The numerical solution of non-singular integral and integro-differential equations by iteration with Chebyshev series

By M. A. Wolfe*

The method of Clenshaw and Norton for the solution of ordinary differential equations is applied to certain integral and integrodifferential equations. Some equations considered by other authors are solved.

(Received July 1968)

1. Introduction

In this paper the numerical solution of the equations

$$y(x) = f(x) + \int_{-1}^{1} g(x, t\,;\, y(t))dt \tag{1.1}$$

$$y^{(1)}(x) = f(x, y(x)) + \int_{-1}^{1} g(x, t\,;\, y(t))dt\,;\; y(-1) = Y \tag{1.2}$$

$$y(x) = f(x) + \int_{-1}^{x} g(x, t\,;\, y(t))dt \tag{1.3}$$

and

$$y^{(1)}(x) = f(x, y(x)) + \int_{-1}^{x} g(x, t\,;\, y(t))dt\,;\; y(-1) = Y \tag{1.4}$$

on the interval $[-1, 1]$ is considered, where $y^{(1)}(x)$ denotes the first derivative of $y(x)$. It is assumed that $g(x, t\,;\, y(t))$ has no singularities in the domain $-1 \leqslant x, t \leqslant 1$.

Under well known conditions, (1.1) and (1.3) define operators which are contraction mappings in the Banach space of functions $y(x)$ (Pogorzelski 1966, Willett 1964, Saaty 1967). A concise statement of the corresponding conditions for (1.2) and (1.4) does not seem to be available in the literature ; however, a simple analysis shows that the procedures to be described for their solution are essentially equivalent to Picard iteration. In the sequel it is assumed that the conditions required for the operators defined by (1.1), (1.2), (1.3), and (1.4) to be contraction mappings, are satisfied.

If A is the operator defined by any one of (1.1), (1.2), (1.3) and (1.4), then the convergent sequence $\{y_i(x)\}$ is generated from

$$y_{i+1} = Ay_i \tag{1.5}$$

where for (1.1) and (1.3)

$$y_0 \equiv f(x) \tag{1.6}$$

and for (1.2) and (1.4)

$$y_0 \equiv Y. \tag{1.7}$$

As is well known, the above procedure can be applied at as many points on $[-1, 1]$ as required, but it is easy to use the method of Clenshaw and Norton (1963) to obtain in one iterative procedure a Chebyshev series representation for $y(x)$, which may be readily evaluated at as many points as desired on $[-1, 1]$.

The techniques for the construction, evaluation, differentiation, and integration of Chebyshev series are well known and the results required for this paper are given in Clenshaw and Norton (1963).

By means of a simple linear transformation it is easy to transform equations with arbitrary limits of integration to the limits exhibited in (1.1), (1.2), (1.3), and (1.4), and it is assumed that if this is done, the resulting equations may be solved iteratively according to (1.5).

2. The Fredholm equations

In this section the numerical solution of the Fredholm equations (1.1) and (1.2) is considered.

For (1.1) the iterative sequence $\{y_i(x)\}$ is defined by (1.6) and

$$y_{i+1}(x) = f(x) + \int_{-1}^{1} g(x, t\,;\, y_i(t))dt. \tag{2.1}$$

Suppose that a Chebyshev series representation is known for $y_i(x)$ so that neglecting truncation errors,

$$y_i(x) = \sideset{}{'}\sum_{j=0}^{n} a_{i,j}T_j(x) \tag{2.2}$$

where the prime denotes that the term corresponding to $j = 0$ is to be halved.

Suppose also that a Chebyshev series representation is known for $g(x, t\,;\, y_i(t))$, so that

$$g(x, t\,;\, y_i(t)) = \sideset{}{'}\sum_{k=0}^{m} b_{i,k}(x)T_k(t). \tag{2.3}$$

Then $$\int_{-1}^{1} g(x, t\,;\, y_i(t))dt = \sum_{k=1}^{m+1} B_{i,k}(x)[1-(-1)^k] \tag{2.4}$$

where

$$\left.\begin{aligned} B_{i,k}(x) &= (b_{i,k-1}(x) - b_{i,k+1}(x))/2k,\ (k=1\,..\,m-1) \\ B_{i,m}(x) &= b_{i,m-1}(x)/2m \\ B_{i,m+1}(x) &= b_{i,m}(x)\,/2(m+1). \end{aligned}\right\} \tag{2.5}$$

Then an estimate of $y_{i+1}(x)$ is given by

$$y_{i+1}(x) = f(x) + \sum_{k=1}^{m+1} B_{i,k}(x)[1-(-1)^k]. \tag{2.6}$$

If a Chebyshev series is found for $y_{i+1}(x)$ then $y_{i+2}(x)$ could be estimated and clearly the process could be indefinitely continued.

These results suggest the following method for the numerical solution of (1.1).

Assuming that the values of $y_i(x_l)$ $(l = 0, \ldots n)$ are given, where

$$x_l = \cos\left(\frac{\pi l}{n}\right) (l = 0 \ldots n) \tag{2.7}$$

* *Department of Applied Mathematics, University of St. Andrews, St. Andrews, Fife*

the values of $g(x_l, x_k ; y_i(x_k))$ $(l = 0 \ldots n ; k = 0 \ldots n)$ are computed and from these the $b_{i,k}(x_l)$ are computed as in Clenshaw and Norton (1963). From the $b_{i,k}(x_l)$, the $B_{i,k}(x_l)$ are computed using (2.5), and hence using (2.6) an estimate of the $y_{i+1}(x_l)$ is obtained. Using these values the coefficients $a_{i+1,j}$ of the truncated Chebyshev series for $y_{i+1}(x)$ are computed as in Clenshaw and Norton. The procedure is continued until convergence is reached, and is started by taking for $y_0(x)$ the function $f(x)$.

The method is easily extended to solve (1.2).

Taking the iterative scheme defined by (1.7) and

$$y^{(1)}_{i+1}(x) = f(x, y_i(x)) + \int_{-1}^{1} g(x, t ; y_i(t))dt \tag{2.8}$$

and neglecting truncation errors, an estimate of $y^{(1)}_{i+1}(x_l)$ is given by

$$y^{(1)}_{i+1}(x_l) = f(x_l, y_i(x_l)) + \sum_{k=0}^{m+1} B_{i,k}(x_l)[1 - (-1)^k](l = 0 \ldots n) \tag{2.9}$$

where it is assumed that $y_i(x)$ is given by (2.2) and the $B_{i,k}(x)$ are given by (2.5).

From (2.9), the coefficients $c_{i,l}$ are obtained from the $y^{(1)}_{i+1}(x_l)(l = 0 \ldots n)$ as in Clenshaw and Norton so that neglecting truncation errors,

$$y^{(1)}_{i+1}(x) = \sum_{l=0}^{n}{}' c_{i,l}T_l(x). \tag{2.10}$$

Integrating this series as in Clenshaw and Norton, a Chebyshev series for $y_{i+1}(x)$ is obtained, where the coefficients $a_{i+1,l}$ are given by

$$\left.\begin{aligned} a_{i+1,0} &= c_{i,l}/2 + 2K_i \\ a_{i+1,l} &= (c_{i,l-1} - c_{i,l+1})/2l\,(l = 1 \ldots n-1) \\ a_{i+1,n} &= c_{i,n-1}/2n \\ a_{i+1,n+1} &= c_{i,n}/2(n+1). \end{aligned}\right\} \tag{2.11}$$

In (2.11) K_i is a constant of integration whose value for each iteration is fixed by requiring that the value of $y_{i+1}(-1)$ obtained from the Chebyshev series is equal to Y. This method of solving (1.2) is just an extension of the method of Clenshaw and Norton (1963) for the numerical solution of nonlinear ordinary differential equations with boundary conditions and is in this sense equivalent to Picard iteration.

3. The Volterra equations

In this section the numerical solution of the Volterra equations (1.3) and (1.4) is considered.

For (1.3) the iterative sequence $\{y_i(x)\}$ is defined by (1.6) and

$$y_{i+1}(x) = f(x) + \int_{-1}^{x} g(x, t ; y_i(t))dt. \tag{3.1}$$

The method proposed for the numerical solution of (1.3) is similar to that proposed for (1.1) except that with

$$C_{k,l} = \cos(\pi kl/n)(k, l = 0, \ldots n) \tag{3.2}$$

the result

$$\int_{-1}^{x_l} g(x_l, t ; y_i(t))dt = \sum_{k=0}^{n} b_{i,k}(x_l)d_{k,l} \tag{3.3}$$

where

$$\left.\begin{aligned} d_{0,l} &= [C_{1,l} + 1]/2 \\ d_{1,l} &= [C_{2,l} - 1]/4 \\ d_{k,l} &= [\{C_{k+1,l} - (-1)^{k+1}\}/(k+1) \\ &\quad - \{C_{k-1,l} - (-1)^{k-1}\}/(k-1)]/2 \end{aligned}\right\} \tag{3.4}$$

is used, where the $b_{i,k}(x_l)$ are as in (2.3).

Then an estimate of the $y_{i+1}(x_l)$ is given by

$$y_{i+1}(x_l) = f(x_l) + \sum_{k=0}^{n} b_{i,k}(x_l)d_{k,l}(l = 0 \ldots n). \tag{3.5}$$

A Chebyshev series for $y_{i+1}(x)$ is obtained as in Clenshaw and Norton (1963) and the process repeated until convergence is reached.

This procedure is easily extended to solve (1.4) defining the sequence $\{y_i(x)\}$ by (1.7) and

$$y^{(1)}_{i+1}(x) = f(x, y_i(x)) + \int_{-1}^{x} g(x, t ; y_i(t))dt \tag{3.6}$$

whence an estimate of $y^{(1)}_{i+1}(x)$ is given by

$$y^{(1)}_{i+1}(x_l) = f(x_l, y_i(x_l)) + \sum_{k=0}^{n} b_{i,k}(x_l)d_{k,l}(l = 0 \ldots n). \tag{3.7}$$

A Chebyshev series for $y_{i+1}(x)$ with coefficients $a_{i+1,j}$ is then obtained from the values of $y^{(1)}_{i+1}(x_l)$ by integration as for (1.2).

4. Computational procedure

The simplest procedure in implementing the methods described in Sections 2 and 3 would be to take n in (2.2) to be the same for all values of i, and to take m in (2.3) equal to n. This is computationally convenient but is inefficient because less accuracy is needed in the early iterations than in the later ones ; also it is not evident *a priori* what value of n should be chosen to ensure adequate representation of $(y(x)$ and $g(x, t ; y(t))$. A more computationally economical procedure is not to iterate to convergence with constant n but to increase n by unity after each iteration. Starting with a low value of n the computational labour on the early iterations is reduced. Computational experiments have shown that no advantage is obtained by iterating to convergence before increasing n, and the Chebyshev series obtained on convergence can be iterated with fixed n to ensure maximum accuracy with the final value of n obtained. With m equal to n, this method is not only more efficient than that with fixed n but also automatically obtains an adequate value for n. An *a posteriori* estimate of the accuracy of the final series is obtained by examining the magnitudes of the last few coefficients.

A criterion for the convergence of the procedure which has been found always to give at least the accuracy required is that convergence is reached when

$$|y_i(x_l) - y_{i-1}(x_l)| \leqslant E|y_i(x_l)|(l = 0 \ldots n) \tag{4.1}$$

where E is the given relative error in $y(x)$.

5. Numerical examples

(i) *Fredholm integral equation*

Elliott (1963) has solved Love's equations and the Lichtenstein–Gerschgorin equation using Chebyshev series and these are convenient examples to illustrate the procedure of Section 2, especially as all three equations are amenable to iterative solution when Aitken acceleration is used.

In the notation of Section 2, Love's equations have

$$f(x) \equiv 1 \; ; \; g(x, t; y(t)) = \pi y(t)/[1 + (x - t)^2] \qquad (5.1)$$

and

$$f(x) \equiv 1 \; ; \; g(x, t; y(t)) = - \pi y(t)/[1 + (x - t)^2] \qquad (5.2)$$

respectively.

These equations are referred to as L.1 and L.2 in **Table 1.** The Lichtenstein–Gerschgorin equation has

$$\left.\begin{aligned} f(x) &= 2 \arctan [k \sin \pi x/\{k^2 (\cos \pi x \\ &\qquad + \cos^2 \pi x) + \sin^2 \pi x\}] \\ g(x, t \; ; (t)) &= ky(t)/\{(k^2 + 1) \\ &\qquad - (k^2 - 1) \cos \pi(x + t)\} \end{aligned}\right\} \quad (5.3)^*$$

This equation is referred to as L.G. in the Table. The table shows the results obtained for L.1, L.2, and L.G. using the procedure outlined in Section 2 with $m = n$ and n increasing by unity after each iteration. The procedure gives convergence according to (4.1) with the given value of E in n_i iterations. The number of times $g(x, t, ; y(t))$ is evaluated is n_g, the number of sets of Chebyshev coefficients computed is n_c, and the number of evaluations of $y(x)$ using its current Chebyshev series is n_y. The maximum absolute error allowed by the convergence criterion on $[-1, 1]$ is denoted by e. In most cases the actual absolute error obtained is less than e.

(ii) *Fredholm integrodifferential equation*

There does not seem to be a suitable example of this type of equation in the literature. Indeed the occurrence of nonlinear nonsingular Fredholm integrodifferential equations in applications is very rare. As an example to illustrate the use of the method, therefore, an equation has been constructed which is amenable to iterative solution, namely the equation for which, in the notation of (1.2)

$$\left.\begin{aligned} Y &= 1 \\ f(x, y(x)) &= y(x) - \{y(x)\}^2/1500 \\ \text{and} \quad g(x, t \; ; y(t)) &= e^{2(x-t)}\{y(t)\}^2/3000 \end{aligned}\right\} \quad (5.4)$$

The analytical solution is $e^{(x+1)}$. With E equal to $0{\cdot}1 \times 10^{-2}$ convergence according to (4.1) is obtained in 8 iterations, the actual maximum absolute error in $y(x)$ on $[-1, 1]$ obtained being $0{\cdot}2 \times 10^{-2}$; starting with n equal to 6, n_g is 924, n_c is 92, and n_y is 184.

* In this paper, $k = 1{\cdot}2$

(iii) *Volterra integral equation*

A number of equations are readily solved using the procedure of Section 3, in particular the equation considered by Laudet and Oules (1960) and by Day (1966), namely

$$y(x) = 1 - x + \int_0^x (xe^{t(x-2t)} + e^{-2t^2})[y(t)]^2 dt \qquad (5.5)$$

whose analytical solution is $y(x) = e^{x^2}$, and whose numerical solution is computed on [0, 1].

On transformation, (5.5) has, in the notation of (1.3),

$$\left.\begin{aligned} f(x) &= (1 - x)/2 \\ g(x, t \; ; y(t)) &= [(x+1)\exp\{(t+1)(x-2t-1)/4\}/2 \\ &\qquad + \exp\{-(t + 1)^2/2\}]\{y(t)\}^2 \end{aligned}\right\} \quad (5.6)$$

with analytical solution $y(x) = \exp\{(x + 1)^2/4\}$.

With n equal to 6, and E equal to $0{\cdot}1 \times 10^{-2}$, convergence was obtained in 10 iterations, the values of n_f, n_c, and n_y being 1405, 126, and 115 respectively, and the actual maximum absolute error in $y(x)$ of (5.5) on [0, 1] is $0{\cdot}6 \times 10^{-3}$.

(iv) *Volterra integrodifferential equation*

A convenient example to illustrate the procedure of Section 3 is the equation which has been solved by several authors, for example Pouzet (1960), Day (1967), Wolfe and Phillips (1968), namely

$$y^{(1)}(x) = 1 + 2x - y(x) + \int_0^x x(1 + 2x)e^{t(x-t)}y(t)dt \qquad (5.7)$$

with $y(0) = 1$.
The analytical solution of this equation is $y(x) = e^{x^2}$. On transformation (5.7) has, in the notation of (1.4),

$$\left.\begin{aligned} Y &= 1 \\ f(x, y(x)) &= (x + 2 - y(x))/2 \\ g(x, t \; ; y(t)) &= (x + 1)(x + 2) \\ &\qquad \exp\{(t + 1)(x - t)/4\}y(t)/8 \end{aligned}\right\} \quad (5.8)$$

with analytical solution $y(x) = \exp\{(x + 1)^2/4\}$.

With n equal to 4 and E equal to $0{\cdot}1 \times 10^{-3}$ convergence is obtained in 6 iterations, the values of n_f, n_c, and n_y being 355, 52, and 50 respectively. The actual maximum absolute error on [0, 1] of $y(x)$ in (5·7) is $0{\cdot}85 \times 10^{-5}$.

The procedure of this paper compares favourably with the higher accuracy predictor-corrector method proposed by Wolfe and Phillips (1968), and with the method of Day (1967).

Table 1

EQUATION	n	n_g	n_c	n_y	n_i	E	e
L.1	10	680	61	138	4	$0{\cdot}1 \times 10^{-2}$	$0{\cdot}2 \times 10^{-2}$
L.2	10	920	78	154	5	$0{\cdot}1 \times 10^{-4}$	$0{\cdot}8 \times 10^{-4}$
L.G.	10	1498	116	240	7	$0{\cdot}1 \times 10^{-2}$	$0{\cdot}1 \times 10^{-2}$

6. Discussion

From consideration of various examples which have been tried, and from those given in this paper (which do not necessarily yield the best results) it is concluded that equations of the general types given by (1.1), (1.2), (1.3), and (1.4) are amenable to solution using Chebyshev series in conjunction with the method of successive approximations, provided that the conditions on the functions f and g are satisfied which makes the operators corresponding to the equations contraction mappings, and provided that f and g are capable of representation by Chebyshev series.

It may be possible to reduce the number of iterations required for convergence in each case by the application of some technique such as Aitken acceleration which is known to succeed for (5.1), (5.2) and (5.3), but the technique used will in general depend upon the convergence properties of the sequence of unaccelerated iterates.

The solutions given in this paper can be improved upon in at least two ways—(i) the solution obtained could be regarded as the initial iterate in a new sequence in which n is kept fixed; (ii) the solution obtained could be regarded as the initial iterate in an iterative procedure of higher order of convergence. These suggestions have not been implemented since it is primarily the purpose of this paper to show that Chebyshev series can be used successfully for the iterative solution of (1.1), (1.2), (1.3), and (1.4); the best method for obtaining a rapidly convergent sequence of iterates will, where the method is applicable, depend in general on the particular equation whose solution is required.

Finally, it is clear that the procedures given in this paper for equations of the types (1.1), (1.2), and (1.3) and (1.4) could in principle be applied to equations of much more general type.

7. Acknowledgement

It is a pleasure to recall the most constructive and encouraging discussions in connection with the subject matter of this paper which were held with Mr. G. M. Phillips of the Mathematical Institute, North Haugh, St. Andrews.

8. Note added in proof

It has been brought to the attention of the author by Professor Elliott of Tasmania that the method of Clenshaw and Norton has been applied to the Fredholm and Volterra integral equations in a different way by Sag (1966). Sag also discusses the special form taken by the method when applied to linear integral equations and to some singular equations.

References

Clenshaw, C. W., and Norton, H. J. (1963). The Solution of Nonlinear Ordinary Differential Equations in Chebyshev Series, *Computer Journal*, Vol. 6, pp. 88–92.

Day, J. T. (1966). *A Starting Method for Solving Nonlinear Volterra Integral Equations*, MRC Report No. 631 A.D., Mathematics Research Centre, University of Wisconsin. This work has also appeared in *Mathematics of Computation*, Vol. 21, pp. 179–188.

Day, J. T. (1967). Note on the numerical solution of integrodifferential equations, *Computer Journal*, Vol. 9, pp. 394–395.

Elliott, D. (1963). A Chebyshev Series Method for the Numerical Solution of Fredholm Integral Equations, *Computer Journal*, Vol. 6, pp. 102–111.

Laudet, M., and Oules, H. (1960). Sur l'Intégration Numérique des Equations Intégrales du Type Volterra, *Symposium on the Numerical Treatment of Ordinary Differential Equations, Integral and Integrodifferential Equations*, Birkhäuser Verlag, 1960, pp. 117–121.

Pogorzelski, W. (1966). *Integral Equations and Their Applications*, Vol. 1, International Series of Monographs on Pure and Applied Mathematics. Pergamon Press, 1966.

Saaty, T. L. (1967). *Modern Nonlinear Equations*, McGraw-Hill Book Company.

Willett, D. (1963). Nonlinear Vector Integral Equations as Contraction Mappings, *Archive for Rational Mechanics and Analysis*, Vol. 15, pp. 79–86.

Wolfe, M. A., and Phillips, G. M. (1968). Some methods for the solution of non-singular Volterra integro-differential equations, *Computer Journal*, Vol. 11, pp. 334–336.

Sag, T. W. (1966). Numerical methods for the solution of integral equations, *Ph.D. Thesis submitted to the University of Manchester, January 1966 (Unpublished).*