Note on a general finite-difference formula for the solution of axially symmetric fields

By T. J. Randall*

A finite-difference formula is derived for use near curved boundaries and in regions with composite media. A method of storing the coefficients economically and with easy access is given so that the formula can be written as a procedure.

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1. Derivation

Consider a three dimensional region with axial symmetry and let the plane of symmetry be covered with a net having a square mesh of length a. If $V = V(r, z)$ is, for example, the electric potential function in the region then V satisfies Laplace's equation

$$\frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} = 0 \qquad (1)$$

where r is the distance from the axis of symmetry and z is the distance from a fixed plane perpendicular to the axis.

Denote $V(r_0, z_0)$ by V_0, $V(r_0 - h_1 a, z_0)$ by V_1, $V(r_0, z_0 + h_2 a)$ by V_2, $V(r_0 + h_3 a, z_0)$ by V_3 and $V(r_0, z_0 - h_4 a)$ by V_4 where $0 < h_1, h_2, h_3, h_4 \leqslant 1$.

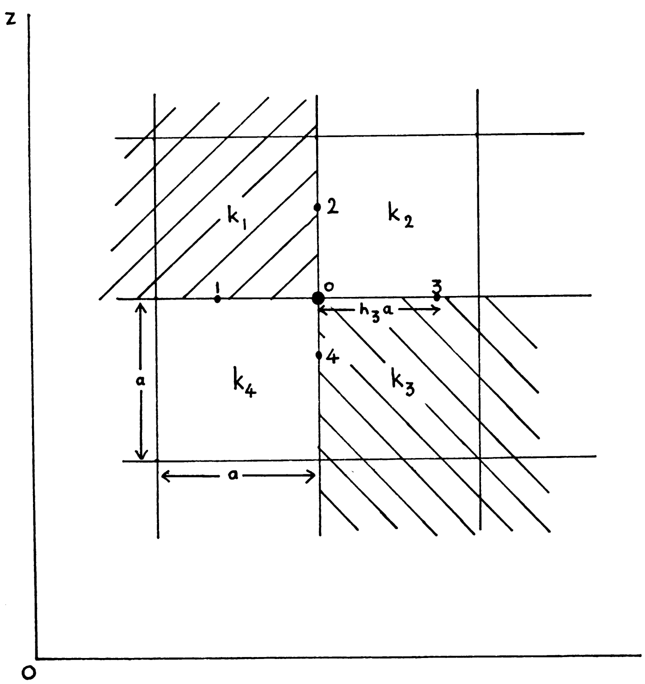

Fig. 1

Divide the plane of symmetry into four regions with dielectric constants K_1, K_2, K_3 and K_4 (see **Fig. 1**).

Using a Taylor's series expansion about the point 0,

$$\left.\begin{aligned} V_1 &= V_0 - h_1 a\frac{\partial V}{\partial r_0} + \frac{h_1^2}{2}a^2\frac{\partial^2 V}{\partial r_0^2} + O(a^3),\\ V_2 &= V_0 + h_2 a\frac{\partial V}{\partial z_0} + \frac{h_2^2}{2}a^2\frac{\partial^2 V}{\partial z_0^2} + O(a^3),\\ V_3 &= V_0 + h_3 a\frac{\partial V}{\partial r_0} + \frac{h_3^2}{2}a^2\frac{\partial^2 V}{\partial r_0^2} + O(a^3),\\ V_4 &= V_0 - h_4 a\frac{\partial V}{\partial z_0} + \frac{h_4^2}{2}a^2\frac{\partial^2 V}{\partial z_0^2} + O(a^3). \end{aligned}\right\} \qquad (2)$$

Applying the generalised form of Gauss's theorem to a small cylindrical surface surrounding the point 0 gives

$$\left.\begin{aligned} &(K_1 + K_2)(E_2)_0 = (K_3 + K_4)(E_4)_0\\ \text{and}\quad &(K_1 + K_4)(E_1)_0 = (K_2 + K_3)(E_3)_0 \end{aligned}\right\} \qquad (3)$$

where $(E_1)_0$, $(E_2)_0$, $(E_3)_0$ and $(E_4)_0$ are the gradients at 0 on the same side of the interfaces as the nodes 1, 2, 3 and 4, respectively. From (1) we obtain the equations

$$\left.\begin{aligned} \frac{\partial E_2}{\partial z_0} + \frac{\partial E_1}{\partial r_0} + \frac{(E_1)_0}{r_0} &= 0,\\ \frac{\partial E_2}{\partial z_0} + \frac{\partial E_3}{\partial r_0} + \frac{(E_3)_0}{r_0} &= 0,\\ \frac{\partial E_4}{\partial z_0} + \frac{\partial E_3}{\partial r_0} + \frac{(E_3)_0}{r_0} &= 0,\\ \frac{\partial E_4}{\partial z_0} + \frac{\partial E_1}{\partial r_0} + \frac{(E_1)_0}{r_0} &= 0. \end{aligned}\right\} \qquad (4)$$

Multiplying each of equations (2) by a factor and adding gives

$$\begin{aligned} \sum_{c=1}^{4} C_i V_i = V_0 \sum_{c=1}^{4} C_i &+ C_2 h_2 a(E_2)_0 - C_4 h_4 a(E_4)_0\\ &+ C_3 h_3 a(E_3)_0 - C_1 h_1 a(E_1)_0\\ &+ \tfrac{1}{2}\left(C_2 h_2^2 a^2 \frac{\partial E_2}{\partial z_0} + C_4 h_4^2 a^2 \frac{\partial E_4}{\partial z_0}\right) \end{aligned}$$

* *Department of Physics and Mathematics, John Dalton College of Technology, Manchester.*

$$+ \tfrac{1}{2}\left(C_3 h_3^2 a^2 \frac{\partial E_3}{\partial r_0} + C_1 h_1^2 a^2 \frac{\partial E_1}{\partial r_0}\right) + O(a^3).$$

Using equations (3) and (4) and making the substitutions

$$C_1 = (K_1 + K_4)d_1/h_1,$$
$$C_2 = (K_1 + K_2)d/h_2,$$
$$C_3 = (K_2 + K_3)d_3/h_3,$$
$$C_4 = (K_3 + K_4)d/h_4,$$

gives

$$\sum_{i=1}^{4} C_i V_i = V_0 \sum_{i=1}^{4} C_i + (E_3)_0\{(K_2 + K_3)d_3 a - (K_2 + K_3)d_1 a + d_1 h_1 a^2[(K_1 + K_4) - (K_2 + K_3)]/2r\} + \tfrac{1}{2}\{(K_2 + K_3)h_3 d_3 + (K_1 + K_4)h_1 d_1\}a^2 \frac{\partial E_3}{\partial r_0} + \tfrac{1}{2}\{(K_3 + K_4)h_4 + (K_1 + K_2)h_2\}a^2 d \frac{\partial E_2}{\partial r_0} + O(a^3)$$

i.e. $$\sum_{i=1}^{4} C_i V_i = V_0 \sum_{i=1}^{4} C_i + A\frac{\partial E_2}{\partial z_0} + B\frac{\partial E_3}{\partial r_0} + C(E_3)_0 + O(a^3).$$

From (4), $A\frac{\partial E_2}{\partial z_0} + B\frac{\partial E_3}{\partial r_0} + C(E_3)_0 = 0$

if $A/1 = B/1 = C/\frac{1}{r} = K$ (say).

Solving for d_1, d_3 and d and hence for C_1, C_2, C_3 and C_4 we obtain the finite-difference formula

$$V_0 \simeq \sum_{i=1}^{4} C_i V_i \Big/ \sum_{i=1}^{4} C_i \qquad (5)$$

where

$$C_1 = \frac{2(K_1 + K_4)(1 - h_3 a/2r)}{h_1\{h_1(K_1 + K_4)(1 - h_3 a/2r) + h_3(K_2 + K_3)(1 + h_1 a/2r)\}},$$

$$C_2 = \frac{2(K_1 + K_2)}{h_2\{h_4(K_3 + K_4) + h_2(K_1 + K_2)\}},$$

$$C_3 = \frac{2(K_2 + K_3)(1 + h_1 a/2r)}{h_3\{h_1(K_1 + K_4)(1 - h_3 a/2r) + h_3(K_2 + K_3)(1 + h_1 a/2r)\}},$$

$$C_4 = \frac{2(K_3 + K_4)}{h_4\{h_4(K_3 + K_4) + h_2(K_1 + K_2)\}}.$$

If $K_1 = K_2 = K_3 = K_4 = h_1 = h_2 = h_3 = h_4 = 1$, equation (5) reduces to the well known formula

$$V_0 \simeq \tfrac{1}{4}(1 - a/2r)V_1 + \tfrac{1}{4}V_2 + \tfrac{1}{4}(1 + a/2r)V_3 + \tfrac{1}{4}V_4.$$

2. Storage of coefficients

Equation (5) is general enough to deal adequately with a large variety of field problems. The only limitation is that boundaries between media must coincide with lines of the net. The author has used it to obtain the potential distributions for a recessed dielectric slab placed between a sphere and a plane (Binns and Randall, 1967) using the method of S.O.R. It has the advantage that all points near curved boundaries or boundaries between media can be relaxed by means of a procedure in a program which has C_1, C_2, C_3 and C_4 as parameters. The main problem involved is that of storing the coefficients for each special node efficiently and with easy access. This can be achieved as follows.

(1) Tag all nodes of the net requiring special treatment, e.g. by making the values negative and working with the modulus (Forsythe and Wasow, 1960).

(2) Count these nodes and declare, dynamically, four arrays $C1$, $C2$, $C3$ and $C4$, say, of length equal to this number of nodes.

(3) Set the count i to zero and traverse the net in the same order as when relaxing the values at the nodes. Each time a negative value is encountered, increase the count by one, calculate C_1, C_2, C_3 and C_4 and store them in $C1$(i), $C2$(i), $C3$(i) and $C4$(i) respectively.

In this way they are stored in arrays with no wasted space and are readily accessible by using the count in each iteration, calling the procedure whenever a negative value is encountered. By setting r to the machine capacity the procedure can be used for two-dimensional fields also.

References

FORSYTHE, G. E., and WASOW, W. R. (1960). *Finite-Difference Methods for Partial Differential Equations*, John Wiley and Sons, p. 359.

BINNS, D. F., and RANDALL, T. J. (1967). Calculation of potential gradients for a dielectric slab placed between a sphere and a plane, *Proc. I.E.E.*, Vol. 114, No. 10, October 1967, p. 1521.