Numerical studies of prototype cavity flow problems*

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A new digital computer method is developed for the Navier-Stokes equations. Finite differences, smoothing and a special boundary technique are fundamental. The method converges in practice for all Reynolds numbers. Examples illustrate both primary and secondary vortices and show the development of selected double-spiral equivorticity curves as the Reynolds number becomes infinite. As a special case, the method applies easily to biharmonic problems.

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1. Introduction

The flow of a gas or of a liquid in a closed cavity has long been of interest in applied science (see, e.g., references [1, 2, 4, 7–12, 14] and the additional references contained therein). In this paper we will apply the power of the high speed digital computer to study prototype, steady state, two dimensional problems for such flows. The numerical methods to be developed will be finite difference methods and will be described in sufficient generality so as to be applicable to nonlinear coupled systems similar in structure to the Navier–Stokes equations.

2. The eddy problem in a rectangle

The class of problems to be studied, called eddy problems in a rectangle, can be formulated as follows. For $d > 0$, let the points (0, 0), (1, 0), (1, d) and (0, d) be denoted by A, B, C and D, respectively (see **Fig. 1**). Let S be the rectangle whose vertices are A, B, C, D and denote its interior by R. On R the equations of motion to be satisfied are the Navier–Stokes equations, that is

$$\Delta\psi = -\omega \tag{2.1}$$

$$\Delta\omega + \mathscr{R}\left(\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right) = 0, \tag{2.2}$$

where ψ is the stream function, ω is the vorticity and $\mathscr{R}$ is the Reynolds number. On S the boundary conditions to be satisfied are

$$\psi = 0, \frac{\partial\psi}{\partial x} = 0, \text{ on } AD \tag{2.3}$$

$$\psi = 0, \frac{\partial\psi}{\partial y} = 0, \text{ on } AB \tag{2.4}$$

$$\psi = 0, \frac{\partial\psi}{\partial x} = 0, \text{ on } BC \tag{2.5}$$

$$\psi = 0, \frac{\partial\psi}{\partial y} = -1, \text{ on } CD \tag{2.6}$$

The analytical problem is defined on $R + S$ by (2.1)–(2.6) and is shown diagrammatically in Fig. 1.

In general, boundary value problem (2.1)–(2.6) cannot be solved by means of existing analytical techniques. Physical solutions have been produced in the laboratory by Pan and Acrivos (1967), while numerical methods which 'converge', but only for small $\mathscr{R}$, have been developed by Burggraf (1966) and Runchal, Spalding and Wolfshtein (1968). A numerical method which converges for all $\mathscr{R}$, but which has been run only for relatively large values of the grid size, has been developed by the writer (see Greenspan, 1968).

We shall describe next a modified, somewhat faster form of the latter method and apply it to a selection of difficult problems which are of wide interest. Among our major objectives will be the construction of secondary vortices and the study of vorticity for large Reynolds number.

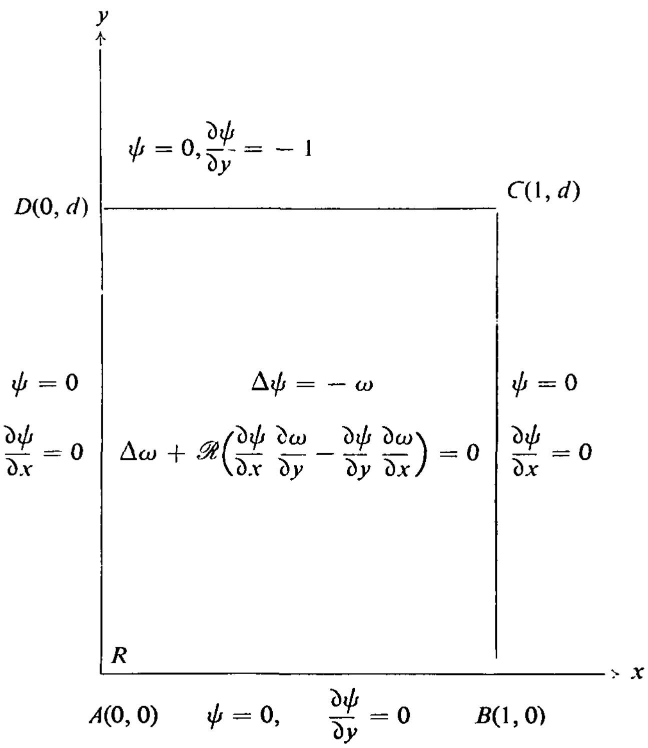

Fig. 1

3. The general numerical method

For a fixed positive integer n, set $h = \frac{1}{n}$. Assume, for simplicity, that d is an integral multiple of h. (If d is

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not an integral multiple of h, the method is easily modified as shown in Greenspan, 1968.) Starting at $(0, 0)$ with grid size h, construct and number in the usual way the set of interior grid points R_h and the set of boundary grid points S_h.

For given tolerances ϵ_1 and ϵ_2, we will show first how to construct on R_h a sequence of discrete stream functions

$$\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots \tag{3.1}$$

and on $R_h + S_h$ a sequence of discrete vorticity functions

$$\omega^{(0)}, \omega^{(1)}, \omega^{(2)}, \ldots \tag{3.2}$$

such that for some integer k both the following are valid:

$$|\psi^{(k)} - \psi^{(k+1)}| < \epsilon_1, \text{ on } R_h \tag{3.3}$$

$$|\omega^{(k)} - \omega^{(k+1)}| < \epsilon_2, \text{ on } R_h + S_h. \tag{3.4}$$

Initially, set

$$\psi^{(0)} = C_1, \text{ on } R_h \tag{3.5}$$

$$\omega^{(0)} = C_2, \text{ on } R_h + S_h, \tag{3.6}$$

where C_1 and C_2 are constants.

To produce the second iterate $\psi^{(1)}$ of sequence (3.1) proceed as follows. At each point of R_h of the form (h, ih), $i = 2, \ldots, n-2$, approximate (2.3) by

$$\psi(h, ih) = \frac{\psi(2h, ih)}{4}. \tag{3.7}$$

At each point of R_h of the form (ih, h), $i = 1, 2, \ldots, n-1$, approximate (2.4) by

$$\psi(ih, h) = \frac{\psi(ih, 2h)}{4}. \tag{3.8}$$

At each point of R_h of the form $(1-h, ih)$, $i = 2, 3, \ldots, n-2$, approximate (2.5) by

$$\psi(1-h, ih) = \frac{\psi(1-2h, ih)}{4}. \tag{3.9}$$

At each point of R_h of the form $(ih, 1-h)$, i $= 1, 2, \ldots, n-1$, approximate (2.6) by

$$\psi(ih, 1-h) = \frac{h}{2} + \frac{\psi(ih, 1-2h)}{4}. \tag{3.10}$$

And at each remaining point of R_h write down the difference analogue

$$-4\psi(x, y) + \psi(x+h, y) + \psi(x, y+h) + \psi(x-h, y) + \psi(x, y-h) = -h^2\omega^{(0)}(x, y) \tag{3.11}$$

of (2.1). Solve the linear algebraic system generated by (3.7)–(3.11) by the generalised Newton's method (Greenspan, 1968) with over-relaxation factor r_ψ and denote this solution by $\bar{\psi}^{(1)}$. Note that for linear systems the generalised Newton's method reduces to S.O.R. Then, on R_h, $\psi^{(1)}$ is defined by the smoothing formula

$$\psi^{(1)} = \rho\psi^{(0)} + (1-\rho)\bar{\psi}^{(1)}, 0 \leqslant \rho \leqslant 1. \tag{3.12}$$

To produce the second iterate $\omega^{(1)}$ of sequence (3.2) proceed as follows. At each point of S_h of the form $(ih, 0)$, $i = 0, 1, 2, \ldots, n$, set

$$\bar{\omega}^{(1)}(ih, 0) = -\frac{2\psi^{(1)}(ih, h)}{h^2}; \tag{3.13}$$

at each point of S_h of the form $(0, ih)$, $i = 1, 2, \ldots, n-1$, set

$$\bar{\omega}^{(1)}(0, ih) = -\frac{2\psi^{(1)}(h, ih)}{h^2}; \tag{3.14}$$

at each point of S_h of the form $(1, ih)$, $i = 1, 2, \ldots, n-1$, set

$$\bar{\omega}^{(1)}(1, ih) = -\frac{2\psi^{(1)}(1-h, ih)}{h^2}; \tag{3.15}$$

and at each point of S_h of the form $(ih, 1)$, $i = 0, 1, 2, \ldots, n$ set

$$\bar{\omega}^{(1)}(ih, 1) = \frac{2}{h} - \frac{2\psi^{(1)}(ih, 1-h)}{h^2}. \tag{3.16}$$

Note that (3.13)–(3.16) are derived from (2.1).

Next, at each point (x, y) in R_h set

$$\alpha = \psi^{(1)}(x+h, y) - \psi^{(1)}(x-h, y)$$
$$\beta = \psi^{(1)}(x, y+h) - \psi^{(1)}(x, y-h)$$

and approximate (2.2), appropriately, by

$$\left(-4 - \frac{\alpha\mathscr{R}}{2} - \frac{\beta\mathscr{R}}{2}\right)\omega(x, y) + \omega(x+h, y) + \left(1 + \frac{\alpha\mathscr{R}}{2}\right)\omega(x, y+h) + \left(1 + \frac{\beta\mathscr{R}}{2}\right)\omega(x-h, y) + \omega(x, y-h) = 0; \quad \text{if} \quad \alpha \geqslant 0, \quad \beta \geqslant 0, \tag{3.17}$$

$$\left(-4 - \frac{\alpha\mathscr{R}}{2} + \frac{\beta\mathscr{R}}{2}\right)\omega(x, y) + \left(1 - \frac{\beta\mathscr{R}}{2}\right)\omega(x+h, y) + \left(1 + \frac{\alpha\mathscr{R}}{2}\right)\omega(x, y+h) + \omega(x-h, y) + \omega(x, y-h) = 0; \quad \text{if} \quad \alpha \geqslant 0, \quad \beta < 0, \tag{3.18}$$

$$\left(-4 + \frac{\alpha\mathscr{R}}{2} - \frac{\beta\mathscr{R}}{2}\right)\omega(x, y) + \omega(x+h, y) + \omega(x, y+h) + \left(1 + \frac{\beta\mathscr{R}}{2}\right)\omega(x-h, y) + \left(1 - \frac{\alpha\mathscr{R}}{2}\right)\omega(x, y-h) = 0; \quad \text{if} \quad \alpha < 0, \ \beta \geqslant 0, \tag{3.19}$$

$$\left(-4 + \frac{\alpha\mathscr{R}}{2} + \frac{\beta\mathscr{R}}{2}\right)\omega(x, y) + \left(1 - \frac{\beta\mathscr{R}}{2}\right)\omega(x+h, y) + \omega(x, y+h) + \omega(x-h, y) + \left(1 - \frac{\alpha\mathscr{R}}{2}\right)\omega(x, y-h) = 0; \quad \text{if} \quad \alpha < 0, \ \beta < 0. \tag{3.20}$$

Solve the linear algebraic system generated by (3.17)–(3.20) by the generalised Newton's method with over-relaxation factor r_ω and denote the solution by $\bar{\omega}^{(1)}$. Finally, on all of $R_h + S_h$ define $\omega^{(1)}$ by the smoothing formula

$$\omega^{(1)} = \mu\omega^{(0)} + (1-\mu)\bar{\omega}^{(1)}, \quad 0 \leqslant \mu \leqslant 1.$$

Proceed next to determine $\psi^{(2)}$ on R_h from $\omega^{(1)}$ and $\psi^{(1)}$ in the same fashion as $\psi^{(1)}$ was determined from $\omega^{(0)}$ and $\psi^{(0)}$. Then construct $\omega^{(2)}$ on $R_h + S_h$ from $\omega^{(1)}$ and $\psi^{(2)}$ in the same fashion as $\omega^{(1)}$ was determined from $\omega^{(0)}$ and $\psi^{(1)}$. In the indicated fashion, construct the sequences (3.1) and (3.2). Terminate the computation when (3.3) and (3.4) are valid.

Finally, when $\psi^{(k)}$ and $\omega^{(k)}$ are verified to be solutions of the difference analogues of (2.1) and (2.2), they are taken to be the numerical approximations of $\psi(x, y)$ and $\omega(x, y)$, respectively.

4. Examples

Consider first the boundary value problem defined by (2.1)–(2.6) with $d = 1$. This problem was solved on the CDC 3600 for $\mathscr{R} = 200$ with $h = \frac{1}{20}$, $\epsilon_1 = 1$, $\epsilon_2 = 10^{-4}$, $\rho = 0{\cdot}1$, $\mu = 0{\cdot}7$, $r_\psi = 1{\cdot}8$, $r_\omega = 1{\cdot}0$, $C_1 = C_2 = 0$, and also for $\mathscr{R} = 500$, 2,000 and 15,000 with the same parameter values except for $\epsilon_2 = 10^{-3}$. Convergence was achieved for $\mathscr{R} = 200$ in 14 minutes with 341 outer iterations, for $\mathscr{R} = 500$ in 11 minutes with 96 outer iterations, for $\mathscr{R} = 2{,}000$ in 4 minutes with 80 outer iterations, and for $\mathscr{R} = 15{,}000$ in $3\frac{1}{2}$ minutes with 40 outer iterations. The resulting stream curves exhibited only primary vortices and are shown in **Fig. 2.** The resulting equivorticity curves exhibited the double spiral development shown in Greenspan (1968) and are given in **Fig. 3.**

With an aim toward producing secondary vortices and toward studying vorticity for large Reynolds numbers, boundary value problems (2.1)–(2.6) were considered again with $d = 1$. The problem was solved for $\mathscr{R} = 50$, 10,000 and 100,000 with $h = \frac{1}{40}$. For $\mathscr{R} = 50$ the remaining input parameters were chosen to be $\epsilon_1 = 10^{-4}$,

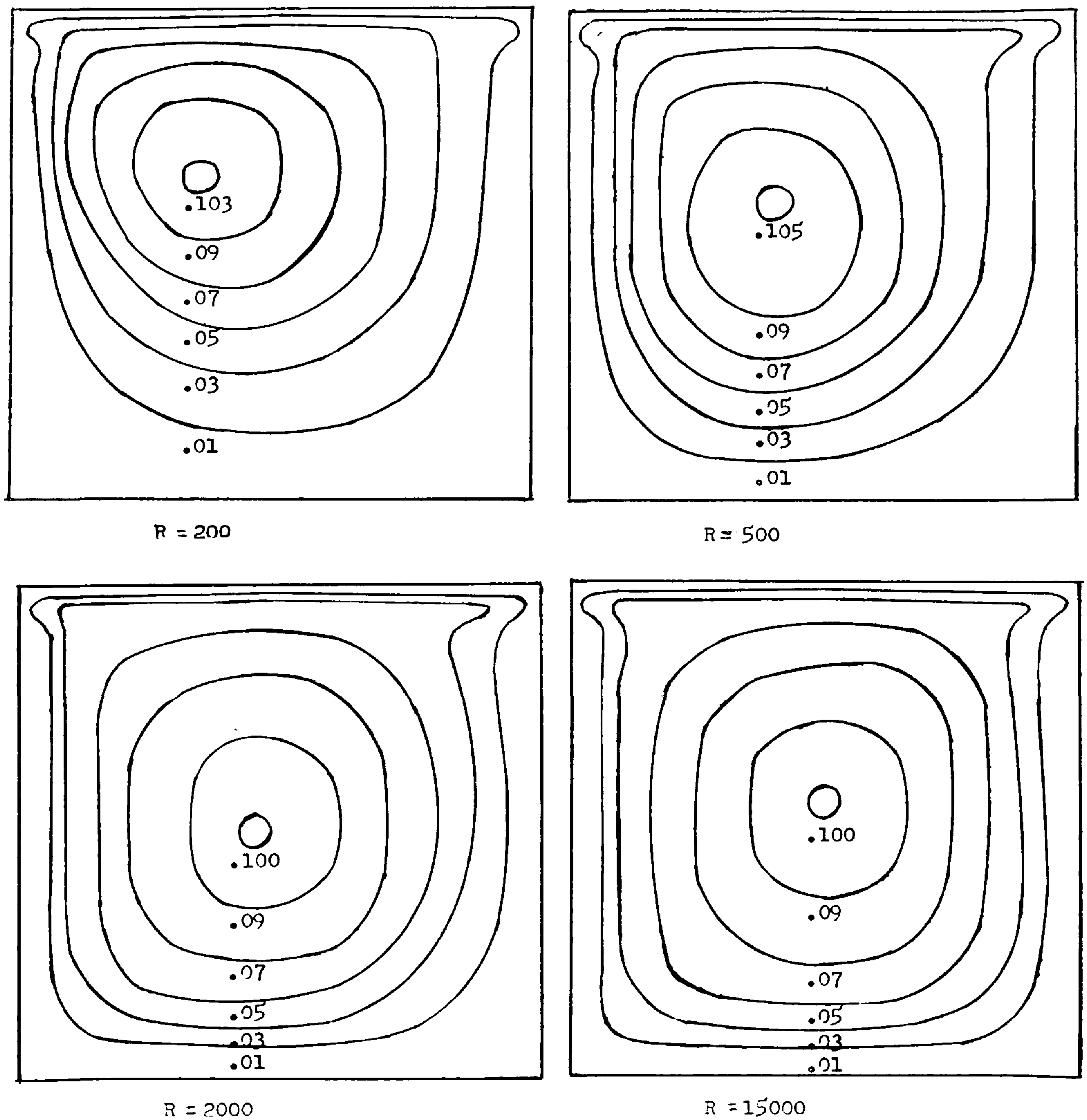

Fig. 2. Typical streamlines for $h = 1/20$

$\epsilon_2 = 10^{-3}$, $\rho = 0{\cdot}03$, $\mu = 0{\cdot}90$, $r_\psi = 1{\cdot}8$, $r_\omega = 1{\cdot}8$, $C_1 = C_2 = 0$. Convergence was achieved in 60 minutes with 100 outer iterations. The resulting flow with the secondary vortices is shown in **Fig. 4.** For $\mathscr{R} = 10{,}000$ the remaining input parameters were chosen to be $\epsilon_1 = 0{\cdot}004$, $\epsilon_2 = 0{\cdot}03$, $\rho = 0{\cdot}03$, $\mu = 0{\cdot}95$, $r_\psi = 1{\cdot}8$, $r_\omega = 1$, $C_1 = C_2 = 0$. After 183 outer iterations, μ was changed to $0{\cdot}85$. Convergence was achieved in 260 minutes with a total of 226 outer iterations. The resulting flow with a single secondary vortex is shown in **Fig. 5.** For $\mathscr{R} = 100{,}000$, the remaining input parameters were chosen to be $\epsilon_1 = 10^{-4}$, $\epsilon_2 = 0{\cdot}005$, $\rho = 0{\cdot}03$, $\mu = 0{\cdot}95$, $r_\psi = 1{\cdot}8$, $r_\omega = 1$, but $\psi^{(0)}$ and $\omega^{(0)}$ were taken to be the 57th outer iterates of the run for $\mathscr{R} = 10{,}000$. Convergence was achieved in 135 minutes with 386 outer iterations. The flow is shown in **Fig. 6** and contains no secondary vortices. The equivorticity curve $\omega = 1{\cdot}630$, with its double-spiral, space filling characteristics is shown in **Fig. 7.** Numerical evidence of Batchelor's result that the vorticity in a large subregion of R converges to a constant as $R \to \infty$ is exhibited in Fig. 7 by setting crosses on those points at which the vorticity is between $1{\cdot}6$ and $1{\cdot}7$.

Finally, consider boundary value problems (2.1)–(2.6) with $d = 2$ and $R = 10$. This problem was solved with $h = \frac{1}{40}$, $\epsilon_1 = 10^{-4}$, $\epsilon_2 = 10^{-3}$, $\rho = 0{\cdot}05$, $\mu = 0{\cdot}90$, $r_\psi = 1{\cdot}8$, $r_\omega = 1{\cdot}80$, $C_1 = C_2 = 0$. Convergence was achieved in 32 minutes with 102 outer iterations. The

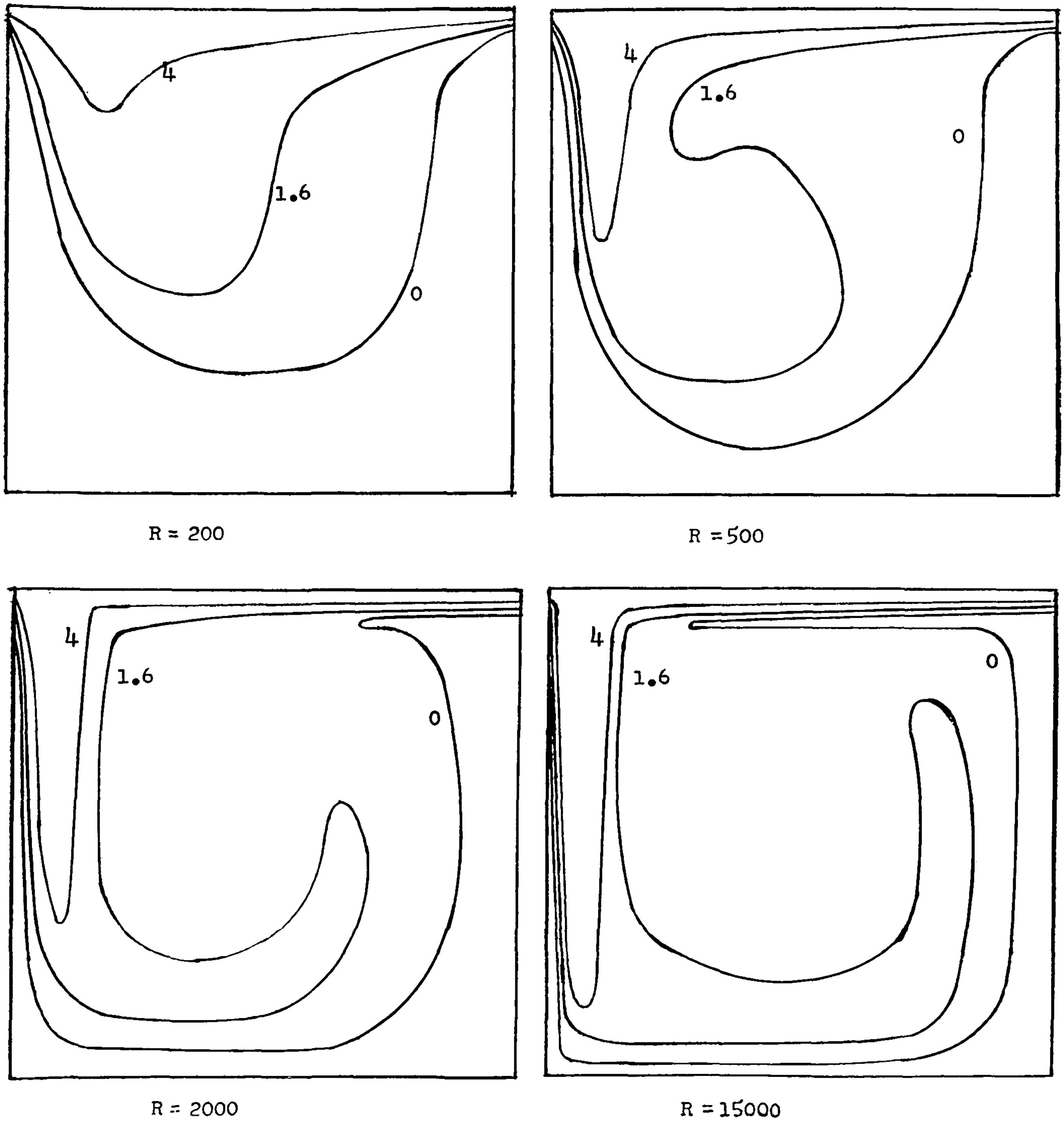

Fig. 3. Selected equivorticity curves for $h = 1/20$

resulting flow, with its two primary and two secondary vortices, is shown in **Fig. 8.**

5. Remarks

From the many examples run in addition to those described in Section 4, the following observations and heuristic conclusions resulted. Divergence or exceptionally slow convergence usually followed if any one of the following choices were made: $0{\cdot}4 \leqslant \rho \leqslant 1$, $0 \leqslant \mu \leqslant 0{\cdot}6$, $r_\psi < 1$, $r_\omega \nless 1$. The choice $\rho = \mu = 0$ yields convergence only for large grid sizes and small Reynolds numbers. The choice $r_\psi = 1{\cdot}8$ was consistently good. For grid sizes larger than or equal to $\frac{1}{20}$, sequence (3.1) converged so much faster than (3.2) that very little attention had to be directed toward the choice of ϵ_1, but for grids smaller than $\frac{1}{20}$ this was not the case and attention had to be directed to the choices of both ϵ_1 and ϵ_2. Deletion of all or even of some of the special formulas (3.7)–(3.10) and substitution with (3.11) always led to divergence for large Reynolds numbers ($\mathscr{R} \sim 10{,}000$), but often did yield secondary vortices for $h = \frac{1}{20}$ for small Reynolds numbers ($\mathscr{R} \sim 50$). The difference equations for $\psi^{(k)}$ and $\omega^{(k)}$ were always satisfied to much smaller tolerances than those imposed in (3.3) and (3.4), respectively.

Several possible modifications of the method of this paper which should be explored if one wishes to speed up the convergence include allowing some or all of

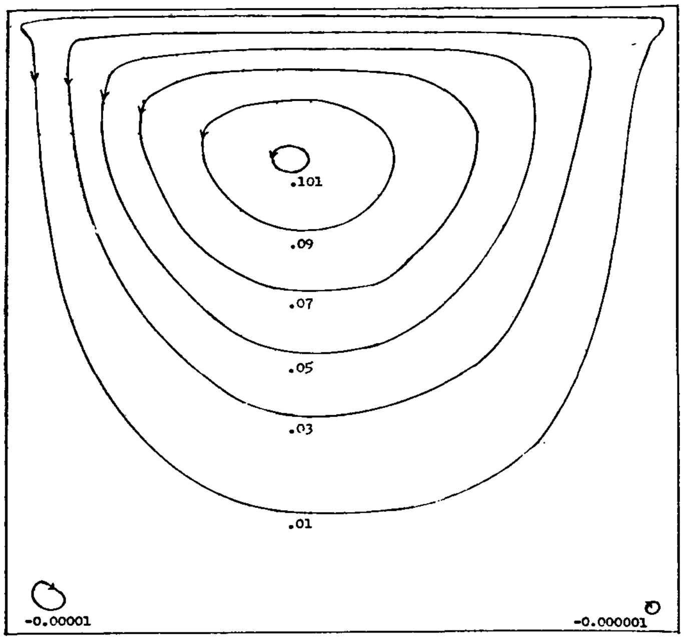

Fig. 4. **Streamlines for Reynolds number 50 with $h = 1/40$**

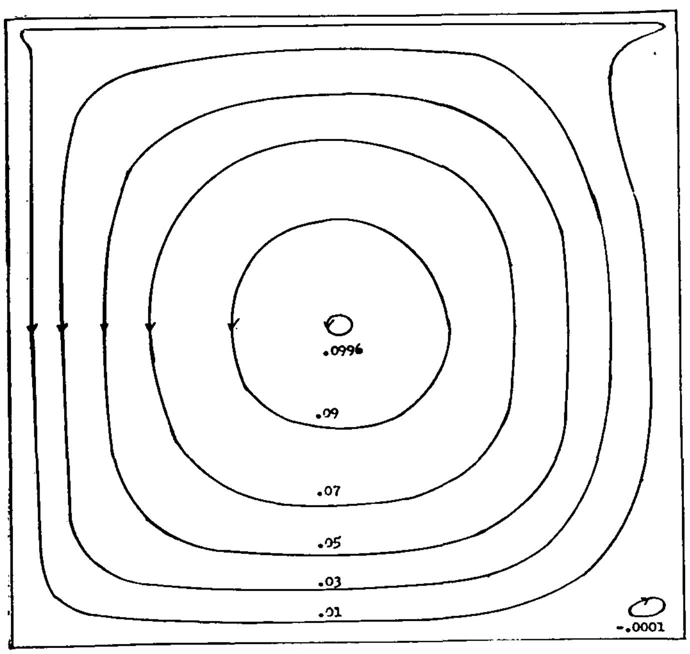

Fig. 5. **Streamlines for Reynolds number 10,000 with $h = 1/40$**

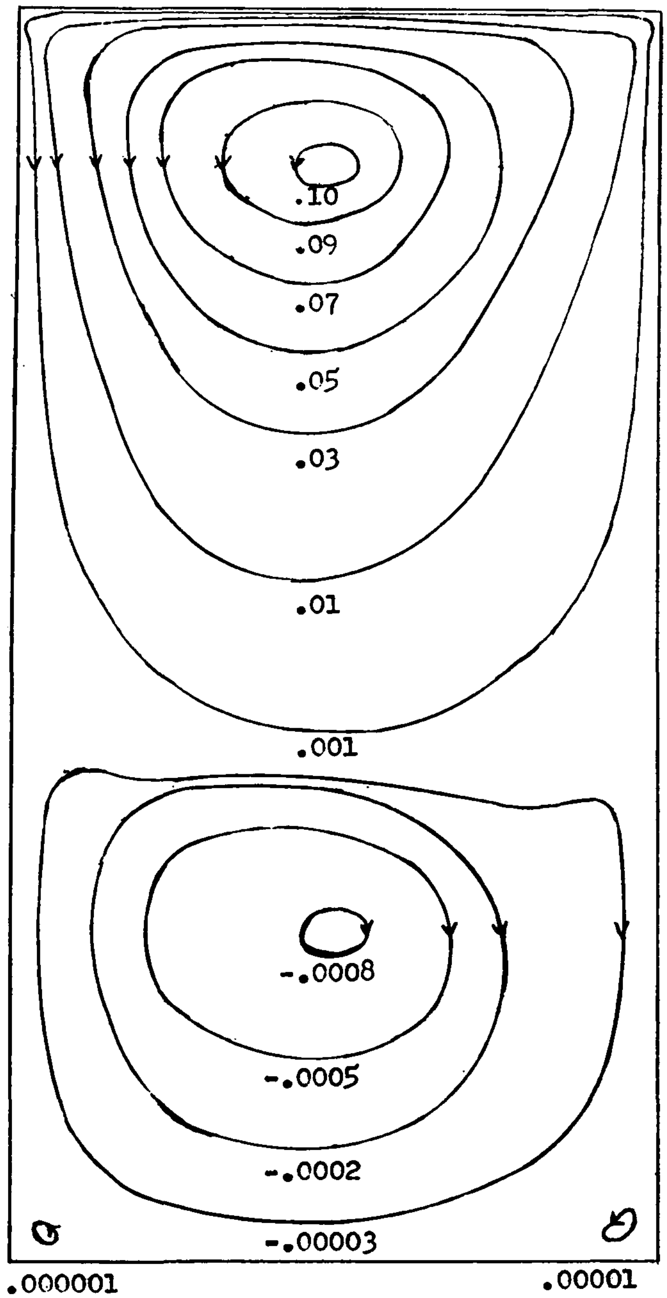

Fig. 8. **Streamlines for Reynolds number 10 with $h = 1/40$ for a 2 by 1 rectangular cavity**

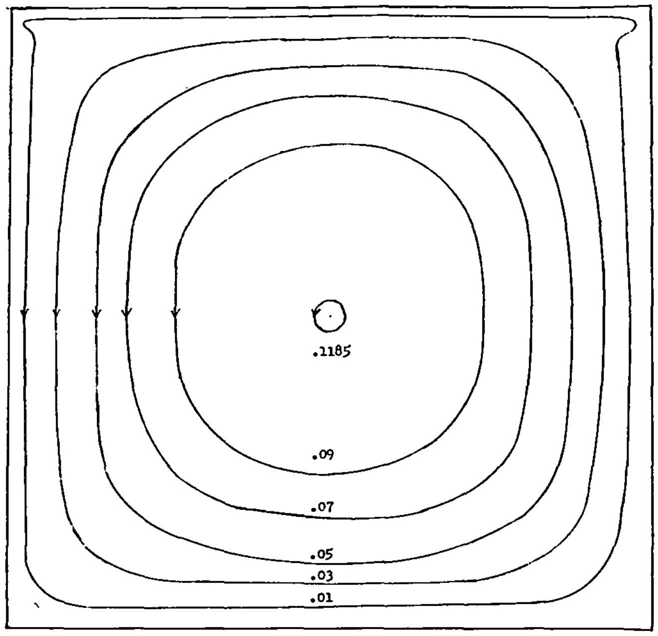

Fig. 6. Streamlines for Reynolds number 100,000 with $h = 1/40$

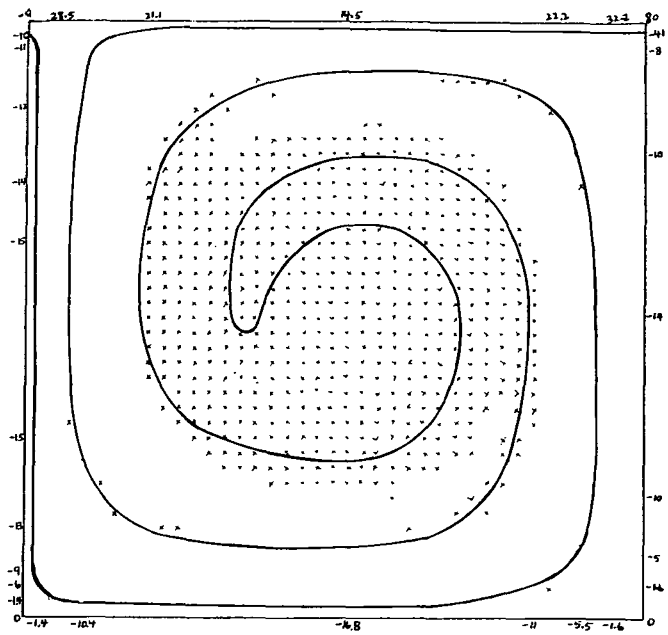

Fig. 7. Equivorticity curves $\omega = 1 \cdot 630$ for Reynolds number 100,000 and $h = 1/40$. At crossed points vorticity is between $1 \cdot 6$ and $1 \cdot 7$

ρ, μ, r_ψ and r_ω to be variable (Carré, 1961), using line over relaxation (Varga, 1962), and choosing $\psi^{(0)}$ and $\omega^{(0)}$ in a more judicious manner than that prescribed in (3.5)–(3.6).

Observe also that the method of Section 3 applies directly to biharmonic problems (i.e. to the case $\mathcal{R} = 0$) and initial computations verify that it extends in a natural way to free convection problems (Batchelor, 1954).

Finally, note that theoretical support for the method of this paper is now beginning to appear for very special cases—see references 3, 5, 13.

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