Neville's method for trigonometric interpolation

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In this note, the well-known interpolation methods of Aitken and Neville are adapted to trigonometric interpolation.

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1. Introduction

Suppose that $f(x)$ is a periodic function of x, specified by its values $y_0, y_1, \ldots, y_n$ at $(n+1)$ distinct points $x = x_0, x_1, \ldots, x_n$. The problem in trigonometric interpolation is to approximate $f(x)$ by a trigonometric polynomial, $\bar{f}(x)$, say, which agrees with it at the data-points. If we assume, for convenience, that the period of $f(x)$ is 2π, and that all the values x_i $(i = 0, 1, \ldots, n)$, lie in the range $-\pi < x_i \leqslant \pi$, the interpolation polynomial takes the form

$$\bar{f}(x) = a_0 + \sum_{r=1}^{m} (a_r \cos rx + b_r \sin rx). \qquad (1)$$

The choice of m in (1) depends on the form of the required fit; this point will be discussed below.

A number of formulae for trigonometric interpolation have been devised—see, e.g. Whittaker and Robinson (1944), Berezin and Zhidkov (1965). The object of this paper is to adapt the well-known interpolation methods of Aitken (1932) and Neville (1934) to the problem. We shall express the results in terms of Neville's form of the method.

In what follows, we shall denote by $f_{ij}(x)$, where $i \leqslant j$, the trigonometric polynomial of the required form which assumes the values $y_i, y_{i+1}, \ldots, y_j$ when $x = x_i, x_{i+1}, \ldots, x_j$. Thus

$$\bar{f}(x) = f_{0n}(x). \qquad (2)$$

2. Half-range series

The simplest form of the algorithm occurs when all the data-points x_i lie in the range $(0, \pi)$, and a fit containing cosine terms only, or sine terms only, is required. We consider the two cases separately.

Case 1. Cosine series

We assume here that all the x_i lie in the range $0 \leqslant x_i \leqslant \pi$. If we stipulate that $m = n$, it can be shown (see, e.g. Berezin and Zhidkov, 1965) that there is a unique polynomial of the form

$$\bar{f}(x) = \sum_{r=0}^{n} a_r \cos rx \qquad (3)$$

which agrees with $f(x)$ at the data-points. This polynomial may be generated as follows.

First, set

$$f_{ii}(x) = y_i, \quad (i = 0, 1, \ldots, n). \qquad (4)$$

Then generate the approximations $f_{ij}(x)$ by the relation

$$f_{ij}(x) = \frac{(\cos x_j - \cos x)f_{i,j-1}(x) - (\cos x_i - \cos x)f_{i+1,j}(x)}{\cos x_j - \cos x_i}. \qquad (5)$$

The calculation may be conveniently set out in tabular form. **Table 1** illustrates the case $n = 4$. After the first column each entry in this table is obtained from a pair of consecutive entries in the preceding column, using (5).

Note that this process is equivalent to applying to the independent variable x the transformation $\xi = \cos x$, and then applying Neville interpolation to the resulting function of ξ.

Case 2. Sine series

Here we suppose that all the x_i lie in the range $0 < x_i < \pi$. If we stipulate that $m = n + 1$, there is a unique interpolation polynomial of the form

$$\bar{f}(x) = \sum_{r=1}^{n+1} b_r \sin rx. \qquad (6)$$

Again, the polynomials $f_{ij}(x)$ are generated by equation (5), but now the initial approximations are given by

$$f_{ii}(x) = y_i \sin x / \sin x_i. \qquad (7)$$

3. Full-range series

If the points x_i are spaced throughout the range $-\pi < x_i \leqslant \pi$, the situation is more complicated, and a variety of formulae exist.

Table 1

Calculation for half-range series

f_{00}	f_{01}	f_{02}	f_{03}	f_{04}
f_{11}	f_{12}	f_{13}	f_{14}	
f_{22}	f_{23}	f_{24}		
f_{33}	f_{34}			
f_{44}				

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Table 2

Calculation for full-range series

f_{00}			
f_{11}	f_{02}		
f_{22}	f_{13}	f_{04}	
f_{33}	f_{24}	f_{15}	f_{06}
f_{44}	f_{35}	f_{26}	
f_{55}	f_{46}		
f_{66}			

Table 3

x	y
1·0	0·946 083
1·1	1·028 685
1·2	1·108 047
1·3	1·183 958
1·4	1·256 227
1·5	1·324 684
1·6	1·389 180
1·7	1·449 592

However, if we stipulate that $n = 2m$ (so that the number of data-points is odd), it can be proved that there is a unique approximation of the form (1). This is provided by Gauss's formula:

$$\bar{f}(x) = \sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{\sin \frac{1}{2}(x - x_j)}{\sin \frac{1}{2}(x_i - x_j)} \tag{8}$$

(see, e.g. Berezin and Zhidkov, 1965, chapter 2, §10). A result equivalent to (8) may be obtained by a method similar to Neville's. For this method we again use (4) to give the initial approximations. Further approximations are then obtained from the relation

$$f_{i-1,j+1}(x) = \frac{\alpha_{j,j+1} f_{i-1,j-1}(x) - \alpha_{i-1,j+1} f_{ij}(x) + \alpha_{i-1,i} f_{i+1,j+1}(x)}{\sin \frac{1}{2}(x_j - x_{i-1}) \sin \frac{1}{2}(x_{j+1} - x_{i-1}) \sin \frac{1}{2}(x_{j+1} - x_i)} \tag{9}$$

where

$$\alpha_{rs} = \sin \tfrac{1}{2}(x_s - x_r + x_j - x_i) \sin \tfrac{1}{2}(x - x_r) \sin \tfrac{1}{2}(x - x_s). \tag{10}$$

After some manipulation it can be shown from (10) that

$$\alpha_{j,j+1} - \alpha_{i-1,j+1} + \alpha_{i-1,i} = \sin \tfrac{1}{2}(x_j - x_{i-1}) \sin \tfrac{1}{2}(x_{j+1} - x_{i-1}) \sin \tfrac{1}{2}(x_{j+1} - x_i). \tag{11}$$

Using this result, it is quite easy to prove (9) inductively.

The calculation may be set out as in **Table 2** (for the case $n = 6$). After the first column, each entry in the table is obtained by combining *three* entries from the preceding column, using equations (9) and (10).

If the restriction $n = 2m$ is relaxed, the uniqueness of the interpolation polynomial (1) is lost. A number of formulae with $m = n$ or $n + 1$ are listed in Whittaker and Robinson (1944), §140. These formulae are all liable to fail under some circumstances. For example, Poisson's formula

$$\bar{f}(x) = \sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{\sin (x - x_j)}{\sin (x_i - x_j)} \tag{12}$$

will fail if two of the abscissae, x_i and x_j, say, differ by π, although this situation is, perhaps, unlikely to arise in practice.

Poisson's polynomial may be generated by formulae like (9) and (10), with the factors $\frac{1}{2}$ omitted. Here if n is even the initial approximations are given by (4), while if n is odd they are given by

$$f_{i-1,i}(x) = \frac{y_i \sin (x - x_{i-1}) - y_{i-1} \sin (x - x_i)}{\sin (x_i - x_{i-1})}, \quad (i = 1, 2, \ldots, n). \tag{13}$$

There appears to be no formula similar to (5) for Poisson's approximation.

4. Alternative method for equally-spaced data

If the points x_i are equally-spaced, a simpler alternative method can be used. For this we first move the origin to the mean of the abscissae; that is, we change the independent variable to

$$\xi = x - \bar{x} \tag{14}$$

where $\bar{x}$ is the mean of the x_i. We then separate $f(\xi)$ into even and odd components:

$$f(\xi) = u(\xi) + v(\xi) \tag{15}$$

Table 4

ξ_i	$\cos \xi_i - \cos \xi$	$f_{ii}(\xi)$			
0·05	−0·0007 9977	1·2417 7898	420 7324	0 7302	01
0·15	−0·0107 7895	1·2381 0739	420 8138	0 7345	
0·25	−0·0306 3761	1·2307 8588	421 1774		
0·35	−0·0601 7732	1·2198 6010			

where

$$u(\xi) = \tfrac{1}{2}[f(\xi) + f(-\xi)],$$
$$v(\xi) = \tfrac{1}{2}[f(\xi) - f(-\xi)]. \qquad (16)$$

Once this has been done, the negative values of ξ can be ignored, and separate interpolations carried out for $u(\xi)$ and $v(\xi)$, using equations (4) and (5) for $u(\xi)$ and (7) and (5) for $v(\xi)$.

If n is odd, so that the number of data-points is even, the two calculations described above may be combined, by setting

$$f_{ii}(\xi) = u(\xi_i) + v(\xi_i) \sin \xi / \sin \xi_i. \qquad (17)$$
$$(i = \tfrac{1}{2}(n+1) \text{ to } n),$$

and then generating the later approximations by (5), with x replaced by ξ. The resulting polynomial contains cosine terms up to degree $\frac{1}{2}(n-1)$ and sine terms up to degree $\frac{1}{2}(n+1)$. If n is even we cannot combine the two calculations in this way, due to the difficulty of assigning a value for the second term on the right in (17) when $i = \frac{1}{2}n$, so that $\xi_i = 0$. However, the final result obtained in this case agrees with that obtained by Gauss's formula (8).

5. Numerical example

To illustrate the convergence of the methods, a numerical example is given below.

Example Estimate the value of y when $x = 1 \cdot 38$ from the values in **Table 3**.

We shall use equations (17) and (5). Here $\bar{x} = 1 \cdot 35$, $\xi = 0 \cdot 03$. The calculation is set out in **Table 4**. To 6 decimals, it gives $y = 1 \cdot 242\ 073$.

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