

Methods of profile optimisation by iterative analogue computation

By N. W. Bellamy and M. J. West*

This paper describes two distinct methods of optimising the profile of a cantilever beam to achieve minimum deflection at its free end. Both methods use iterative analogue computation techniques and are suitable for use on present day semi-hybrid computers. Several arguments are given to show the advantage of applying Pontryagin's maximum principle over the usual hill-climbing methods.

(Received March 1968, revised October 1968)

For some years, analogue iterative techniques have been extensively used in the design of homogeneous structural shapes. The majority of these design problems involve the solution of high order differential equations with defined boundary values (Wilby and Bellamy, 1962; Paul, 1965). Recently, however, the availability of hybrid computers has led to the development of more advanced iterative techniques opening the way to the relatively difficult problems of optimisation.

Attention has been focused for a long time on methods of optimisation for use on digital computers. Axial iteration techniques, the method of steepest descent, and the methods given by Rosenbrock (1960), Fletcher and Powell (1963), and Powell (1964) have been used for hill climbing, according to their merit for a specific problem solution. Of these only the axial iteration technique is really suitable to iterative analogue computation although all the other methods are feasible with full hybrid machines.

A radically different method for optimisation introduced by Pontryagin, Boltyanskii, Gamkrelidze and Mischenko (1962) shows distinct advantages over standard hill-climbing techniques, but its mathematical complexity appears to be severely limiting its use. At first sight, the simple-boundary value equations given by Pontryagin's method ideally match the iterative techniques available on analogue machines. Unfortunately the equations tend to have a number of non-linear terms which emphasise the inaccuracies of analogue computing elements.

One class of problems of general interest is the optimisation of a structural shape or profile to a given performance index. This paper describes the solution of a problem of this nature using iterative analogue techniques. Both hill-climbing and Pontryagin methods are used to determine the optimum profile of a cantilever beam for that beam to undergo minimum deflection at its free end.

Statement of the problem

Consider the cantilever beam shown in Fig. 1 and Fig. 2. The beam has constant width a and density ρ and is symmetrical about its central axis. It is required to determine the profile of the beam for minimum end deflection under its own weight with the maximum and minimum height restrictions, h_1 and h_2 respectively.

* Lanchester College of Technology, Coventry

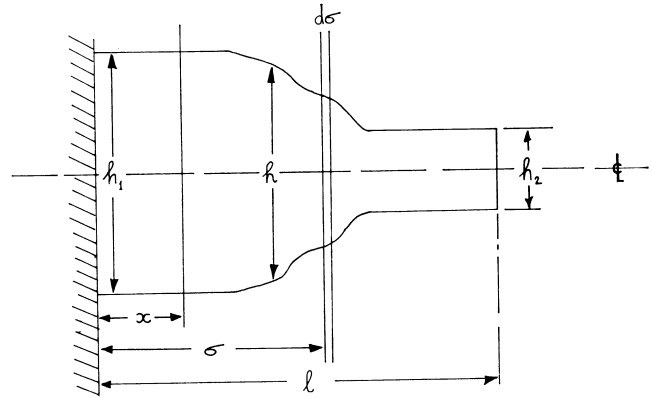


Fig. 1. Beam parameters for Pontryagin's method

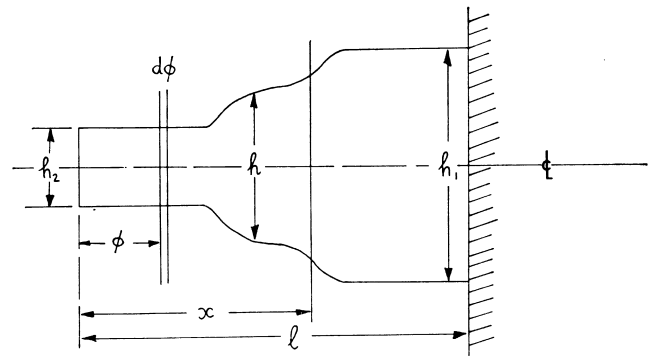


Fig. 2. Beam parameters for hill-climbing method

The beam deflects according to the equation

$$M(x) = -EI \frac{d^2y}{dx^2} \quad (1)$$

where
$$I = \frac{ah^3}{12} \quad (2)$$

$M(x)$ is the bending moment at a point x along the beam, E is Young's Modulus and I is the second moment of area.

If x is taken to increase from the wall to the free end of the beam as in Fig. 1, then

$$M(x) = \int_x^l \rho a [h(\sigma)] (\sigma - x) d\sigma \quad (3)$$

or, if x is taken to increase from the free end to the wall as in Fig. 2, then

$$M(x) = \int_0^x \rho a[h(\phi)](x - \phi) d\phi \quad (4)$$

where h in each case indicates height of the beam as a function of distance along the beam. Note that in either case

$$\frac{d^2 M(x)}{dx^2} = \rho a h(x). \quad (5)$$

Hill climbing method

The deflection $y(x)$ for a given beam profile can be computed using standard analogue techniques by solving equations (1) and (3) or equations (1) and (4) for the given boundary conditions. Both deflection and slope of the beam are zero at the wall, whilst bending moment M , and $\frac{dM}{dx}$ are zero at the free end of the beam. It is therefore advantageous to compute along the beam from its free end towards the wall (equation 4), since this involves only the evaluation of a single boundary condition, namely slope at the free end, in order to obtain deflection of the free end for a given beam profile. Computing in the other direction (equation 3) necessitates solving for two boundary conditions before deflection can be evaluated. After the development of the analogue model of the beam, the problem resolves itself into the generation and optimisation of parameters describing the profile of the beam.

With the inherent high speed iterative operation of analogue computers, rapid convergence of parameters to their optimum values is possible without resort to the more sophisticated hill-climbing methods. Furthermore, an investigator can observe the performance of the iterative solution and make adjustments according to his logic and intuition. Unfortunately two of the well-known limitations of analogue computers, inaccuracy and drift, can introduce errors and in some cases prevent correct optimisation.

In order to control even simple axial multi-parameter hill-climbing on an analogue computer, relatively complex digital mode control facilities are required. Present day analogue computers are usually equipped with a complement of digital logic capable of performing this control function. The authors had the use of a machine of this type, a Solartron 247 system, linked to a comprehensive sequential program control unit (Bellamy and Hulton, 1968). With this machine, control sequences of the type given in this paper could be patched directly without the need of a tedious logic interpretation.

Preliminary investigations into the problem showed that the optimum profile of the beam would have three distinct sections (Fig. 1). At the fixed and free ends the maximum and minimum height restrictions would ensure that the beam height at and near these points would be h_1 and h_2 respectively. Between these two constant height sections the profile would probably be a continuous function $h(x)$, from $h = h_1$ to $h = h_2$.

Describing the exact form of the function $h(x)$ over the mid-section of the beam in terms of parameters that can be optimised is, of course, the root of the problem. Of the methods of generating continuous functions on analogue computers, piece-wise linear approximation is

C

usually the most satisfactory. In this case it is convenient to generate straight line segments making up the mid-section profile of the beam and to use the slopes of these segments as the parameters to be optimised.

For the sake of simplicity, the first attempts at the problem on the analogue computer used only two parameters to describe the mid-section profile of the beam. One parameter was the length l_1 of the constant height section at the free end and the other was the slope α of a straight line approximation of the mid-section. Fig. 3 shows the profile considered for this simple two-parameter optimisation.

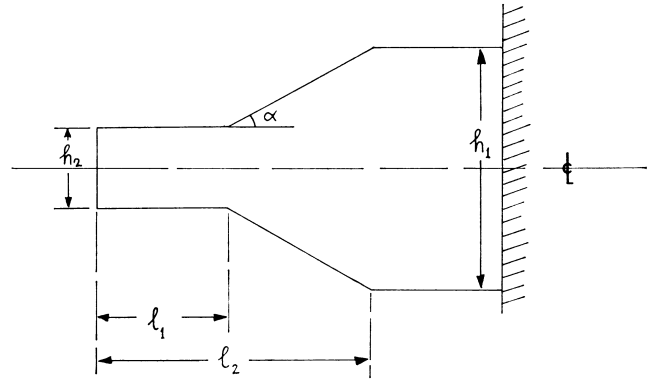


Fig. 3. Beam profile for two parameter optimisation

Hill climbing on the analogue computer

The complete analogue flow diagram for the profile optimisation using the two parameters mentioned is given in Fig. 4. For the purpose of reference the system can be considered to consist of three groups of integrators, A, B and C with two separately controlled track-store units. Integrators in group A are used to compute the bending moment and deflection of the beam according to equation (4). The independent variable x is given by integrator A1 which also feeds comparator 1 to indicate the end of the beam. Unfortunately there is a boundary condition problem in that the slope at the free end of the beam is not known and needs to be found before the deflection can be computed.

A double iteration sequence has to be adopted to overcome this problem. Integrator A4 generates slope along the beam and its value is tracked on track-store T1. At the end of the first computing run, this value is stored by T1 and used as the correct initial condition of slope $\frac{dy}{dx}(0)$ for a second run which evaluates the required deflection.

Integrator B1 generates the beam profile determined by the parameters l_1 and α . It is initially held with a value corresponding to the height h_2 at the free end of the beam and is only put into the compute mode at $x = l_1$ given by comparator 2. The voltage on B1 is then driven at a constant rate equivalent to the slope parameter α , until comparator 3 detects that $h = h_1$, where B1 is again held constant over the remaining length of the beam.

The integrators C1 and C2 hold the values of l_1 and α respectively and can be updated during each computing run, depending on the optimising procedure. At the

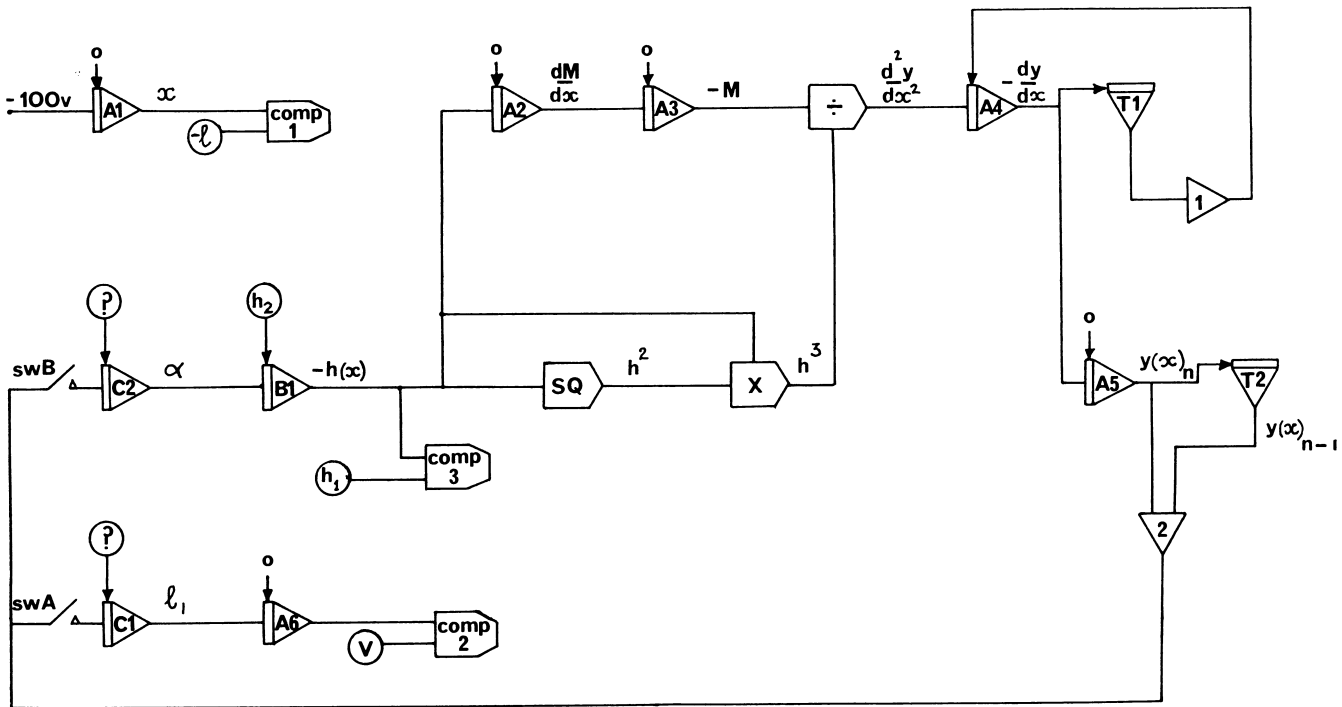


Fig. 4. Analogue flow diagram for hill-climbing method

end of each run integrator A5 holds the value of deflection $y(x)_n$ at the free end of the beam for the given profile. This is compared with the previous deflection $y(x)_{n-1}$ on track-store T2 to give the difference in deflection of the past two runs. By integrating this difference $y(x)_{n-1} - y(x)_n$ on integrators C1 or C2 for a short time, new updated parameters are produced. Before the next run, T2 is allowed to track A5 in order to store the value $y(x)_n$ ready for the next iteration. The magnitude of $y(x)_{n-1} - y(x)_n$ is dependent upon the previous parameter change, the distance from the top of the hill, and the gain of the optimising loop, whilst its sign is dependent on whether the deflection has been decreased or increased by the last parameter change.

It was found that if each parameter was adjusted several times in turn, a reasonably quick convergence was achieved. Even so, the system tended to hunt around the optimum, especially when the ratio h_2/h_1 exceeded 0.5. It is in fact shown later that as h_2/h_1 increases, the hill becomes very flat topped, so one must expect increased hill climbing difficulty.

A detailed investigation of the hunting indicated that the main contributory factor was drift of the computing units, particularly in hold and store modes, causing the small errors generated as the optimum is approached to be modified so as to initiate false error corrections. In the particular system used, iteration speeds were typically 2 seconds per run and it is essential that integrators and track-store units should drift as little as possible during successive runs. Because of the time constant associated with it, T1 cannot instantaneously track voltages fed into it from A4. Therefore, at the end of a first run, a known time must pass before one can use the voltage on T1. Since T1 stores only for the time it takes to reset A4, it will not contribute much error due to drift at say 10 mV/sec (a typical value in this system). Similarly,

T2 does not track instantaneously but as it stores continuously, and is updated every two runs, it can be an important source of error. It is possible to reduce drift with a backing-off current, and this was done as a matter of course, but further reduction is achieved by including a large capacitor in the feedback of T2. Unfortunately updating T2 in the track mode must then take longer. The overall effect is that drift elsewhere in the system takes place over a longer period of time, though T2 drifts at a slower rate. In a similar manner, the drift of C1 and C2 can be reduced, but only at the expense of increased updating time, during which T2 drifts for an increased period. Obviously, one must effects reach a compromise which best balances out the two.

It is worth mentioning that the authors, in investigating the role drift played in the optimisation, deliberately varied the drift of T2 between +10 mV/sec and -10 mV/sec. The result was a change in the optimum profile corresponding to changes of up to 1% in the values of l_1 and l_2 in cases where the hill was tending to become rather flat.

The sequential mode control program used to control the analogue iteration is well worth mentioning. **Table 1** shows a digital control program of the operations involved in the iterations; in the main it should be self explanatory. One point to note, however, is the two operational loops for the alternate computation of slope and deflection.

The two-parameter optimisation described is, of course, only for a straight line approximation of the beam profile. Further three-parameter computations using two straight-line segments to make up the mid-section of the beam showed that a single-line approximation of the mid-section is very near to the correct solution. This fact was later substantiated by the solutions given by Pontryagin's method.

Table 1**Sequence of digital control instructions for hill-climbing**

INSTRUCTION NO.	OPERATION
1.	Reset initial conditions of all integrators.
2.	Set track-store units to track.
3.	Wait for start.
4.	Compute group A integrators.
5.	Compute group B integrators when $x = l_1$ (comparator 2).
6.	Hold group B integrators when $h = h_1$ (comparator 3).
7.	Hold group A integrators when $x = l$ (comparator 1).
8.	Track T1 for 100 msec; then store.
9.	Jump to 12 if deflection has been evaluated.
10.	Reset group A and B integrators for 100 msec.
11.	Jump to instruction 4.
12.	Compute C integrators for one second; then hold.
13.	Track T2 for one second then store.
14.	Reset group A and B integrators.
15.	Track T1.
16.	Jump to instruction 4.

In view of the many difficulties encountered, it was concluded that, for this particular problem, hill-climbing on an analogue computer was not entirely satisfactory. Some other more profitable method would have to be found which would avoid the analogue computing limitations mentioned, and lead to a more direct and precise solution.

Pontryagin's principle applied to the beam equations

The Pontryagin maximum principle provides a very elegant approach to optimisation problems without resort to hill-climbing techniques. A concise review of the principle is to be found in Leitmann (1962) in which the proof is developed and examples given.

When applying Pontryagin's principle to the equations of the beam there is no advantage in regarding x as increasing from the beam free end rather than from the wall. In either case one must solve two simple boundary conditions to obtain an optimum profile, and the equations which must be considered are similar. In fact, x has been taken to increase from the wall, and so equation (3), referring to Fig. 1, is used in the following mathematical derivation. This choice is influenced solely by the fact that the analogue representation is slightly neater than it would be if equation (4) had been used.

Pontryagin's principle states that in a given system, the pay-off function will be minimised if the control variable u is adjusted at all times so as to maximise the Hamiltonian function.

The differential equations of the system will be in the form

$$\dot{x}_i = f_i(x_0 \dots x_n, u, t) \text{ where } i = 0, 1, \dots, n.$$

The x_i are designated state variables, and it is required that a pay-off function of the form

$$S = \sum_{i=0}^n c_i x_i(t)$$

be minimised with respect to the control variable $u(t)$.

In the case of the cantilever beam, let $y(l)$, the deflection of the beam at $x = l$, be introduced as a dimensionless parameter x_0 such that

$$\frac{y(l)}{l} = x_0. \quad (6)$$

Define $\tau = \frac{x}{l}$ as dimensionless distance along the beam, and the control variable $u = \frac{h}{l}$.

x_0 , the pay-off function, must be minimised and so

$$S = \sum_{i=0}^n c_i x_i(t_f) = x_0, \quad t_f \text{ referring to the point } \tau = 1,$$

from which it follows that $c_0 = +1$ and

$$c_1 = c_2 = \dots = c_n = 0.$$

Introducing x_0 as a state variable, and defining \dot{x} to mean the differentiation of x with respect to τ

$$\dot{x}_0 = \frac{dy}{dx} = x_1 \text{ say} \quad (7)$$

$$\dot{x}_1 = \frac{x_2}{u^3} \text{ where } x_2 = -\frac{12M}{Eal^2} \quad (8)$$

$$\dot{x}_2 = -\frac{12\dot{M}}{Eal^2} = x_3 \text{ say} \quad (9)$$

$$\dot{x}_3 = Au \text{ where } A \text{ is a constant.} \quad (10)$$

The Hamiltonian, defined as $H = \sum_{i=0}^n p_i f_i$ can now be formed:

$$H = p_0 x_1 + \frac{p_1 x_2}{u^3} + p_2 x_3 + p_3 Au. \quad (11)$$

Since $\dot{p}_i = -\frac{\partial H}{\partial x_i}$, the Hamiltonian yields the following equations for the adjoint variables:

$$\dot{p}_0 = 0 \quad (12)$$

$$\dot{p}_1 = -p_0 \quad (13)$$

$$\dot{p}_2 = -\frac{p_1}{u^3} \quad (14)$$

$$\dot{p}_3 = -p_2. \quad (15)$$

Maximising the Hamiltonian to find the control variable equation for minimum deflection, and setting the control variable in this case equal to u^*

$$\frac{\partial H}{\partial u} = -\frac{3p_1 x_2}{u^4} + Ap_3 = 0 \quad (16)$$

$$u = u^* = \left[\frac{3p_1 x_2}{Ap_3} \right]^{1/4} \quad (17)$$

This maximises the Hamiltonian since $\frac{\partial^2 H}{\partial u^2}$ for this value

of u^* is less than zero in value for all τ .

A general expression (see Leitmann, 1962) can now be used to evaluate the adjoint variable boundary conditions. This is

$$-\sum_{i=0}^n c_i \Delta x_i(t_f) = \sum_{i=0}^n [p_i(t_f) \Delta x_i(t_f) - p_i(t_0) \Delta x_i(t_0)] \quad (18)$$

from which the following are obtainable

$$p_0(1) = -1 \quad (19)$$

$$p_1(1) = 0 \quad (20)$$

$$p_2(0) = 0 \quad (21)$$

$$p_3(0) = 0. \quad (22)$$

From (13) and (19) it follows that

$$p_1 = \tau - 1 \quad (23)$$

and hence $\dot{p}_2 = \left(\frac{1-\tau}{u^3} \right)$ from (14) (24)

Summarising, the following are the necessary equations and boundary conditions suitable for evaluating an optimum profile by analogue computation.

$$p_1 = -(1 - \tau) \quad p_1(1) = 0$$

$$\dot{p}_2 = \frac{(1 - \tau)}{u^3}$$

$$\dot{p}_3 = -p_2 \quad p_2(0) = 0, p_3(0) = 0$$

$$\dot{x}_2 = x_3 \quad x_2(1) = 0, x_3(1) = 0$$

$$\dot{x}_3 = Au^* \quad \text{where } u^* = \left[\frac{3p_1 x_2}{Ap_3} \right]^{1/4}.$$

Provided all boundary conditions are fulfilled the u^* curve produced in the analogue simulation is the optimum profile of the beam over its mid-section. Maximum and minimum height restrictions can be introduced into the simulation itself.

Analogue computation of equations derived by Pontryagin

Fig. 5 shows the analogue flow diagram representing the equations derived using Pontryagin's method. Two integrator groups, A and B, operating in conjunction with two track-store units, together with three analogue switches and eight non-linear elements comprise the essential hardware. The single-loop digital control program is shown in Table 2 which should be self-explanatory.

Table 2

Sequence of digital control instructions for Pontryagin's method

INSTRUCTION NO.	OPERATION
1.	Reset initial conditions of all integrators.
2.	Set track-store units to track.
3.	Wait for start.
4.	Close analogue switch <i>a</i> , open analogue switches <i>b</i> and <i>c</i> .
5.	Compute group A integrators.
6.	Open analogue switch <i>a</i> and close analogue switch <i>b</i> when $u^* = u_1$ (comparator 2).
7.	Open analogue switch <i>b</i> and close analogue switch <i>c</i> when $u^* = u_2$ (comparator 3).
8.	Hold group A integrators when $(1 - \tau) = 0$ (comparator 1).
9.	Track T1 and T2 for 100 msec; then store.
10.	Compute group B integrators for 100 msec; then hold.
11.	Reset group A integrators for 100 msec.
12.	Set track-store units to track.
13.	Jump to instruction 4.

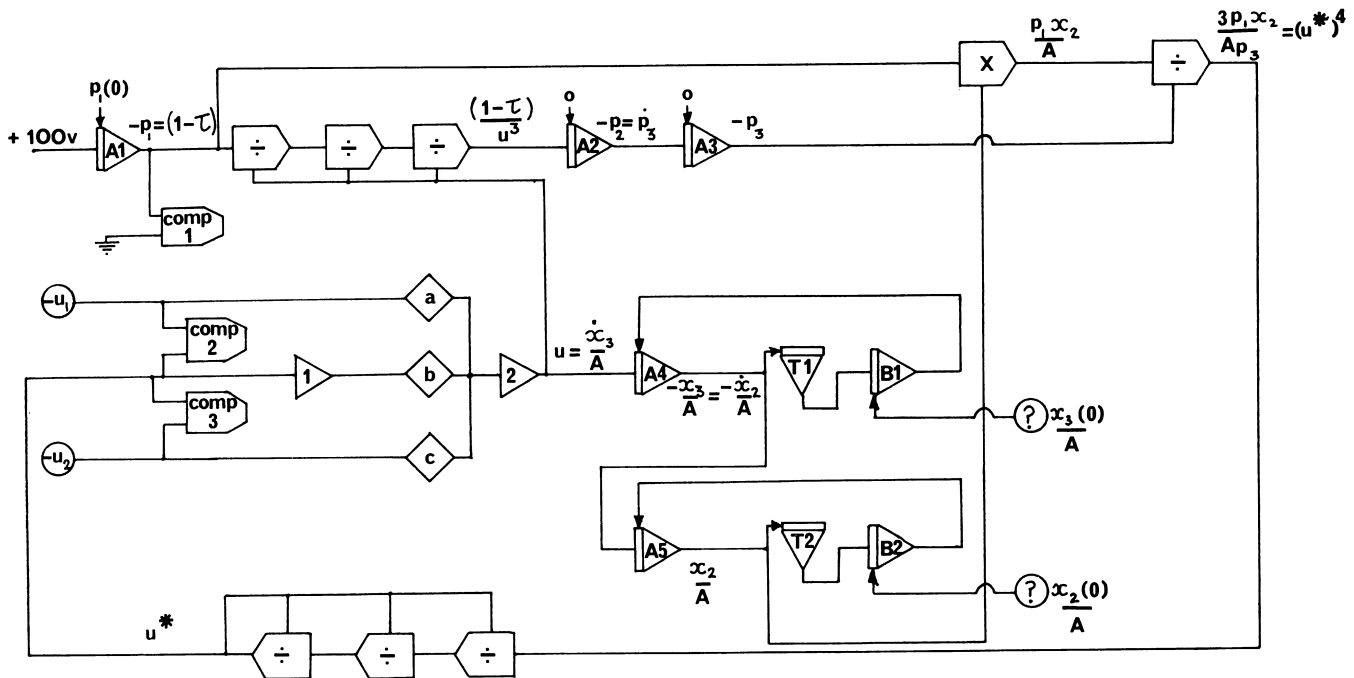


Fig. 5. Analogue flow diagram for Pontryagin's method

Considering the analogue flow diagram in detail, the integrator A1 generates the function $(1-\tau)$ as a ramp until comparator 1 detects when $(1-\tau)$ falls to zero which is at the free end of the beam. This is convenient in that the boundary condition $p_0(1) = 0$ is thus deliberately made to be fulfilled.

Amplifier 2 output, u , is initially fixed through the analogue switches at u_1 , which is proportional to h_1 the fixed end height of the beam. During a computing run, u^* begins at infinity since $p_3(0) = 0$ and then rapidly decreases until comparator 2 detects when $u^* = u_1$. At this point the analogue switches make $u = u^*$ in order to generate the mid-section profile of the beam. As u^* decreases further comparator 3 detects when $u^* = u_2$ which is proportional to h_2 the height of the free end of the beam. This constant height is maintained for the remaining portion of the beam.

The function u^* generated by the system is only optimum if all boundary conditions are satisfied. The initial conditions are all known with the exception of $p_1(0)$, $x_2(0)$ and $x_3(0)$. As previously stated, $p_1(0)$ is set to an arbitrary value and decreased until it becomes zero at the free end. This necessitates $x_2(1)$ and $x_3(1)$ becoming zero at the free end of the beam to satisfy the final conditions.

In order to satisfy the boundary conditions, guessed values of $x_3(0)$ and $x_2(0)$ held on integrators B1 and B2 are first preset into integrators A4 and A5 respectively. At the end of each iteration, the error voltages on A4 and A5 are stored on the track-store units T1 and T2 and then used to adjust the initial guesses of $x_3(0)$ and $x_2(0)$ by setting B1 and B2 into the compute mode for a brief period. Provided the iterative-loop gains are reasonable, the initial conditions $x_3(0)$ and $x_2(0)$ will converge rapidly to their correct values to comply with the condition that $x_2(1) = x_3(1) = 0$.

The first attempts at computations produced solutions which were unsatisfactory from the point of view of uniformity of the u^* curve. The reasons for this are twofold: first, the large number of non-linear elements in a loop, and second, the wide range of values over which each variable extended. The difficulties due to non-linear effects were largely overcome by using time-division multipliers instead of quarter square multipliers. This avoided the prominence in the solutions of the discontinuities due to the break points and straight line segments which are a feature of quarter square multipliers. The problem of the dynamic range of variables was difficult to overcome since scaling is limited by the maximum excursion a variable has to make. In particular, the term $(u^*)^4$ varied over an excessive range and in one instance went below 0.5 volt, at which point it was very difficult to take an accurate fourth root.

Beam profile solutions

No solutions are given for the beam profile obtained by hill-climbing since for reasons already given this technique was abandoned in favour of Pontryagin's method. Some of the beam profile solutions computed by the latter method are shown in Fig. 6. It is interesting to note that the mid-section of the beam is almost a straight line and only when h_2/h_1 ratios are less than 0.5 is there any slight sign of curvature. If such a beam was manufactured, the practical approach would obviously

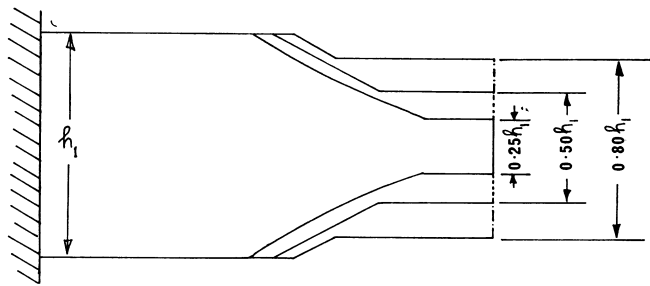


Fig. 6. Optimum beam profiles

be to make the structure so as to include three straight edges for ease of production.

If the mid-section of the beam is considered straight, optimum values of l_1/l and l_2/l become the only parameters necessary to describe the beam profile. These optimum values vary according to the value of h_2/h_1 and the graphs of Fig. 7 serve to illustrate this effect. Interpretation of these graphs shows that as h_2/h_1 increases, the mid-section of the beam profile reduces in length and increases in slope. Careful examination of Fig. 7 shows that the extrapolated point of intersection of the two curves occurs when $h_2/h_1 = 1.0$ as one would expect. Furthermore, the curve l_1/l aims for the origin which is again a sensible result if one considers that l_1 must be zero if h_2 is zero. What is not clear is the meaning of the suggested intersection of the curve l_2/l with the vertical axis.

The object of the exercise is to determine the profile of the beam for minimum deflection of its free end. It is therefore of interest to examine the variation of deflection for different beam profiles to establish the shape of the hill. Fig. 8 shows the variation of a one

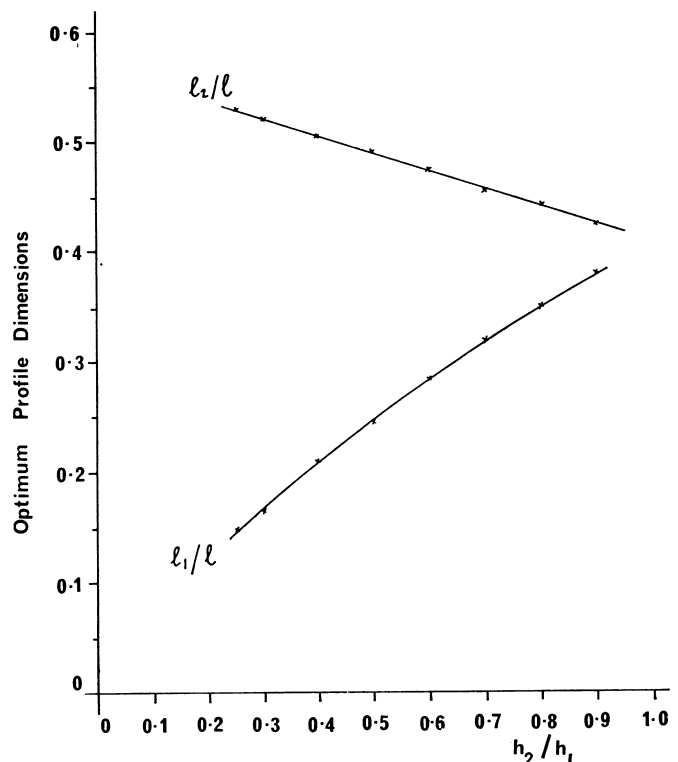


Fig. 7. Graph showing optimum values given by Pontryagin's method

parameter hill as a set of curves for different values of h_2/h_1 . To obtain these curves, the assumption was made that the optimum mid-section profile of the beam is straight edged. On this basis, optimum α was evaluated from the Pontryagin results (Fig. 7) for a particular h_2/h_1 ratio, and set into the hill climbing system as a fixed value. End deflections were then noted for a range of values of l_1/l . The flat nature of the inverted hill curves near the optimum accounts for difficult behaviour of the hill-climbing attempts, particularly when the value of h_2/h_1 is high. In addition, a nominal percentage change in the value of l_1/l or in the value of h_2/h_1 produces a much smaller percentage change in deflection. This effect should be kept in mind when estimating the accuracy of the computing results.

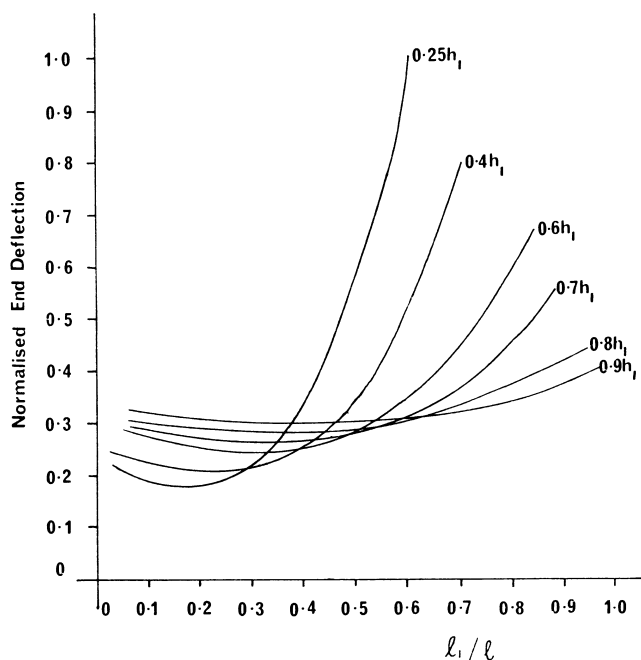


Fig. 8. One parameter 'hills'

Conclusions

Two entirely different methods of optimising a beam profile using an analogue computer have been described. The first method, hill-climbing, is considered unsatisfactory, since it only produces an approximation to the profile of the beam and correct optimisation proved difficult to achieve. The second more direct method using Pontryagin's maximum principle generates the

exact optimum profile with only two simple boundary conditions to solve. This latter method is certainly powerful in its application even though its non-linear expressions are liable to introduce errors in analogue computations.

No mention has been made so far on the solution of the problem by digital computation. Dixon (1967) attempted this, but made a basic error in using a wrong beam equation. He corrected this in a later paper (1968), and the results closely resemble those quoted by the authors in the present paper. Dixon has confirmed that solution by analogue computation provides a solution many times faster than is possible by digital methods, and it is felt that the fast iterative solutions on an analogue computer in many ways compensate for the slow but accurate digital computations. Perhaps some compromise may be made by using the analogue solutions as a starting point for accurate digital computations and so taking advantage of both types of machine.

It would clearly be possible to extend this optimisation in many ways, for instance, by not fixing the width of the beam to be a constant. This would certainly increase the difficulty of deriving the appropriate analogue simulation equations but should not be beyond the scope of competent mathematicians. Though this paper deals with a relatively simple set of equations, there is no reason why the profile methods discussed should not have wide application. In fact, the authors are at present working on the problems of optimisation of structures governed by higher order differential equations, involving more parameters. The authors feel that piece-wise linear approximation and hill-climbing is not practicable for such systems, and that solution by digital computer is a tedious alternative with rather prohibitive computing time. Provided one is assured of reasonable accuracy despite the involved non-linear functions, the application of the Pontryagin principle appears to be an expedient choice and, the authors feel, an extremely powerful tool in analogue computation.

Acknowledgements

The authors are grateful for the facilities provided by the Department of Electrical Engineering, Lanchester College, Coventry, for the support given by Mr. H. S. Baldwin, IBM (United Kingdom) Ltd., and the mathematical advice given by Mr. L. C. Dixon, Department of Mathematics, Hatfield College of Technology.

References

- BELLAMY, N. W., and WILBY, C. B. (1962). *Elastic Analysis of Shells by Electronic Analogy*, Edward Arnold.
- PAUL, R. J. A. (1962). *Fundamental Analogue Techniques*, Blackie.
- ROSENBROCK, H. H. (1960). An Automatic Method for Finding the Greatest or Least Value of a Function, *Computer Journal*, Vol. 3, p. 175.
- FLETCHER, R., and POWELL, M. J. D. (1963). A Rapidly Convergent Descent Method for Minimisation, *Computer Journal* Vol. 6, p. 163.
- POWELL, M. J. D. (1964). An Iterative Method for Finding Stationary Values of a Function of Several Variables, *Computer Journal*, Vol. 5, p. 147.
- PONTRYAGIN, BOLTYANSKII, GAMKRELIDZE and MISHCHENKO (1962). *The Mathematical Theory of Optimal Processes*, Interscience.
- BELLAMY, N. W., and HULTON, L. J. (1968). A Sequential Mode Control System for Hybrid Computation, *Control*, Vol. 12, No. 117.
- LEITMANN, G. (1962). *Optimization Techniques*, Academic Press.
- DIXON, L. C. W. (1967). Pontryagin's Maximum Principle Applied to the Profile of a Beam, *Journal of the Royal Aeronautical Society*, Vol. 71, No. 679.
- DIXON, L. C. W. (1968). Further Comments on Pontryagin's Principle Applied to the Profile of a Beam, *Journal of the Royal Aeronautical Society*, Vol. 72, No. 690.