}
x_{2}=h\left(-g_{0}-g_{1}\right) \tag{11}
\end{equation*}
$$

These steps are shown in Fig. 1c.
The recursion relationships required for the continuation of this method are readily inferred from the preceding discussion. Define

$$
\begin{align*}
g_{j} & =g\left(x_{j}\right) \\
\gamma_{i} & =-\sum_{j=-1}^{i} g_{j} ; \quad i=-1,0,1, \ldots  \tag{12}\\
\gamma_{-1} & =-g_{-1}=0
\end{align*}
$$

and let

$$
\begin{equation*}
\hat{g}\left(x_{i}\right)=\gamma_{i-1} ; \quad i=0,1, \ldots \tag{13}
\end{equation*}
$$

Then,

$$
\left.\begin{array}{l}
\gamma_{i}=\gamma_{i-1}-g_{i} ; \quad i=0,1, \ldots  \tag{14}\\
x_{i}=\hat{g}^{-1}\left(\gamma_{i-1}\right)=h\left(\gamma_{i-1}\right)
\end{array}\right\}
$$

At each iteration one evaluation each of $g()$ and $h()$ is required. The quantity $\gamma_{i}$ is referred to as the offset vector. Now we shall discuss circumstances in which the sequence $\left\{x_{0}, x_{1}, \ldots\right\}$ generated by eqns. (12-14) converges to $x_{\infty}$.

## 3. Convergence properties

One expects that the convergence properties of the offset vector method depend upon the accuracy with which $\hat{g}(x)$ approximates $g(x)$. To pursue this reasoning define an error vector $\Delta g(x)$ by

$$
\begin{equation*}
g(x)=\hat{g}(x)+\Delta g(x) \tag{15}
\end{equation*}
$$

Substituting $x_{i}$ for $x$, we have

$$
\begin{equation*}
g\left(x_{i}\right)=\hat{g}\left(x_{i}\right)+\Delta g\left(x_{i}\right) \tag{16}
\end{equation*}
$$

Into eqn. (16) we can substitute for $\hat{g}\left(x_{i}\right)$ and $x_{i}$ from eqns. (13) and (14), producing

$$
\begin{equation*}
g\left(x_{i}\right)=g_{i}=\gamma_{i-1}+\Delta g\left[h\left(\gamma_{i-1}\right)\right] \tag{17}
\end{equation*}
$$

Rearranging terms and substituting for the quantity
( $\gamma_{i-1}-g_{i}$ ) from eqn. (14) yields

$$
\begin{equation*}
\gamma_{i}=-\Delta g\left[h\left(\gamma_{i-1}\right)\right] . \tag{18}
\end{equation*}
$$

Equation (18) is equivalent to eqn. (14) and is the recursion for solving

$$
\begin{equation*}
\gamma=-\Delta g[h(\gamma)] . \tag{19}
\end{equation*}
$$

by successive approximations. The solution, $\gamma_{\infty}$, to eqn. (19) is the limit of the sequence of offset vectors $\left\{\gamma_{0}, \gamma_{1}, \ldots\right\}$. Viewed another way, it is the value of $\hat{g}\left(x_{\infty}\right)$. (See Fig. 1.)
Sufficient conditions for the convergence of the sequence $\left\{\gamma_{i}\right\}$ are known for successive approximation iteration methods. For example, convergence is assured (Todd, 1962) if $\Delta g[h()]$ satisfies the Lipschitz condition

$$
\begin{align*}
\max \left|\Delta g\left[h\left(\gamma^{\prime}\right)\right]-\Delta g\left[h\left(\gamma^{\prime \prime}\right)\right]\right|<k \max \left|\gamma^{\prime}-\gamma^{\prime \prime}\right| & ; \\
0 & <k<1 \tag{20}
\end{align*}
$$

for all $\gamma^{\prime}$ and $\gamma^{\prime \prime}$ in a neighbourhood of $\gamma_{\infty}$ containing $\gamma_{-1}=0$.

Alternatively, a recursion relationship for $x_{i}$ can be derived from eqn. (14). Substituting for $\gamma_{i-1}$ and $\gamma_{i-2}{ }_{3}^{3}$ from respectively eqns. (14) and (13), we have

$$
\begin{equation*}
x_{i}=h\left[-\Delta g\left(x_{i-1}\right)\right] . \tag{21}
\end{equation*}
$$

The solution of this expression with $x_{i}$ and $x_{i-1}$ replaced by $x$ is the value of $x=x_{\infty}$ that renders $g\left(x_{\infty}\right)=0$ and $\hat{g}\left(x_{\infty}\right)=\gamma_{\infty}$.

A third way of viewing the iterative procedure is that the sequence $\left\{g_{0}, g_{1}, \ldots\right\}$ of evaluations of $g\left(x_{i}\right)$ is being $\frac{\text { 에 }}{}$ driven to a limit of zero. This is perhaps the most natural point of view for the applications to be considered subsequently. From eqns. (12)-(14) it is evident $\stackrel{\sim}{\circ}$ that $g\left(x_{i}\right)$ is a nonlinear function of all $g\left(x_{j}\right), j<i$, of the form

$$
\begin{equation*}
g_{i}=g\left[h\left(0-g_{0}-g_{1}-\ldots-g_{i-1}\right)\right] . \tag{22}
\end{equation*}
$$

Similarly,

$$
\begin{equation*}
g_{i+1}=g\left[h\left(0-g_{0}-g_{1}-\ldots-g_{i}\right)\right] . \tag{23}
\end{equation*}
$$

Linearising $g_{i+1}$ about $g_{i}$ with substitution from eqns. $\stackrel{\rightharpoonup}{3}$ (12)-(14) we have

$$
\begin{equation*}
g_{i+1} \cong g_{i}-G\left(x_{i}\right) H\left(\gamma_{i-1}\right) g_{i} ; \quad i=0,1, \ldots \tag{24}
\end{equation*}
$$

where

$$
\begin{equation*}
G(x) \equiv \frac{\partial g(x)}{\partial x} ; \quad H(\gamma) \equiv \frac{\partial h(\gamma)}{\partial \gamma} . \tag{25}
\end{equation*}
$$

Equation (24) indicates that

$$
\begin{array}{cc} 
& \lim _{i \rightarrow \infty} g_{i}=0 \\
\text { if } & \|I-G H\|<1 \tag{26}
\end{array}
$$

in some sufficiently small region about $x_{\infty}$ such that the linearisation is valid. Note that if $g()=\hat{g}(), G H=I$.

These convergence properties provide a comparison
between the offset vector method and other procedures that can be employed for finding $x_{\infty}$. Considering eqn. (19), perhaps the most significant observation is that the method does not possess second-order convergence because the gradient matrix corresponding to eqn. (19),

$$
\left.\frac{\partial \Delta g[h(\gamma)]}{\partial \gamma}\right|_{\gamma_{\infty}} \neq 0
$$

in general (Todd, 1962). Thus a Newton-Raphson technique, beginning at $x_{0}$ may require fewer iterations to approach $x_{\infty}$ within a desired accuracy. However, the offset vector method possesses two advantages that motivate its use in certain situations.

First, applications arise in which $g(x)$ cannot be expressed in closed form, such as the solutions of many two-point boundary value problems. In these cases every evaluation of $g(x)$ requires numerical integration of differential equations. In addition, for Newton-Raphson-type procedures the gradient matrix must also be computed numerically, requiring additional complete integrations of the appropriate differential equation, for each iteration. Hence, if the approximation $\hat{g}(x)$ is sufficiently accurate, one may conceivably reach a point sufficiently close to $x_{\infty}$ with an offset vector technique before a higher order method gets started. The offset vector method has proved sufficiently rapid in situations of this kind to be incorporated in a real-time airborne guidance system (Price et al., 1964). An example of such an application is included in Section 6.

Second, the offset vector method is a reasonable starting procedure for a higher order method in situations where many iterations are required. The points $x_{0}, x_{1}, \ldots$ and associated values $g_{0}, g_{1}, \ldots$ can be stored to provide corrections, based on finite differences, to subsequent evaluations of $x_{i}$. A possible method for accomplishing this is described in Section 7.

There is the disadvantage that some means must exist for finding an appropriate $\hat{g}(x)$. Whether this can be done depends upon the particular problem and the analyst's ingenuity; for this reason the concept of offset vectors does not provide a ready-made numerical algorithm for attacking all parameter selection problems. The fact that applications are known (see the references mentioned in Section 1 and the example of Section 6) where the method can be applied is a testimonial to its usefulness.

## 4. Example 1

To illustrate the offset vector method, a simple onedimensional example is presented using equation numbers corresponding to those expressions in preceding sections which are exemplified.

Given

$$
\begin{equation*}
g(x)=1+x+\epsilon x^{3}=0 \tag{1}
\end{equation*}
$$

Let

$$
\begin{equation*}
\hat{g}(x)=1+x \tag{2}
\end{equation*}
$$

Then

$$
\begin{gather*}
\gamma_{i}=\gamma_{i-1}-g_{i} \\
x_{i}=h\left(\gamma_{i-1}\right)=\gamma_{i-1}-1 \tag{14}
\end{gather*}
$$

Using the criterion for convergence provided by eqn. (26), we find that

$$
\begin{gather*}
G(x)=1+3 \epsilon x^{2} ; \quad H(\gamma)=1  \tag{25}\\
3 x^{2}|\epsilon|<1 \tag{26}
\end{gather*}
$$

Furthermore, from eqn. (24)

$$
\begin{equation*}
\left|\frac{g_{i+1}}{g_{i}}\right| \cong 3 x_{i}^{2}|\epsilon| \tag{24}
\end{equation*}
$$

which provides a measure of the convergence rate.
It should be emphasised again that the offset vector method is not promoted especially for a high convergence rate. In general, and for this example in particular, it converges more slowly than Newton's method. The main advantage is the relative simplicity with which each iteration can be performed. This is illustrated by observing that the recursion relationships in eqn. (14) for this example require two subtractions and one evaluation of $g(x)$ per iteration. On the other hand, Newton's formula,

$$
x_{i+1}=x_{i}-\frac{g\left(x_{i}\right)}{g^{\prime}\left(x_{i}\right)} ; \quad g^{\prime}\left(x_{i}\right)=\left.\frac{d g(x)}{d x}\right|_{x=x_{i}}
$$

requires one subtraction, one division, one evaluation of $g(x)$, and one evaluation of $d g(x) / d x$ per iteration; clearly this entails significantly more computation. The total time required to obtain an acceptably accurate solution for $x_{\infty}$ is less for the offset vector method if $|\epsilon|$ is sufficiently small so that only one iteration of either method is required.

In situations where $g(x)$ has several dimensions and a complicated functional form, the computational advantages offered by an offset vector method are more significant. As mentioned previously, it is competitive with higher order techniques when a sufficiently good approximate solution can be obtained. In applications where the problem must be solved repeatedly, as in rocket guidance systems, considerable computational saving may be gained. This is illustrated by the example in Section 6.

## 5. The two-point boundary-value problem

The use of offset vectors to develop iterative techniques for solving two-point boundary-value problems is a numerical procedure that has been applied to near-earth (Godal, 1961), (Price et al., 1964) and interplanetary guidance (Battin, 1964a), (Slater et al., 1966) problems. In this section it is shown that the convergence properties can be stated in the same terms as for the parameter selection problem.

A two-point boundary-value problem is posed by assuming a given dynamical system described by $n$-dimensional vector differential equations

$$
\begin{equation*}
\dot{x}=f(x, t) \tag{27}
\end{equation*}
$$

with prescribed end conditions

$$
\left.\begin{array}{l}
\Phi\left[x\left(t_{0}\right), t_{0}\right]=0  \tag{28}\\
\psi\left[x\left(t_{f}\right), t_{f}\right]=0
\end{array}\right\}
$$

where $t_{0}$ and $t_{f}$ are initial and final times, $x$ is an $n$-dimensional state vector, $\Phi$ and $\psi$ are respectively $l$ - and $m$-dimensional vectors, with $l+m=n+2$. It is assumed that a solution exists which cannot be determined in closed form, requiring the use of numerical techniques.

We shall regard the solution to eqn. (27) known when the complete set of initial conditions $x\left(t_{0}\right), t_{0}$ is determined such that eqns. (27) and (28) are satisfied. The explicit dependence upon eqn. (27) is conceptually eliminated by writing the solution as

$$
\begin{equation*}
x(t)=x\left[x\left(t_{0}\right), t_{0}, t\right] \tag{29}
\end{equation*}
$$

so that eqn. (28) becomes

$$
g\left[x\left(t_{0}\right), t_{0}, t_{f}\right]=\left[\begin{array}{l}
\Phi\left[x\left(t_{0}\right), t_{0}\right]  \tag{30}\\
\psi\left\{x\left[x\left(t_{0}\right), t_{0}, t_{f}\right], t_{f}\right\}
\end{array}\right]=0
$$

Equation (30) has the form of eqn. (1) where the parameters to be determined are $x\left(t_{0}\right), t_{0}$, and $t_{f}$.

The offset vector method is implemented in a manner analogous to that described in Section 2. Approximate solvable relations

$$
\begin{equation*}
\hat{g}\left[x\left(t_{0}\right), t_{0}, t_{f}\right]=0 \tag{31}
\end{equation*}
$$

are derived, often by means of a simplified set of differential equations

$$
\begin{equation*}
\dot{x}=\widehat{f}(x, t) \tag{32}
\end{equation*}
$$

subject to eqn. (28). For example, eqn. (27) may describe motion in a many-body gravitational field and eqn. (32) may represent an approximating two-body model with eqn. (28) specifying the initial and final positions at specified times. The solutions $x_{0}\left(t_{0_{0}}\right), t_{0_{0}}$, and $t_{f_{0}}$ of eqn. (31) are entered as initial conditions into eqn. (27), and the differential equations are integrated from $t_{0_{0}}$ to $t_{f_{0}}$ producing

$$
\begin{equation*}
x_{0}\left(t_{f_{0}}\right) \equiv x\left[x_{0}\left(t_{0_{0}}\right), t_{0_{0}}, t_{f_{0}}\right] \tag{33}
\end{equation*}
$$

Substitution of $t_{0_{0}}, t_{f_{0}}$ and $x_{0}\left(t_{0_{0}}\right)$ for $t_{0}, t_{f}$, and $x\left(t_{0}\right)$ in eqn. (30) yields

$$
g\left[x_{0}\left(t_{0_{0}}\right), t_{0_{0}}, t_{f_{0}}\right]=g_{0} \neq 0
$$

in general. Defining the vector

$$
z^{T}=\left[x\left(t_{0}\right)^{T}, t_{0}, t_{f}\right]
$$

the iterative computation of the sequence $\left\{z_{0}, z_{1}, \ldots\right\}$ proceeds just as in Section 2 with the understanding that each evaluation of

$$
g\left[x_{i}\left(t_{0_{i}}\right), t_{0_{i}}, t_{f_{i}}\right]=g_{i}
$$

requires integration of eqn. (27).
The motivation for using offset vectors is now more apparent. Vis-à-vis higher order methods it may be of
considerable computational advantage to obtain even an algebraically complex form of eqn. (31) if computation of the gradient of $g\left[x\left(t_{0}\right), t_{0}, t_{f}\right]$ is thereby avoided. A practical multidimensional example of this type is considered in the next section. Observe that the projectile problem discussed in the Introduction can also be formulated as a two-point boundary-value problem and its solution obtained in the manner described above.

## 6. Example 2

This section discusses an application of the offset vector method to a practical two-point boundary-value problem. Equation numbers denote those expressions in previous sections which are exemplified.

Consider the motion of a body in a planar orbit in the earth's gravitational field. If the earth's rotation and atmospheric friction are neglected,* the equations of motion are reasonably accurately represented by

$$
\left.\begin{array}{rl}
\dot{x} & =v_{x} \\
\dot{v}_{x} & =-\frac{x}{r^{3}}\left[E+\frac{J E A^{2}}{r^{2}}-\frac{5 J E A^{2} z^{2}}{r^{4}}\right]  \tag{27}\\
\dot{z} & =v_{z} \\
\dot{v}_{z} & =-\frac{z}{r^{3}}\left[E+\frac{J E A^{2}}{r^{2}}-\frac{5 J E A^{2} z^{2}}{r^{4}}\right]-\frac{2 J E A^{2} z}{r^{5}}
\end{array}\right\}
$$

where $A$ is the equatorial radius, $J$ and $E$ are constants, $r=\sqrt{ }\left(x^{2}+z^{2}\right)$, and $x$ and $z$ are position coordinates in an orthogonal coordinate system with the $z$ axis along the earth's polar axis. Because the orbit is polar, only two dimensions need be considered. Equations (27) describe the gravitational accelerations including the effects of the earth's slightly elliptical shape. Let us pose the problem of finding the initial velocity components, $v_{x}\left(t_{0}\right)$ and $v_{z}\left(t_{0}\right)$, required to transfer a body from a given initial position at time $t_{0}=0$ to a given final position at a specified final time. Hence

$$
\left.\begin{array}{ll}
t_{0}=0 & t_{f}-T_{f}=0 \\
x\left(t_{0}\right)-a_{x}=0 & x\left(t_{f}\right)-b_{x}=0  \tag{28}\\
z\left(t_{0}\right)-a_{z}=0 & z\left(t_{f}\right)-b_{z}=0
\end{array}\right\}
$$

where $a_{x}, a_{z}, b_{x}, b_{z}$, and $T_{f}$ are given.
For the case where the earth's oblate effects are neglected ( $J=0$ in eqn. (27)), the task of finding the initial velocities subject to the given conditions is the familiar Lambert's problem of classical mechanics. For this case eqn. (27) can be integrated analytically by changing the independent variable; several methods of obtaining explicit expressions for $g(x)$ are known (Battin, 1964b). For $J \neq 0$, there is no known method of integrating eqn. (27) analytically; hence a numerical technique is required.

The offset vector method is naturally adapted to this application by using the known solution to Lambert's

[^2]

Fig. 2. Computational flow diagram of the $i$ th iteration in Example 2
problem with $J=0$ as an approximation. Introducing $J=0$ into eqn. (27) produces a set of equations represented by eqn. (32) in Section 5. For the terminal conditions prescribed by eqn. (28), one form of $\hat{g}(x)$ due to Godal (Battin, 1964b) is given by

$$
\left.\begin{array}{l}
v_{x}(0)-C_{1}\left(b_{x}-C_{2} a_{x}\right)=0 \\
v_{z}(0)-C_{1}\left(b_{z}-C_{2} a_{z}\right)=0 \\
C_{1}-\frac{\sqrt{ }(E P)}{r_{f} r_{0} \sin \theta}=0 \\
C_{2}-1+\frac{r_{f}}{P}(1-\cos \theta)=0 \\
r_{0}-\sqrt{ }\left(a_{x}^{2}+a_{z}^{2}\right)=0 \\
r_{f}-\sqrt{ }\left(b_{x}^{2}+b_{z}^{2}\right)=0 \\
\theta-\cos ^{-1}\left[\left(a_{x} b_{x}+a_{z} b_{z}\right) / r_{0} r_{f}\right]=0  \tag{31}\\
P-\frac{\sqrt{ }\left(r_{0} r_{f}\right) \sin ^{2} 0 \cdot 5 \theta}{(B-\cos \alpha) \cos 0 \cdot 5 \theta}=0 \\
B-\left(r_{0}+r_{f}\right) / 2 \sqrt{ }\left(r_{0} r_{f}\right) \cos 0 \cdot 5 \theta=0 \\
T_{f}-2\left\{\left(\sqrt{ }\left(r_{0} r_{f}\right) \cos 0 \cdot 5 \theta\right)^{1 \cdot 5} \sqrt{\left(\frac{B-\cos a}{E}\right.}\right) \\
\left.\quad\left[1+\frac{(B-\cos \alpha)(2 \alpha-\sin 2 \alpha)}{2 \sin ^{3} \alpha}\right]\right\}=0
\end{array}\right\}
$$

The solutions to eqns. (31) are the proper initial velocities to achieve the conditions in eqns. (28), neglecting the oblateness of the earth. Observe that eqns. (31) are transcendental in $\alpha$; therefore their solution must be obtained numerically. This represents a situation where eqns. (2) cannot be inverted analytically.

The offset vector method proceeds by carrying out the following steps:

1. Denote the solutions of eqn. (31) as $v_{x 0}\left(t_{0}\right)$ and $v_{z 0}\left(t_{0}\right)$; these are obtained by any convenient numerical method. Newton's method has been used in this simulation.
2. Integrate eqn. (27) from $t=0$ to $t=T_{f}$ using $a_{x}, a_{z}, v_{x 0}\left(t_{0}\right)$ and $v_{z 0}\left(t_{0}\right)$ as initial conditions. Denote position on this trajectory by $x_{0}(t)$ and $z_{0}(t)$.
3. Evaluate the left hand sides of eqn. (28) for the integrated trajectory. Define

$$
\begin{aligned}
& \Delta x_{0}\left(T_{f}\right) \equiv x_{0}\left(T_{f}\right)-b_{x} \\
& \Delta z_{0}\left(T_{f}\right) \equiv z_{0}\left(T_{f}\right)-b_{z} .
\end{aligned}
$$

4. Recompute the initial velocities from eqn. (31) by requiring

$$
\begin{aligned}
x\left(T_{f}\right)-b_{x} & =-\Delta x_{0}\left(T_{f}\right) \\
z\left(T_{f}\right)-b_{z} & =-\Delta z_{0}\left(T_{f}\right) .
\end{aligned}
$$

This implies that eqn. (31) undergoes the changes of variable,

$$
\begin{gathered}
b_{x} \rightarrow b_{x}-\Delta x_{0}\left(T_{f}\right) \equiv b_{x_{0}} \\
b_{z} \rightarrow b_{z}-\Delta z_{0}\left(T_{f}\right) \equiv b_{z 0} .
\end{gathered}
$$

Denote the solutions as $v_{x 1}\left(t_{0}\right)$ and $v_{z 1}\left(t_{0}\right)$.
5. Repeat steps 2 through 4 in an iterative fashion. The functional diagram in Fig. 2 illustrates the steps at the $i$ th iteration.
For this simulation the following parameter values are used:

$$
\begin{array}{ll}
a_{x}=2.093 \times 10^{7} \text { feet } & E=1.407645 \times 10^{16} \\
a_{z}=0.0 \text { feet } & J=1.62345 \times 10^{-3} \\
b_{x}=0.0 \text { feet } & A=2.093 \times 10^{7} \text { feet } \\
b_{z}=3.0 \times 10^{7} \text { feet } & T_{f}=2400.0 \text { seconds }
\end{array}
$$

This roughly represents insertion into a 2000 -mile altitude orbit at a point above the pole from a point on the equator. The computation was performed in double precision arithmetic on an IBM 360/65 computer. Newton's method is applied to solve Lambert's problem and a Gill-modified Runge-Kutta integration technique is used to integrate eqn. (27) with a 20 second time step. The values of terminal position error, $\Delta x_{i}\left(T_{f}\right)$ and $\Delta z_{i}\left(T_{f}\right)$, for two iterations are given in Table 1:

Table 1

## Position error data from simulation

(Rounded off to 3 significant figures)


Fig. 3. Progress of finite-difference modification of offset vector method in two dimensions

Adequate accuracy is obtained in one iteration for many applications. For these cases any other numerical method that has an equal or greater convergence rate can be compared on the basis of the computational complexity of each iteration.

In this simulation the time required to solve Lambert's problem with sufficient accuracy is approximately 0.01 seconds whereas that required for integrating eqn. (27) is 0.30 seconds. Because the latter* dominates

[^3]the former, any method that requires more differential equations to be integrated is at a competitive disadvantage with the offset vector method. For NewtonRaphson type procedures, the gradient matrix of $g()$ with respect to $v_{x i}\left(t_{0}\right)$ and $v_{z i}\left(t_{0}\right)$ must be obtained. This can be obtained numerically by perturbing each velocity component separately and integrating eqn. (27) to determine the effect on the end conditions. Obtaining the complete gradient matrix by this procedure requires $n$ additional complete integrations of eqn. (27) per iteration; this results in tripling the amount of integration required in this example, effectively tripling the computation time for each iteration. The gradient matrix can also be obtained by integrating the linear variational equations associated with eqn. (27); however, the increased computation is of the same order as that required to obtain the matrix by the perturbation technique.

These comparisons indicate that the offset vector method is superior to higher order methods in some problems. The example considered here has application to rocket guidance for which the thrust is directed so that the vehicle's velocity matches the values of $v_{x i}\left(t_{0}\right)$ and $v_{z i}\left(t_{0}\right)$ in Fig. 2. The two-point boundary-value problem must be solved many times in rapid succession because the initial time and the rocket's position are constantly changing. For 'real-time' computation of this sort, speed is a primary consideration.

## 7. Modified offset vector method

In Section 3 it is pointed out that the offset vector method can serve as a starting procedure for higher-order techniques. The possibility for doing this is evident at the $(n+1)$ th step after the sequences $\left\{x_{0}, x_{1}, \ldots x_{n}\right\}$ and $\left\{g_{0}, g_{1}, \ldots g_{n}\right\}$ have been computed. Defining

$$
\left.\begin{array}{l}
\Delta x_{i} \equiv x_{i}-x_{i-1}  \tag{34}\\
\Delta g_{i} \equiv g_{i}-g_{i-1}
\end{array}\right\}
$$

we have sufficient information to derive an approximate gradient matrix (or its inverse) provided the $\Delta x_{i}$ 's (or $\Delta g_{i}$ 's) are independent. For example,

$$
\begin{equation*}
\frac{\partial g}{\partial x} \cong \tilde{G}=X^{-1} \mathscr{G} \tag{35}
\end{equation*}
$$

where $\mathscr{G}$ and $X$ are matrices whose $i$ th columns are respectively $\Delta g_{i}$ and $\Delta x_{i}$. Faster convergence may possibly be obtained by continuing the numerical procedure with a Newton-Raphson-like technique using $\widetilde{G}$ to determine new values of $x$ according to

$$
\begin{equation*}
x_{i+1}=-\widetilde{G}_{i}^{-1} g_{i} \tag{36}
\end{equation*}
$$

where $\widetilde{G_{i}}$ depends upon the last $n$ values of $\Delta g$ and $\Delta x$. In this section we shall describe a recursive method whereby the gradient information available at each stage is utilised to adjust the offset vector computation, producing results analogous to eqn. (36).

Consider the first two steps in the offset vector method after which $x_{0}, x_{1}, g_{0}$ and $g_{1}$ are known. These 'points' are indicated for a two-dimensional case in Figs. 3a and 3b. With $\Delta x_{1}$ and $\Delta g_{1}$ thereby determined, we can calculate the required first order change $\Delta x_{1}^{\prime}$ in $x$ to produce a desired change $\Delta g^{\prime}$ in $g$ in the direction of $\Delta g_{1}$ :

$$
\begin{equation*}
\Delta x_{1}^{\prime}=\frac{\Delta g^{\prime}}{\left|\Delta g_{1}\right|} \Delta x_{1} \tag{37}
\end{equation*}
$$

Note that $\Delta g^{\prime}$ is a scalar that may be either positive or negative. Out objective being to drive $g$ to zero, to first order (approximately*), we can remove that component in the direction parallel to $\Delta g_{1}$ by defining

$$
\left.\begin{array}{rl}
\Delta g_{1}^{\prime} & =-\left(g_{1} \cdot i_{\Delta g_{1}}\right) i_{\Delta g_{1}} ; \quad i_{\Delta g_{1}} \equiv \frac{\Delta g_{1}}{\left|\Delta g_{1}\right|}  \tag{38}\\
\Delta x_{1}^{\prime} & =-\left(g_{1} \cdot i_{\Delta g_{1}}\right) \Delta x_{1} /\left|\Delta g_{1}\right| \\
x_{1}^{\prime} & =x_{1}+\Delta x_{1}^{\prime} \\
g_{1}^{\prime} & =g_{1}+\Delta g_{1}
\end{array}\right\}
$$

These quantities are illustrated in Fig. 3. Note that $-\left(g_{i} \cdot i_{\Delta_{g 1}}\right)$ in eqn. (38) plays the role of $\Delta \mathrm{g}^{\prime}$ in eqn. (37).

There is as yet no gradient information available in the direction normal to $\Delta g_{1}$ so, at this point, return to the offset vector algorithm. First, using eqn. (13) calculate the value $\gamma_{0}^{\prime}$ of the offset vector that corresponds to $x_{1}^{\prime}$ :

$$
\begin{equation*}
\gamma_{0}^{\prime}=\hat{g}\left(x_{1}^{\prime}\right) \tag{39}
\end{equation*}
$$

Assume that

$$
\begin{equation*}
g\left(x_{1}^{\prime}\right) \cong g_{1}+\Delta g_{1}^{\prime} \equiv g_{1}^{\prime} \tag{40}
\end{equation*}
$$

note that exact equality does not hold because $\Delta g_{1}^{\prime}$ is computed from a linearised analysis. Now let

$$
\left.\begin{array}{l}
\gamma_{1}=\gamma_{0}^{\prime}-g_{1}^{\prime}  \tag{41}\\
x_{2}=h\left(\gamma_{1}\right) \\
g_{2}=g\left(x_{2}\right)
\end{array}\right\}
$$

This completes a new step in the iteration process. Observe that the same number of evaluations of $g(x)$ are required as for the offset vector method. The difference is that $x_{2}$ is computed with the aid of an intermediate value $x_{1}^{\prime}$ that is calculated by a finite difference projection.

From $x_{2}$ and $g_{2}$ the quantities

$$
\begin{equation*}
\Delta x_{2}=x_{2}-x_{1} ; \quad \Delta g_{2}=g_{2}-g_{1} \tag{42}
\end{equation*}
$$

are calculated as illustrated in Fig. 3. In the twodimensional case $\Delta x_{1}, \Delta x_{2}, \Delta g_{1}$ and $\Delta g_{2}$ provide sufficient information to continue the search for $x_{\infty}$ by a finite difference method alone, provided the $\Delta x$ 's and $\Delta g$ 's are independent. In higher dimensions we can proceed as before, calculating an intermediate $x_{2}^{\prime}$ based upon finite difference projections in both $\Delta g_{1}$ and $\Delta g_{2}$

[^4] in the direction $\Delta g_{1}$ is computed from a finite difference.


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Fig. 4. Illustration of orthogonalisation of the vectors $\Delta g_{i}$ with the associated transformation on the $\boldsymbol{\Delta} \boldsymbol{x}_{\boldsymbol{i}}$
directions and using the offset vector to find corrections to $x_{2}^{\prime}$ in the remaining directions. Here we shall derive a recursion based upon orthogonalisation of the vectors $\Delta g_{1}$.

Suppose $\Delta g_{1}, \Delta g_{2}, \Delta x_{1}$ and $\Delta x_{2}$ are given as shown in Fig. 4. The component of $\Delta g_{2}$ orthogonal to $\Delta g_{1}$ is given by

$$
\delta g_{2}=\Delta g_{2}-\left(\Delta g_{2} . i_{\Delta g_{1}}\right) i_{\Delta_{g 1}}
$$

According to eqn. (37), the associated change in $x$ required to accomplish the increment $\delta g_{2}$ is given by

$$
\delta x_{2}=\Delta x_{2}-\left(\Delta g_{2} \cdot i_{\Delta_{1}} /\left|\Delta g_{1}\right|\right) \Delta x_{1}
$$

Defining $\delta x_{1}=\Delta x_{1}$ and $\delta g_{1} \equiv \Delta g_{1}$, we can calculate the change $\Delta g_{2}^{\prime}$ required to drive the projection of an $n$-dimensional vector $g_{2}$ on the space of the orthogonal vectors $\delta g_{1}$ and $\delta g_{2}$ to zero. Requiring

$$
\left(\Delta g_{2}^{\prime}+g_{2}\right) \cdot \delta g_{i}=0 ; \quad i=1,2
$$

we have

$$
\Delta g_{2}^{\prime}=-\left(g_{2} . i_{\delta_{g_{1}}}\right) i_{\Delta g_{1}}-\left(g_{2} . i_{\Delta g_{2}}\right) i_{\delta_{g 2}}
$$

The associated change in $x, \Delta x_{2}^{\prime}$, is given by

$$
\Delta x_{2}^{\prime}=-\left(g_{2} . i_{\partial g_{1}} /\left|\delta g_{1}\right|\right) \delta x_{1}-\left(g_{2} . i_{\delta g_{2}} /\left|\delta g_{2}\right|\right)_{\delta x_{2}}
$$

Having $\Delta g_{2}^{\prime}$ and $\Delta x_{2}^{\prime}$, we can calculate $x_{1}^{\prime}, g_{2}^{\prime}, \gamma_{1}, \gamma_{2}, x_{3}$, and $g_{3}$ from eqns. (38), (39) and (41) by increasing the value of each subscript by one.

This reasoning leads to the following set of recursion relationships for deriving $x_{i+1}$, having $\left\{x_{0}, x_{1}, \ldots x_{i}\right\}$, $\left\{g_{0}, g_{1}, \ldots g_{i}\right\}$, orthogonal directions $\left\{\delta g_{1}, \delta g_{2}, \ldots \delta g_{i-1}\right\}$, and the corresponding set of 'influence' directions $\left\{\delta x_{1}, \delta x_{2}, \ldots \delta x_{i-1}\right\}:$

$$
\begin{aligned}
g_{i} & =g\left(x_{i}\right) \\
\Delta x_{i} & =x_{i}-x_{i-1} \\
\Delta g_{i} & =g_{i}-g_{i-1} \\
\delta g_{i} & =\Delta g_{i}-\sum_{j=1}^{i-1}\left(\Delta g_{i} \cdot i_{\delta g_{j}}\right) i_{\delta g_{j}} \\
\delta x_{i} & =\Delta x_{i}-\sum_{j=1}^{i-1}\left(\Delta g_{i} \cdot i_{\delta g_{j}} /\left|\delta g_{j}\right|\right) \Delta x_{j} \\
\Delta g_{i}^{\prime} & =-\sum_{j=1}^{i}\left(g_{i} \cdot i_{\delta g j}\right) i_{\delta g j} \\
\Delta x_{i}^{\prime} & =-\sum_{j=1}^{i}\left(g_{i} \cdot i_{\delta g_{j}} /\left|\delta g_{j}\right|\right) \delta x_{j} \\
x_{i}^{\prime} & =x_{i}+\Delta x_{i}^{\prime} \\
g_{i}^{\prime} & =g_{i}+\Delta g_{i}^{\prime} \\
\gamma_{i-1}^{\prime} & =h\left(x_{i}^{\prime}\right) \\
\gamma_{i} & =\gamma_{i-1}^{\prime}-g_{i}^{\prime} \\
x_{i+1} & =h\left(\gamma_{i}\right)
\end{aligned}
$$

To start the process, two iterations of the unmodified offset vector method are performed to provide values of $x_{0}, x_{1}, g_{0}$ and $g_{1}$. For $i \geq n$, we can discard all $\delta x_{j}$ and $\delta g_{j}$ for $j \leq i-n$; one set of directions is then effectively removed at each step to be replaced by $\delta g_{i}$ and $\delta x_{i}$. Furthermore, for $i \geqq n$ the last four expressions of eqn. (43) can be disregarded if we let

$$
\begin{equation*}
x_{i ; 1} \equiv x_{i}^{\prime} ; \quad i \geq n . \tag{44}
\end{equation*}
$$

That is, a Newton-Raphson-like procedure, using approximate derivatives can be substituted for the offset vector method at the $n$th step.

## 8. Summary and conclusions

The offset vector method presented here is one that has been utilised to solve mathematical problems arising from special applications. The technique has evolved in this fashion because it requires knowledge of an approximate solution whose availability is dependent upon the physical situation. The purpose of this paper is to give the method more formal status as a numerical technique by presenting a recipe for its implementation, by developing criteria for convergence, and by illustrating its advantages through examples. It is found that the convergence rate is generally slower than that of second and higher order methods, but each iteration is relatively rapid to perform. Possible applications are those where few iterations are required or as a starting procedure for higher order methods when many iterations are necessary.

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[^0]:    * Staff Member, Experimental Astronomy Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts. This research has been sponsored by NASA ERC Contract No. NGR 22-009-207.

[^1]:    *This is not to say that $x_{0}$ need be determined by an explicit formula; the solution to eqn. (2) may also have to be obtained numerically. An example of this kind is given in Section 6.

[^2]:    * It is recognised that neglect of the earth's rotation contradicts the intent of treating a practical example. However, this effect can be included without changing the qualitative interpretation of the numerical results; it is omitted only to reduce the complexity of the discussion.

[^3]:    * An integration step three or four times larger than 20 seconds would give terminal position accuracy better than 100 feet in this example.

[^4]:    * This is not an exact first order calculation because the gradient

