```
    for j := 1 step 1 until k - 1 do
      B[k, i] := B[k, i] - dot[j] × xi[j, i];
      mod := mod + B[k, i] ↑ 2
    end;
  mod := sqrt (mod);
  for i := 1 step 1 until n do xi[k, i] := B[k, i]/mod
    end;
```

This requires $n(n + \frac{1}{2})(n + 1)$ additions, etc., n^3 multiplications, n^2 divisions and n square root determinations, while the working stage requirement is $(n^2 + n + 1)$ real variables. These figures neglect the preliminary reordering process which is necessary, since this is approximately counterbalanced by the reduction in the number of ξ_k^1 which then have to be calculated.

The new procedure is thus seen to have considerable advantages over Swann's (which is itself an improvement on Rosenbrock's in respect of the univariate search) in terms of speed, economy of storage, and ability to deal with the case $d_k = 0$. Specifically, the number of additions, etc., and the working storage requirement are reduced by a factor of the order of n, and the number of multiplications by a factor of the order of $n/2$.

Appendix: Inductive proof of equation (9)

If equations (9) and (10) for B_k and $|B_k|$ respectively, and equation (11) for ξ_k^1 are assumed to be valid for a particular value of $k(>1)$, then using the basic equations (2) and (3) and the explicitly derived equation (5) we have

$$\begin{aligned} B_{k+1} &= A_{k+1} - \sum_{j=1}^{k} (A_{k+1} \cdot \xi_j^1)\xi_j^1 \\ &= A_{k+1} - (A_{k+1} \cdot \xi_1^1)\xi_1^1 - \sum_{j=2}^{k} (A_{k+} \xi_{kj}^{1})\xi_j^1 \\ &= A_{k+1} - \left(A_{k+1} \cdot \frac{A_1}{|A_1|}\right) \frac{A_1}{|A_1|} \\ &\quad - \sum_{2=j}^{k} \left(A_{k+1} \cdot \frac{A_j|A_{j-1}|^2 - A_{j-1}|A_j|^2}{|A_{j-1}||A_j|\sqrt{(|A_{j-1}|^2 - |A_j|^2)}}\right) \\ &\quad \left(\frac{(A_j|A_{j-1}|^2 - A_{j-1}|A_j|^2}{|A_{j-1}|\,|A_j|\sqrt{(|A_{j-1}|^2 - |A_j|^2)}}\right) \end{aligned}$$

Now
$$\begin{aligned} A_{k+1} \cdot A_1 &= \sum_{k+1} d_i \xi_i^0 \cdot \sum_1 d_i \xi_i^0 \\ &= \sum_{k+1} d_i^2 = |A_{k+1}|^2 \text{ (since } k > 1) \end{aligned}$$

and similarly
$$\begin{aligned} A_{k+1} \cdot A_j &= A_{k+1} \cdot A_{j-1} \\ &= |A_{k+1}|^2 \text{ (since } k + 1 > j). \end{aligned}$$

Hence
$$\begin{aligned} B_{k+1} &= A_{k+1} - \frac{A_1|A_{k+1}|^2}{|A_1|^2} \\ &\quad - \sum_{j=2}^{k} \left\{\frac{|A_{k+1}|^2(|A_{j-1}|^2 - |A_j|^2)}{|A_{j-1}|^2|A_j|^2}\right\} \\ &\qquad \times \left\{\frac{A_j|A_{j-1}|^2 - A_{j-1}|A_j|^2}{|A_{j-1}|^2 - |A_j|^2}\right\} \\ &= A_{k+1} - \frac{A_1|A_{k+1}|^2}{|A_1|^2} \\ &\quad - |A_{k+1}|^2 \sum_{j=2}^{k} \frac{A_j|A_{j-1}|^2 - A_{j-1}|A_j|^2}{|A_{j-1}|^2|A_j|^2} \\ &= A_{k+1} - |A_{k+1}|^2 \left\{\sum_{j=2}^{k} \left(\frac{A_j}{|A_j|^2} - \frac{|A_{j-1}|}{|A_{j-1}|^2}\right) + \frac{A_1}{|A_1|^2}\right\}. \end{aligned}$$

But
$$\begin{aligned} \sum_{j=2}^{k} \left(\frac{A_j}{|A_j|^2} - \frac{A_{j-1}}{|A_{j-1}|^2}\right) &= \sum_{j=2}^{k} \frac{A_j}{|A_j|^2} \\ &\quad - \sum_{j=1}^{k-1} \frac{A_j}{|A_j|^2} = \frac{A_k}{|A_k|^2} - \frac{A_1}{|A_1|^2}. \end{aligned}$$

Thus
$$B_{k+1} = A_{k+1} - \frac{A_k|A_{k+1}|^2}{|A_k|^2} = \frac{A_{k+1}|A_k|^2 - A_k|A_{k+1}|^2.}{|A_k|^2} \tag{12}$$

Now (12) is formally the same as (9), with k replaced by $(k + 1)$, so that if (9) is valid for a given value of k, it is also valid for the next higher value of k. But we have already shown in equation (6) that (9) is valid for the case $k = 2$, hence (9) is valid for all k such that $2 \leqslant k \leqslant n$, and consequently (10) and (11) are also valid in this range.

References

ROSENBROCK, H. H. (1960). An Automatic Method for finding the Greatest or Least Value of a Function, *Computer Journal* Vol. 4, pp. 175–184.

SWANN, W. H. (1964). *Report on the Development of a new Direct Search Method of Optimisation*, Imperial Chemical Industries Ltd., Central Instrument Laboratory Research Note 64/3.

Book Review

Semi-Groups of Operators and Approximation, by Paul L. Butzer and Hubert Berens, 1967; 318 pages. (Springer-Verlag, $14.)

This book is concerned with the mathematical aspects of semi-group theory and in particular those aspects which are connected in some way or other with approximation. This theory is of significance in our understanding of the underlying theory of such topics as classical approximation theory, the solutions of partial differential equations and the theory of singular integrals, but is somewhat far removed from the everyday needs of the computing fraternity.

Chapter 1 gives a straightforward presentation of the standard theory of semi-groups of operators. Chapter 2 presents basic approximation theorems for semi-group operators with a study in particular of Dirichlet's problem for the unit disc and Fourier's problem of the ring. Chapter 3 is devoted to the incorporation of approximation theorems for semi-group operators into the theory of intermediate spaces (intermediate between the initial Banach space and the domain of definition of the powers of the infinitesimal generator of the semi-group) and to deep generalisations in the new setting. The last chapter outlines and discusses applications of the previous general theory, including the semi-group of left translations, the singular integrals of Abel-Poisson for periodic functions and of Cauchy-Poisson for functions on the real line, and the singular integral of Gauss-Weierstrass on Euclidean n-space in connection with Sobolev and Besov spaces. There is also a helpful appendix summarising the material in functional analysis that is assumed.

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