Asymptotic estimates for the error of the Gauss-Legendre quadrature formula

By M. M. Chawla*

Estimates for the error of the Gauss–Legendre quadrature are obtained, for large n, by Davis' method employing the double as well as the line integral norms. It is shown that the error estimate derived through the line integral norm is better than that obtained through the double integral norm; in fact, the latter estimate over-estimates the error nearly by a factor of $(2n + 1)^{1/2}$.

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1. Introduction

Let $A(\epsilon_\rho) = \{f : f$ is analytic on $[-1, 1]$ and continuable analytically so as to be single valued and regular in the closure of an ellipse ϵ_ρ with foci at $z = \pm 1$ and whose sum of semi-axes is $\rho(\rho > 1)\}$.

Let $E_{G_n}(f)$ denote the error of the n-point Gauss–Legendre quadrature over $[-1, 1]$. Since E_{G_n} is a bounded linear functional over $L^2(\epsilon_\rho)$, the L^2 completion of $A(\epsilon_\rho)$, its double integral norm is given (see Davis (1963), p. 346) by

$$||E_{G_n}||_D^2 = \frac{4}{\pi}\sum_{k=2n}^{\infty}(k+1)\frac{|E_{G_n}(U_k)|^2}{\rho^{2k-2} - \rho^{-2k-2}} \tag{1}$$

where $U_n(z) = (1 - z^2)^{-1/2}\sin[(n+1)\arc\cos z]$ = Chebyshev polynomial of the second kind, and the double integral norm is derived from the inner product

$$(f, g) = \int\int_{\epsilon_\rho} f(z)\overline{g(z)}dxdy. \tag{2}$$

The corresponding line integral norm of E_{G_n} has not been given explicitly, nor has it been used to estimate errors.

The line integral norm is derived from the inner product

$$(f, g) = \int_{\epsilon_\rho}\frac{f(z)\overline{g(z)}ds}{|1 - z^2|^{1/2}} \quad (|dz| = ds). \tag{3}$$

Since the Chebyshev polynomials of the first kind

$$\left.\begin{aligned} p_n^*(z) &= \sqrt{\left(\frac{2}{\pi}\right)}(\rho^{2n} + \rho^{-2n})^{-1/2}T_n(z) \\ p_0^*(z) &= (2\pi)^{-1/2} \\ T_n(z) &= \cos(n \arc\cos z) \end{aligned}\right\} \tag{4}$$

form a complete orthonormal sequence for the space $H^2(\epsilon_\rho)$, the completion of $A(\epsilon_\rho)$ under the inner product (3), therefore the line integral norm of the error-functional E_{G_n} over $H^2(\epsilon_\rho)$ can be easily given in terms of this system of orthonormal polynomials (see Davis (1963), pp. 240–41, Ex. 3, and equation (9.3.13)) as

$$||E_{G_n}||_L^2 = \sum_{k=2n}^{\infty}|E_{G_n}(p_k^*)|^2$$

$$= \frac{2}{\pi}\sum_{k=2n}^{\infty}\frac{|E_{G_n}(T_k)|^2}{\rho^{2k} + \rho^{-2k}}. \tag{5}$$

Error estimates are obtained by using Schwarz inequality:

$|E_{G_n}(f)| \leqslant ||E_{G_n}||\ ||f||$, where $||f||$ is a norm of f. Observe that the double integral norm of f may be estimated from

$$||f||_D \leqslant M(\epsilon_\rho)\left(\int\int_{\epsilon_\rho}dxdy\right)^{1/2}$$
$$= \tfrac{1}{2}(\pi(\rho^2 - \rho^{-2}))^{1/2}M(\epsilon_\rho) \tag{6}$$

where $M(\epsilon_\rho) = \max|f(z)|$ on ϵ_ρ; while the line integral norm may be estimated from

$$||f||_L \leqslant M(\epsilon_\rho)\left(\int_{\epsilon_\rho}\frac{ds}{|1 - z^2|^{1/2}}\right)^{1/2} = (2\pi)^{1/2}M(\epsilon_\rho). \tag{7}$$

2. Asymptotic error estimates

In Chawla and Jain (1968), we noted the following asymptotic formula for $E_{G_n}(f)$. For $f \epsilon A(\epsilon_\rho)$, and for large n,

$$E_{G_n}(f) \simeq -i\int_{\epsilon_\rho}\frac{f(z)dz}{[z + \sqrt{(z^2 - 1)}]^{2n+1}} \tag{8}$$

where the sign of the square-root is chosen so that $|z + (z^2 - 1)^{1/2}| > 1$.

Let $z = \frac{1}{2}(\xi + \xi^{-1})$, $\xi = \rho e^{i\theta}$ $(0 \leqslant \theta \leqslant 2\pi)$. Since on ϵ_ρ,

$$T_n(z) = \tfrac{1}{2}(\xi^n + \xi^{-n}) \tag{9}$$

substituting (9) in (8) we obtain, for large n,

$$E_{G_n}(T_k) \simeq -\frac{i}{4}\int_{C_\rho}\frac{(\xi^{2k+2} + \xi^2 - \xi^{2k} - 1)d\xi}{\xi^{2n+k+3}} \quad (C_\rho : |\xi| = \rho) \tag{10}$$

where $k \geqslant 2n$. Applying the residue theorem we obtain

$$E_{G_n}(T_k) \simeq \left.\begin{aligned} &\frac{\pi}{2}\text{ if } k = 2n \\ -&\frac{\pi}{2}\text{ if } k = 2n + 2 \end{aligned}\right\} \tag{11}$$

* *Department of Mathematics, Indian Institute of Technology, Hauz Khas, New Delhi-29, India.*

and zero for other values of $k > 2n$. Substituting (11) in (5) we obtain

$$||E_{G_n}||_L \simeq \sqrt{\left(\frac{\pi}{2}\right)}\,[(\rho^{4n} + \rho^{-4n})^{-1} + (\rho^{4n+4} + \rho^{-4n-4})^{-1}]^{1/2}. \quad (12)$$

In view of the inequality

$$(\rho^k + \rho^{-k})^{-1} < \rho^{-k} \quad (\rho > 1)$$

(12) may be bounded by

$$||E_{G_n}||_L \leqslant \sqrt{\left(\frac{\pi}{2}\right)}\,\rho^{-2n}(1 + \rho^{-4})^{1/2}(1 + o(1)), \quad n \to \infty. \quad (13)$$

Combining (13) and (7), the line integral norm leads to the following estimate of error.

THEOREM 1. Let $f \epsilon A(\epsilon_\rho)$, $\rho > 1$. Then

$$|E_{G_n}(f)| \leqslant \pi \frac{M(\epsilon_\rho)}{\rho^{2n}}(1 + \rho^{-4})^{1/2}(1 + o(1)), \; n \to \infty \quad (14)$$

Again, since on ϵ_ρ,

$$U_n(z) = \frac{\xi^{n+1} - \xi^{-n-1}}{\xi - \xi^{-1}} \quad (15)$$

substituting (15) in (8), we have for large n,

$$E_{G_n}(U_k) \simeq -\frac{i}{2}\int_{C_\rho} \frac{(\xi^{2k+2} - 1)d\xi}{\xi^{2n+k+3}} \quad (16)$$

where $k \geqslant 2n$. Applying the residue theorem we obtain

$$E_{G_n}(U_k) \simeq \begin{matrix} \pi \text{ if } k = 2n \\ 0 \text{ if } k \geqslant 2n + 1 \end{matrix} \quad (17)$$

Substituting (17) in (1) we obtain the double integral error norm, for large n,

$$||E_{G_n}||_D \simeq 2((2n + 1)\pi)^{1/2}(\rho^{4n+2} - \rho^{-4n-2})^{-1/2}. \quad (18)$$

In view of the inequality

$$(\rho^{k+\alpha} - \rho^{-k-\alpha})^{-1} < (\rho^\alpha - \rho^{-\alpha})^{-1}\rho^{-k} \quad (\alpha > 0, \rho > 1)$$

(18) may be bounded by

$$||E_{G_n}||_D \leqslant 2((2n + 1)\pi)^{1/2}(\rho^2 - \rho^{-2})^{-1/2}\rho^{-2n}(1 + o(1)), \quad n \to \infty \quad (19)$$

Combining (19) and (6), the double integral norm leads to the following error estimate.

THEOREM 2. Let $f \epsilon A(\epsilon_\rho)$, $\rho > 1$. Then

$$|E_{G_n}(f)| \leqslant \pi(2n + 1)^{1/2}\frac{M(\epsilon_\rho)}{\rho^{2n}}(1 + o(1)), \quad n \to \infty. \quad (20)$$

Clearly, for large n, the estimate (20) overestimates the error given by the estimate (14) nearly by a factor of $(2n + 1)^{1/2}$.

References

DAVIS, P. J. (1963). *Interpolation and Approximation*, Blaisdell, New York.

CHAWLA, M. M., and JAIN, M. K. (1968). Asymptotic Error Estimates for the Gauss Quadrature Formula, *Math. Comp.*, Vol. 22, pp. 91–97.

Book Review

Graphs, Dynamic Programming and Finite Games, by A. KAUFMANN, 1967; 484 pages. (New York: *Academic Press*, 116s.)

The topics treated in this book have applications in economics, engineering, industry, town and transport planning, psychology and other practical areas. The contents are as follows:

There is also a bibliography and a subject index.

The method of presentation is completely original. In the first part of the book the concepts are introduced and treated by means of simple concrete examples, for which only a very elementary knowledge of mathematics is needed, avoiding abstraction. In the second part the same subject matter is presented in a more theoretical manner starting from first principles, but leaving out some proofs. The text is illustrated throughout by numerous diagrams, and examples play a prominent role in the second part of the book also. The intention of the book clearly is to help readers who have little knowledge of mathematics to learn to apply the techniques which are handled.

In the opinion of the reviewer the purpose of the book might have been better achieved if its approach had been closer to that of D. König's book, 'Theorie der endlichen und unendlichen Graphen', of course with much more emphasis on application and with numerous illustrative examples.

G. A. DIRAC (Swansea)