Implicit integration processes with error estimate for the numerical solution of differential equations

By C. F. Haines†

Numerical integration processes originally suggested by Rosenbrock (1963) are modified and developed to give a practical method for the solution of sets of stiff differential equations. A control scheme for these processes is developed and a comparison, based on numerical experiments, between this method and other commonly used methods is given.

(Received July 1968, revised December 1968)

Numerical methods for solving the set of first order differential equations

$$\dot{x} = \phi(x) \tag{1}$$

are considered.* In this paper chief interest lies in those systems (1) which are termed stiff differential systems. That is to say, when the equations (1) are linearised the time constants of the solution have a wide spread in magnitude. It is well-known that when certain numerical integration methods are used to solve this type of system problems of stability arise in the computation of the solution. These stability difficulties restrict the size of step length which may be used during the computation, thus increasing the computer time required. In extreme cases the time taken to obtain the solution may become so large as to restrict the use which can be made of a model.

Implicit numerical integration methods can be made completely stable at least for linear sets of equations (1) although this stability may fail to carry over the non-linear systems. In practice, integration methods are developed which are highly stable when applied to linear systems. It is then hoped that some degree of stability will still be retained when the methods are applied to nonlinear systems.

One such method comprises the processes of Rosenbrock (1963),

$$x_{r+1} = x_r + R_1 k_r + R_2 l_r + R_3 m_r + \ldots \tag{2}$$

$$k_r = h_r(\phi(x_r) + a_1 A(x_r) k_r) \tag{3}$$

$$l_r = h_r(\phi(x_r + b_1 k_r) + a_2 A(x_r + c_1 k_r) l_r) \tag{4}$$

$$m_r = h_r(\phi(x_r + b_2 k_r + d_1 l_r) + a_3 A(x_r + c_2 k_r + e_1 l_r) m_r) \tag{5}$$

. . . . . .

where $A = (A_{ij}) = \left(\frac{\partial \phi_i}{\partial x_j}\right)$ and R_1, R_2, R_3, a_1, a_2, a_3, b_1, b_2, c_1, c_2, d_1, and e_1 are constants.

These processes, which will be called 'processes (B)', have certain advantages over other integration methods. One advantage is that because the equations (3), (4) and (5) may be solved successively to yield k_r, l_r and m_r no iteration scheme is involved in this method. Other implicit integration methods require iteration at each step. It has been found (Huckaba and Danley, 1960) that restrictions as severe as those imposed by stability in the explicit Runge–Kutta processes may be placed on the step length, h_r, if certain of these iterative methods are used.

In an 's' stage process (B) the exponentials of the solution (in linear problems) are replaced by the approximations

$$\zeta_i(\lambda_i h_r) = \frac{1 + \alpha_1(\lambda_i h_r) + \alpha_2(\lambda_i h_r)^2 + \ldots + \alpha_s(\lambda_i h_r)^s}{1 + \beta_1(\lambda_i h_r) + \beta_2(\lambda_i h_r)^2 + \ldots + \beta_s(\lambda_i h_r)^s} \tag{6}$$

during one time interval. In the above expression α_i and β_i are constants and functions of the process (B) coefficients and λ_i are the constants in the exponentials, $e^{\lambda_i t}$, of the solution. In order to satisfy the conditions for stiff stability (Gear, 1967) it is necessary to examine the locus $|\zeta_i(\lambda_i h_r)| = 1$ in the complex plane. The coefficient α_s is made zero to make the method stable at infinity.

As an example of the function $\zeta_i(\lambda_i h_r)$, the function arising from one of the 3rd order processes (B) (equations (37) through (43)) is examined. In this case

$$\zeta_i(\lambda_i h_r) = \frac{1 - \frac{8}{3}(\lambda_i h_r) + \frac{2}{9}(\lambda_i h_r)^2 + \frac{1}{3}(\lambda_i h_r)^3}{1 - \frac{11}{3}(\lambda_i h_r) + 5(\lambda_i h_r)^2 - 3(\lambda_i h_r)^3 + \frac{2}{3}(\lambda_i h_r)^4} \tag{7}$$

Now writing $\lambda_i h_r = \sigma + i\omega$, a plot similar to those obtained by, for example, Widlund (1967) of the stable regions of the process (B) in the complex plane is given in **Fig. 1**.

It can be seen that this process (B) is unstable only for a small region of the left half plane and is in fact stable for the wedge $|\omega| < 4|\sigma|$. Many of the problems arising from distillations or heat exchangers have constants λ_i which lie near to the negative real axis. For the purposes of obtaining a general stability condition it is assumed that λ_i are real, for although the complex plane analysis is very useful, a quicker method for obtaining stability conditions for the process (B) is preferable. In fact it will be seen that the stability conditions developed using the assumption that λ_i are real yield satisfactory results. Writing $\lambda_i = -k_i$ the expression (6) becomes

* Time dependent systems may be considered by writing $x_{n+1} = t$ and transforming the n equation system $\dot{x} = \Phi(x, t)$ into the $n + 1$ equation system $\dot{x} = \phi(x)$.

† *Control Systems Centre, University of Manchester, Institute of Science and Technology; now with Rolls-Royce Ltd., Bristol Engines Division, P.O. Box 3, Bristol*

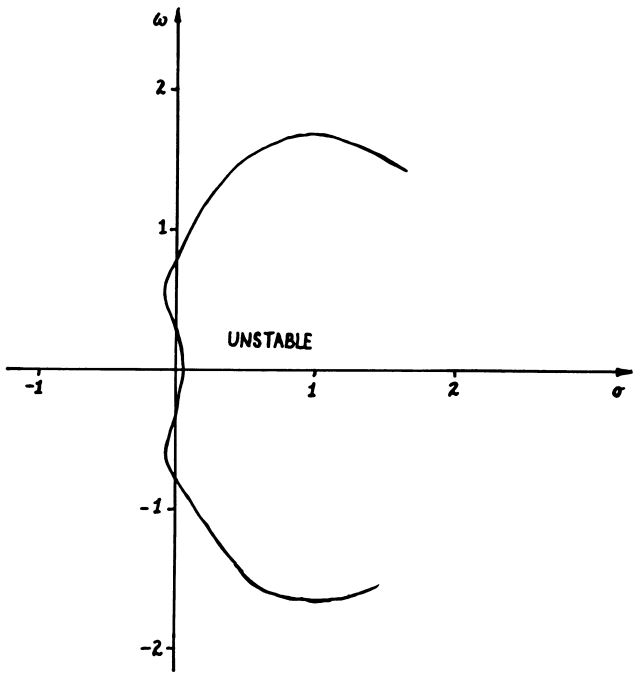

Fig. 1. Stable region of 3rd order process (B)

$$\zeta_i(k_ih_r) = \frac{1-\alpha_1(k_ih_r)+\alpha_2(k_ih_r)^2+\ldots+(-1)^s\alpha_s(k_ih_r)^s}{1-\beta_1(k_ih_r)+\beta_2(k_ih_r)^2+\ldots+(-1)^s\beta_s(k_ih_r)^s} \tag{8}$$

Thus if $|\zeta_i(k_ih_r)| < 1$, for all $(k_ih_r) > 0$*, with $\alpha_s = 0$, stability of the process (B) is ensured in the linear case. The process (B) coefficients are, therefore, chosen such that these two conditions are satisfied as well as the conditions which make the process (B) accurate to same desired order of h_r. Any other freedom in choice of process (B) coefficients may then be used to reduce the computation at each step by eliminating unnecessary matrix inversions or function evaluations.

Unfortunately the processes (B) require rather a large amount of calculation per step. Consequently they are appropriate chiefly for problems where there is an extreme spread of time constants or where the work per step can be reduced. Examples are distillation and heat exchanger problems which are particularly suitable for solution by processes (B) as the matrices $A(x_r)$ are tri-diagonal and also the time constants generally have a wide spread in magnitude. Thus, due to the availability of special matrix techniques the amount of computation per step is reduced and also full use may be made of the good stability properties of the method. The other methods considered do not show as much gain in this type of problem although the method of Gear (1967) does appear to possess similar advantages to those of the processes (B) in such solutions. Gear's method, however, has not been tested here.

Another deficiency of the processes (B) is that no estimate of the error at each step is given. However, this is rectified in the next section where one such estimate is developed. Also a further necessary improvement is some method for evaluating the matrix $A(x_r)$ in equations (3), (4) and (5) other than by analytical means. This is because the functions ϕ might not be easily differentiated in general problems and also it is more convenient to write only the functions ϕ when programming problems involving many equation systems. Finally it is apparent that some reduction in the number of evaluations of $A(x_r)$ at each step would save computing time. Attempts to effect this have proved unsuccessful (Haines, 1968) as much less stable methods are obtained which are not of practical use. An alternative would seem to lie in the choice of the process (B) coefficients so as to make, say, $e_1 = 0$ if this is consistent with other conditions on the process (B) coefficients.

* It is quite apparent that this condition is more easily checked than the complex plane condition.

An error estimate for the processes (B)

An approximation to the matrix $A(x_r)$ for a third order process (B) is:

$$A_{ij}(x_r) = \frac{\phi_i(x_r + \frac{1}{2}\gamma\Delta(x_j)_r) - \phi_i(x_r - \frac{1}{2}\gamma\Delta(x_j)_r)}{\gamma\Delta(x_j)_r}. \tag{9}$$

$\Delta(x_j)_r = (x_j)_r - (x_j)_{r-1}$ is the jth component of some suitable small increment of x_r. It has been found (Haines, 1968) that it is necessary to include the factor γ in the above approximation in order to improve the accuracy of the approximation. The most suitable value of γ is obtained experimentally (Haines, 1968) for each process (B) and is found to be the same for all problems tested. However, it must be stated that for different processes (B) the most suitable values does vary to some degree. The value of γ for the process (B) used in the next section is given there along with the values of the process (B) coefficients.

It is a simple matter to show that the approximation (9) maintains the overall order of the third order process (B) and it can be seen from the numerical experiments of the next section that the approximation (9) is useful in practical examples.*

The general third order processes (B) with error estimate are written

$$x_{r+1} = x_r + R_1k_r + R_2l_r + R_3m_r + R_4n_r \tag{10}$$

$$x_{r+1} = x_r + R_1'k_r + R_2'l_r + R_3'm_r + R_4'n_r \tag{11}$$

$$k_r = h_r(I - a_1A(x_r)h_r)^{-1}\phi(x_r) \tag{12}$$

$$l_r = h_r(I - a_2A(x_r + c_1k_r)h_r)^{-1}\phi(x_r + b_1k_r) \tag{13}$$

$$m_r = h_r(I - a_3A(x_r + c_2k_r + e_1l_r)h_r)^{-1} \phi(x_r + b_2k_r + d_1l_r) \tag{14}$$

$$n_r = h_r(I - a_4A(x_r + c_3k_r + e_2l_r + g_1m_r)h_r)^{-1} \phi(x_r + b_3k_r + d_2l_r + f_1m_r) \tag{15}$$

$$n_r' = h_r(I - a_4'A(x_r + c_3'k_r + e_2'l_r + g_1'm_r)h_r)^{-1}\phi(x_r + b_3'k_r + d_2'l_r + f_1'm_r). \tag{16}$$

The error estimate is $(\epsilon_j)_{r+1} = K((x_j)_{r+1} - (x_j')_{r+1})$ (17)

where K is some constant depending on the values of the process (B) coefficients used.

* It is suggested that the Runge–Kutta method be used, with some suitably small value of h_r, for the initial step as no value of $(x_{-1})_j$ is available for evaluating the approximation (9).

The expansions, in powers of h_r, of (10) and (11) may be compared with the Taylor series. In order to ensure that the two processes (B), x_{r+1} and x'_{r+1}, should agree with the Taylor series up to and including terms of $O(h_r^3)$, the following equations must be satisfied.

$$1 = R_1 + R_2 + R_3 + R_4 \tag{18}$$

$$1 = R'_1 + R'_2 + R'_3 + R'_4 \tag{19}$$

$$\tfrac{1}{2} = R_1a_1 + R_2(a_2 + b_1) + R_3(a_3 + b_2 + d_1) + R_4(a_4 + b_3 + d_2 + f_1) \tag{20}$$

$$\tfrac{1}{2} = R'_1a_1 + R'_2(a_2 + b_1) + R'_3(a_3 + b_2 + d_1) + R'_4(a'_4 + b_3 + d_2 + f_1) \tag{21}$$

$$\tfrac{1}{6} = R_1a_1^2 + R_2(a_2^2 + (a_2 + a_1)b_1) + R_3(a_3^2 + a_3(a_2 + b_1) + a_1b_2 + d_1(a_2 + b_1)) + R_4(a_4^2 + a_4(b_3 + d_2 + f_1) + b_3a_1 + d_2(a_2 + b_1) + f_1(a_3 + b_2 + d_1)) \tag{22}$$

$$\tfrac{1}{6} = R'_1a_1^2 + R'_2(a_2^2 + (a_2 + a_1)b_1) + R'_3(a_3^2 + a_3(a_2 + b_1) + a_1b_2 + d_1(a_2 + b_1)) + R'_4(a_4'^2 + a'_4(b'_3 + d'_2 + f'_1) + b'_3a_1 + d'_2(a_2 + b_1) + f'_1(a_3 + b_2 + d_1)) \tag{23}$$

$$\tfrac{1}{6} = R_2(a_2c_1 + \tfrac{1}{2}b_1^2) + R_3(a_3(c_2 + e_1) + \tfrac{1}{2}(b_2 + d_1)^2) + R_4(\tfrac{1}{2}(b_3 + d_2 + f_1)^2 + a_4(c_3 + e_2 + g_1)) \tag{24}$$

$$\tfrac{1}{6} = R'_2(a_2c_1 + \tfrac{1}{2}b_1^2) + R'_3(a_3(c_2 + e_1) + \tfrac{1}{2}(b_2 + d_1)^2) + R'_4(\tfrac{1}{2}(b'_3 + d'_2 + f'_1)^2 + a'_4(c'_3 + e'_2 + g'_1)). \tag{25}$$

Further restrictions are placed on the process (B) coefficients when the previously mentioned stability conditions, which ensure high stability in linear problems, are applied. For the third order processes (B) with error estimate these conditions require

$$0 = a_1a_2a_3a_4 - R_1a_2a_3a_4 - R_2a_3a_4(a_1 - b_1) - R_3(a_1a_2 + d_1b_1 - a_2b_2 - a_1d_1)a_4 - R_4(a_1a_2a_3 - b_3a_2a_3 - d_2a_1a_3 - f_1a_1a_2 + f_1d_1a_1 + f_1b_2a_2 + d_2b_1a_3 - f_1d_1b_1) \tag{26}$$

$$0 = a_1a_2a_3a'_4 - R'_1a_2a_3a'_4 - R'_2a_3a'_4(a_1 - b_1) - R'_3(a_1a_2 + d_1b_1 - a_2b_2 - a_1d_1)a'_4 - R'_4(a_1a_2a_3 - b'_3a_2a_3 - d'_2a_1a_3 - f'_1a_1a_2 + f'_1d_1a_1 + f'_1b_2a_2 + d'_2b_1a_3 - f'_1d_1b_1) \tag{27}$$

and also that the respective functions $\zeta_i(k_ih_r)$ are such that $|\zeta_i(k_ih_r)| < 1$ for all $(k_ih_r) > 0$*.

The error estimate $(\epsilon_j)_{r+1}$ is obtained after the style of the Kutta–Merson (Merson, 1957) error estimate. This is effected by subtracting the Taylor series x_{r+1}^T, from the expansions of x_{r+1} and x'_{r+1} in turn. The following equations result:

$$x_{r+1} - x_{r+1}^T = \epsilon_{r+1} = (\rho_1x_r^{(i+1,1)} + \rho_2x_r^{(i+1,2)} + \ldots + \rho_px_r^{(i+1,p)})h_r^{(i+1)} + O(h_r^{i+2}) \tag{28}$$

$$x'_{r+1} - x_{r+1}^T = \epsilon'_{r+1} = (\sigma_1x_r^{(i+1,1)} + \sigma_2x_r^{(i+1,2)} + \ldots + \sigma_px_r^{(i+1,p)})h_r^{(i+1)} + O(h_r^{i+2}). \tag{29}$$

ρ_i and σ_j are constants and functions of the process (B) coefficients. $x_r^{(i+1,1)}$ is the sole non-zero term of the $(i+1)$th time derivative of x_r when $\dot{x}_r$ is linear. Otherwise the terms $x_r^{(i+1,q)}$ for $1 < q \leqslant p$ are not assumed to be in a specific order.

From (28) and (29)

$$x_{r+1} - x'_{r+1} = ((\rho_1 - \sigma_1)x_r^{(i+1,1)} + \ldots + (\rho_p - \sigma_p)x_r^{(i+1,p)})h_r^{i+1} + O(h_r^{i+2}). \tag{30}$$

Hence the error estimate for each component of x_{r+1} is

$$(\epsilon_j)_{r+1} = \frac{(\rho_1x_r^{(i+1,1)} + \ldots + \rho_px_r^{(i+1,p)})_j}{((\rho_1 - \sigma_1)x_r^{(i+1,1)} + \ldots + (\rho_p - \sigma_p)x_r^{(i+1,p)})_j}(x_{r+1} - x'_{r+1})_j. \tag{31}$$

Two different error estimates are now possible for the processes (B). The first is precisely analogous to the Kutta–Merson estimate. For this estimate it is necessary to assume that $\dot{x}_r$ is linear over the interval h_r, i.e. $x_r^{(i+1,q)} = 0$ for $1 < q \leqslant p$. The other estimate (Haines, 1968) involves no such assumption and is more accurate but requires more computer time. For most purposes the first mentioned estimate will be satisfactory (Haines, 1968) hence it will now be assumed that $\dot{x}_r$ is linear over the interval, h_r, for the evaluation of the error estimate.

The coefficient K of equation (17) can now be written

$$K = \frac{\rho_1}{(\rho_1 - \sigma_1)}. \tag{32}$$

The general expressions for ϕ_1 and σ_1 are

$$\rho_1 = R_1a_1^3 + R_2(a_2^3 + b_1(a_2^2 + a_1a_2 + a_1^2)) + R_3(a_3^3 + b_2a_1^2 + a_1(a_1b_1 + a_2^2 + a_2b_1) + a_3(b_2a_1 + a_2d_1 + b_1d_1) + a_3^2(b_2 + d_1)) + R_4(a_4^3 + b_3a_1^2 + d_2(a_1b_1 + a_2^2 + a_2b_1) + f_1(b_2a_1 + d_1a_2 + d_1b_1 + a_3^2 + a_3(b_2 + d_1)) + a_4^2(b_3 + d_2 + f_1) + a_4(b_3a_1 + d_2(a_2 + b_1) + f_1(a_3 + b_2 + d_1))) - \frac{1}{24}. \tag{33}$$

$$\sigma_1 = R'_1a_1^3 + R'_2(a_2^3 + b_1(a_2^2 + a_1a_2 + a_1^2)) + R'_3(a_3^3 + b_2a_1^2 + d_1(a_1b_1 + a_2^2 + a_2b_1) + a_3(b_2a_1 + a_2d_1 + b_1d_1) + a_3^2(b_2 + d_1)) + R'_4(a_4'^3 + b'_3a_1^2 + d'_2(a_1b_1 + a_2^2 + a_2b_1) + f'_1(b_2a_1 + d_1a_2 + d_1b_1 + a_3^2 + a_3(b_2 + d_1)) + a_4'^2(b'_3 + d'_2 + f'_1) + a'_4(b'_3a_1 + d'_2(a_2 + b_1) + f'_1(a_3 + b_2 + d_1))) - \frac{1}{24}. \tag{34}$$

This completes all that is necessary for the evaluation of the error estimate (17).

* This condition is found to be satisfied by most sets of process (B) coefficients even if the coefficients are chosen so that only equations (18) through (25) are satisfied.

The control scheme suggested for use in conjunction with the error estimate (17) is as follows. If θ is the maximum error allowed in the solution (θ is specified by the user) then

$$\text{if } \max_j |(\epsilon_j)_{r+1}| > \theta \quad \text{then } h_{r+1} = \tfrac{1}{2}h_r \tag{35}$$

$$\text{if } \max_j |(\epsilon_j)_{r+1}| < \theta/10 \text{ then } h_{r+1} = 2h_r. \tag{36}$$

The control scheme (35) and (36) is used in the next section as indicated.

Experimental results

The solutions to several different nonlinear problems have been computed using the processes (B). Many of these were distillation and chemical kinetics problems. For the purposes of a comparison of different integration methods a typical distillation example is used. Some of the solutions of this problem were obtained using the controlled processes (B) and others result from the application of the processes (B) with a pre-set scheme for varying h_r. The latter give an indication of the size of step, h_r, which may be used consistent with stability in each integration method.

First, the solutions using a pre-set scheme for h_r are considered. The integrations were started at some given initial point and continued towards equilibrium of the distillation system. The step length in each solution was doubled at every fourth step, having started at a suitable initial value ($h_r = 1$ second).

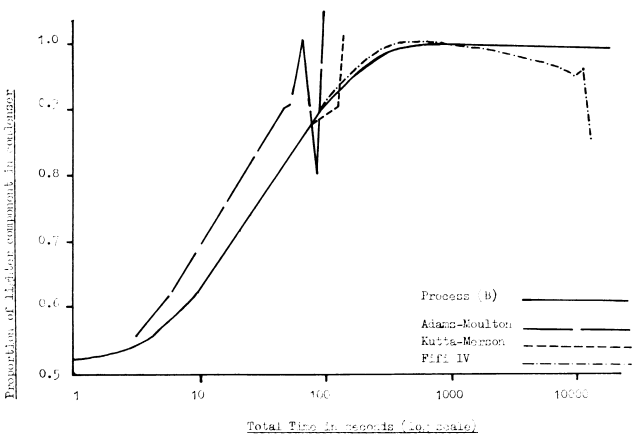

Fig. 2. Solution of a binary distillation problem by various numerical integration methods

Fig. 2 gives a comparison, for a typical component of the distillation example, of the different solutions obtained by using various integration methods. As is to be expected the explicit Kutta–Merson method becomes unstable after only a few increases of the step length (at $h_r = 16$sec.). The solution using the predictor-corrector method FiFi IV (Sumner, 1965) has only a limited accuracy for steps of up to 64sec., at which point it too becomes unstable. The Adams–Moulton predictor-corrector method is seen to be completely unsuitable for this type of problem as steps of 8seconds cause the solution to become unstable. The third order process (B) solution is sufficiently accurate (agrees with a Kutta–Merson solution, using a constant step $h_r = 1$sec., throughout the computation, to five significant figures) and remains stable although steps of 2,048 seconds are used. For this particular problem, it can be seen that significantly larger steps are permitted if process (B) is used.

The third order process (B) with error estimate which is used in the second set of solutions is

$$x_{r+1} = x_r + (19/9)k_r - (43/18)l_r + (28/9)m_r - (11/6)n_r \tag{37}$$

$$x'_{r+1} = x_r + (10/3)k_r - l_r + (7/3)m_r - (11/3)n_r \tag{38}$$

$$k_r = h_r(I - h_rA(x_r))^{-1}\phi(x_r) \tag{39}$$

$$l_r = h_r(I - h_rA(x_r + k_r))^{-1}\phi(x_r + k_r) \tag{40}$$

$$m_r = h_r(I - h_rA(x_r + \tfrac{1}{2}k_r + \tfrac{1}{2}l_r))^{-1}\phi(x_r + \tfrac{1}{2}k_r + \tfrac{1}{2}l_r) \tag{41}$$

$$n_r = h_r(I - \tfrac{2}{3}h_rA(x_r))^{-1}\phi(x_r + (2/99)k_r + (95/99)l_r + (2/99)m_r) \tag{42}$$

$$n'_r = h_r(I - \tfrac{1}{2}h_rA(x_r))^{-1}\phi(x_r + (21/44)k_r + (19/44)l_r + (1/11)m_r). \tag{43}$$

The error estimate is

$$(\epsilon_j)_{r+1} = (29/8)((x_j)_{r+1} - (x'_j)_{r+1}). \tag{44}$$

In the above

$$(A_{ij}) = \frac{(\phi_i(x_r + \tfrac{1}{2}\gamma\Delta(x_j)_r) - \phi_i(x_r - \tfrac{1}{2}\gamma\Delta(x_j)_r))}{\gamma\Delta(x_j)_r} \tag{45}$$

where $$\gamma\Delta(x_j)_r = 5((x_j)_r - (x_j)_{r-1}). \tag{46}$$

Solutions of the distillation problem using the controlled process (B) and the controlled Kutta–Merson method have been computed (Haines, 1968). Both solutions were obtained using a pre-set error bound of 0·0005. In fact the error estimate of the third order process (B) ensured that this degree of accuracy was present in the process (B) solution. As expected over the first few intervals the Kutta–Merson scheme is more efficient using less computer time. However, the process (B) becomes increasingly more efficient as the integration progresses and after a total time of 10,000 seconds the steps are 500 times larger than those permitted in the Kutta–Merson method. At this point of the solution the ratio of computing times for the two methods is approximately 1 : 1. However, the equilibrium values are not attained until after 24 hours and not at 10,000 seconds. These results show that the process (B) is most valuable for obtaining accurate solutions over long ranges of integration, but that for short ranges of integration no significant advantage is gained over the Kutta–Merson scheme as the work per step is much greater for the process (B).

Conclusion

The stability conditions applied to the processes (B) are seen to be effective in practical problems and give the processes (B) a significant advantage over the other methods tested in this respect. Also the accuracy of the

third order process (B) is sufficiently good for most practical purposes and the error estimate (44) is quite adequate for ensuring some desired degree of accuracy during the integration. Any type of problem of the form (1) may be solved using processes (B) but systems involving many equations require a large amount of computer time per step. The processes (B) are most suited to the solution, either (1), of problems in which the matrix $A(x_r)$ is quite sparse (thus allowing the use of special matrix invertion techniques: the processes (B) perhaps benefiting more than other methods from these time saving techniques) and where the time constants have a wide spread in magnitude, or (2), of problems having only a few equations in (1) and where the time constants are as above. Distillation and heat exchanger problems are examples of the former and are well suited to the process (B) solution as the stability properties of the processes (B) are used to their full extent in such problems. The range of problems for which the processes (B) have real practical value would be extended if the amount of calculation per step were reduced: present research is being directed towards this objective.

Acknowledgement

I should like to express my appreciation for the valuable assistance of Professor H. H. Rosenbrock during the course of this work and in the preparation of this paper; and to Mr. A. J. Young and ICI Limited for financial support and assistance.

References

GEAR, C. W. (1967). *The numerical integration of stiff differential equations*, University of Illinois, Department of Computer Science Report 221.

HAINES, C. F. (1968). *Implicit processes for the numerical integration of systems of first order differential equations*, M.Sc. thesis, University of Manchester Institute of Science and Technology.

HUCKABA, C. E., and DANLEY, D. E. (1960). Calculational procedures for binary batch rectification, *A.I.Ch.E. Journal*, Vol. 6, p. 336.

MERSON, R. H. (1957). An operational method for the study of integration processes, *Proc. Symp. on Data Processing*, W.R.E., Salisbury, South Australia.

ROSENBROCK, H. H. (1963). General implicit processes for the numerical solution of differential equations, *Computer Journal*, Vol. 5, p. 329.

SUMNER, H. (1965). *A digital computer code for the solution of sets of first order differential equations*, A.E.E.W.R.-453, (A.E.E. Winfrith).

WIDLUND, O. B. (1967). A note on unconditionally stable linear multistep methods, *BIT*, Vol. 7, p. 65.

Book Review

Direct Analogue Computers, by V. Paschkis and F. L. Ryder, 1968; 400 pages. (*Interscience*, 160s.)

It is rather depressing to see the march of time progressively eroding one's fundamental tenets. This disillusionment must have happened to many of today's engineers. Perhaps the biggest disappointments of all can be awarded to the one-time tinkerers, like myself, who enjoyed many happy hours developing ingenious analogue computers for the solution of problems in the fields of network analysis, circuit theory, control systems, fluid flow, *etc.* One by one, the simple or complex devices are being, or are claimed to have been, replaced by digital machines, and the digital graphical display systems attack the fortress of the rearguard defence, the special intimacy of the operator and his analogue.

The title *Direct Analogue Computers* makes a distinction between two types of analogue computer which I find hard to appreciate. The demarcation line is claimed to be as follows: if two fields of phenomena are described by the same equations, then solution to problems in one field can be obtained by experiments in the other field. This leads to the direct analogue computer or simulator. For example, a resistor capacitor network can be the direct analogue of a heat conduction system. The other class of analogue computer or 'equation solver' is given by noting that certain circuit elements represent certain mathematical operations. The original system is described by finite difference equations, and these equations are represented by appropriate interconnections of circuit elements, and this, if I understand the conclusion, is a clearly distinct method of system analysis.

This book is a spirited plea for the use of a particular class of analogue machine in a wide class of problems in heat conduction, fluid flow and structures. The authors have a wealth of practical experience and this is shown by the many examples in the text, and the emphasis on practical difficulties such as scaling and effects which limit the overall accuracy. It is heartening to learn of the significant applications which have been and are being made by the electrical circuit analogue computers.

Nevertheless, a lot of the material seems to belong to the last generation of analogue computer technology, for good amplifiers do not drift a few millivolts per hour, and transistorised power supplies do not normally use thermionic power devices. As what seems a final fling for direct analogue computers, the authors devote their last chapter to the description of an unbuilt machine with an on-line digital machine connected to a beast with 10^5 relay contacts.

This volume is not likely to be of much appeal to the majority of professional computer users. It can be of use to engineers seriously considering the virtues of special purpose devices and contains information on the pitfalls as well as the advantages of this class of computer. I found it interesting and provocative.

J. L. DOUCE (Coventry)