A general procedure for evaluating the controllability of time delay feedback control systems

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A general algorithm for evaluating the normalised integral-square-error criterion of controllability is presented as a function of the parameters of a feedback control system. Pure time delays are incorporated as Padé polynomial approximations and a matrix solution of the integral using Gaussian elimination avoids the need for standard integral tables.

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1. Introduction

The theoretical determination of optimal control settings for feedback control systems subject to random load disturbances is a well-known problem in control engineering. Numerous forms of criteria for assessing controllability have been proposed in the past, many of which were tested by Graham and Lathrop (1953). However, only the integral of error squared criterion (I.S.E.) has found widespread application and been used successfully for many years. Griffin (1967) describes in detail the advantages of determining optimal three-term control settings using the I.S.E. criterion. The major advantages spring from the fact that if $H(s)$, the effective error-to-load transfer function, is expressed as a rational function of the complex pulsatance s, then the nth order definite integral

$$\text{I.S.E.} = I_n = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} |H(s)|^2 ds \qquad (1.1)$$

is a well-known mathematical form. Useful results can be obtained by evaluating this integral for different values of the system parameters.

For the general problem, $H(s)$ may be written as the ratio of two polynomials, i.e.

$$\left.\begin{aligned} &H(s) = c(s)/d(s) \\ \text{where} \quad & \\ &c(s) = \sum_{k=0}^{n-1} c_k s^k \text{ and } d(s) = \sum_{k=0}^{n} d_k s^k \end{aligned}\right\} \qquad (1.2)$$

Standard tables of expressions for calculating integrals of the form (1.1), when $H(s)$ is a rational function having poles in the left-hand plane, were originally presented by James, *et al.* (1947). In order to use these tables, the order of $d(s)$ must be made one higher than that of $c(s)$. The values of the coefficients c_k and d_k are then substituted into the standard expressions to give the value of I_n.

Using a new method suggested by Dr. A. C. Hall, Booton, *et al.* (1953) re-analysed the problem of evaluating (1.1), gave the general theory in matrix form and presented an extended list of standard tables (Newton, Gould and Kaiser, 1957). The use of these tables has been well-illustrated by Dadachanji (1967), who compared the performances of electronic and pneumatic controllers for various lengths of transmission lines. These lines constitute time delays and increasing their lengths tends to make the system less controllable. Consequently it is important to discover the optimal control parameters which give the best value of controllability.

Unfortunately, the use of standard tables is limited and cannot be recommended for $n > 6$. Dadachanji (1967) showed that expressions become prohibitively complex as the order increases and a different computer procedure is needed for each value of n. Furthermore, time delays in the system are incorporated in (1.1) by the use of polynomial approximations. The accuracy of these approximations increases with the order of the polynomial and consequently large orders are preferred. In any case, standard expressions for $n > 10$ are not available and the order of the integral (and hence the polynomial approximations) is necessarily restricted.

In order to overcome these problems, a new procedure is required which computes the integral coefficients for a control system of any order. The procedure should include polynomial approximations of any order and the integral should be evaluated without resorting to the standard tables. This paper concentrates on deriving expressions for the coefficients of $c(s)$ and $d(s)$ for a general feedback control system and shows how the matrix approach to the evaluation of (1.1) given by Booton, *et al.*, gives the required method for obtaining a general computer solution. The paper finishes with a description of a program which enables the normalised controllability to be plotted as a function of control parameters and typical curves are presented.

2. The feedback control system

Fig. 1 shows a feedback control system, where the subscripts C, P and L refer to controller, process and load respectively. We assume that the load disturbance may be described by a stochastic process which is both stationary and homogeneous, i.e. ergodic. Typical disturbances have a Gaussian probability distribution with the power spectrum

$$\Phi(\omega) = \frac{\Phi_D}{1 + \omega^2 T_D^2} \qquad (2.1)$$

where Φ_D is the constant power per unit bandwidth, ω the angular frequency and T_D the effective disturbance

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time constant. Hence, the power spectrum of the load disturbance is identical to that obtained by passing white noise through a band-limiting filter having the transfer function

$$G_D(s) = \frac{1}{1 + sT_D} \tag{2.2}$$

Let the process transfer function $G_p(s)$ be a linear combination of poles and a time delay, i.e.

$$G_p(s) = \frac{K_p e^{-s\tau_p}}{\prod_{\alpha=1}^{N} (1 + T_{p\alpha} s)} = \frac{K_p e^{-s\tau_p}}{X_p(s)} \tag{2.3}$$

where K_p is the gain, $T_{p\alpha}$ the process time constants and τ_p the pure time delay. Similarly, the load transfer function is expressed as

$$G_L(s) = \frac{K_L e^{-s\tau_L}}{\prod_{\beta=1}^{M} (1 + T_{L\beta} s)} = \frac{K_L e^{-s\tau_L}}{X_L(s)} \tag{2.4}$$

The three-term controller has the transfer function

$$G_c(s) = \frac{K}{s\tau_I}(\tau_I \tau_D s^2 + \tau_I s + 1) \tag{2.5}$$

where K is the gain, τ_I the integral time constant and τ_D the derivative time constant. Finally, a pure time delay

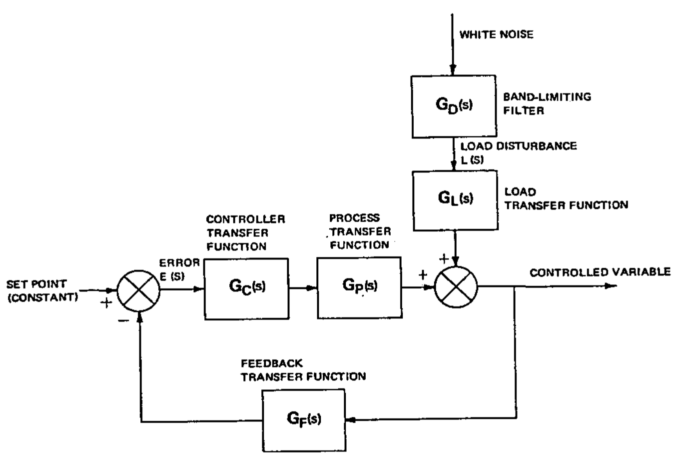

Fig. 1. General feedback control system

τ_F is included in the feedback path, representing the measurement time delay which characterises most practical applications in process control. This block has the transfer function

$$G_F(s) = e^{-s\tau_F} \tag{2.6}$$

Applying block diagram algebra around the loop

$$[E(s)G_c(s)G_p(s) + L(s)G_L(s)]\, G_F(s) = -E(s)$$

$$\therefore G(s) = \frac{E(s)}{L(s)} = -\frac{G_L(s)G_F(s)}{1 + G_c(s)G_p(s)G_F(s)} \tag{2.7}$$

This is the error-to-load transfer function. Substituting for G_L, G_p and G_F using (2.3), (2.4) and (2.6)

$$G(s) = \frac{K_L e^{-s(\tau_L + \tau_F)} X_p(s)}{[X_p(s) + K_p e^{-s(\tau_p + \tau_F)} G_c(s)] X_L(s)} \tag{2.8}$$

Combining $G(s)$ with the band-limiting filter gives the effective error-to-load transfer function

$$H(s) = G(s)G_D(s) \tag{2.9}$$

3. Calculation of the integral coefficients

The nth order integral

$$I_n = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} |G(s)G_D(s)|^2\, ds \tag{3.1}$$

must now be expressed as the ratio of two polynomials. Here, a difficulty arises since the time delays are not given in polynomial form. In the past, Maclaurin, Bessel, Tchebycheff and Padé polynomials have been tried as possible approximations. Following the accounts given by Al-Shaikh and Soliman (1965) and Riley and Walker (1968), we use the all-pass Padé approximation

$$e^{-s\tau} \simeq \frac{1 - a_1 s\tau + a_2 s^2\tau^2 - \ldots + a_r(-s)^r \tau^r}{1 + a_1 s\tau + a_2 s^2 \tau^2 + \ldots + a_r (s)^r \tau^r}$$

where

$$a_j = \frac{(2r - j)!\, r!}{(2r)!\, j!\, (r - j)!} \tag{3.2}$$

As seen in (2.8) we require the combinations

$$e^{-s(\tau_p + \tau_F)} \simeq \sum_{j=0}^{r} a_j (-s)^j (\tau_p + \tau_F)^j \Big/ \sum_{j=0}^{r} a_j (s)^j (\tau_p + \tau_F)^j = \frac{P_1}{P_2} \tag{3.3}$$

i.e.

$$P_1(s) = P_2(-s)$$

and

$$e^{-s(\tau_p + \tau_L)} \simeq \sum_{j=0}^{r} a_j (-s)^j (\tau_p + \tau_L)^j \Big/ \sum_{j=0}^{r} a_j (s)^j (\tau_p + \tau_L)^j = \frac{P_3}{P_4} \tag{3.4}$$

i.e.

$$P_3(s) = P_4(-s)$$

Substituting (2.5), (3.3) and (3.4) into (2.8) gives

$$G(s) = \frac{K_L P_2 P_3 X_p(s) s\tau_I}{[KK_p(1 + s\tau_I + s^2\tau_I\tau_D)P_1 + s\tau_I P_2 X_p(s)] P_4 X_L(s)} \tag{3.5}$$

and from (2.2), (3.1) and (3.5)

$$I_n = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \left| \frac{K_L X_p(s) P_2 P_3 s\tau_I}{[KK_p(1 + s\tau_I + s^2\tau_I\tau_D)P_1 + s\tau_I P_2 X_p(s)]\, P_4 X_L(s)(1 + sT_D)} \right|^2 ds \tag{3.6}$$

By multiplying out the factors of the numerator and denominator in the integrand, we obtain the integral in standard form and hence the coefficients c_k and d_k can be found.

4. Numerical evaluation of the definite integral

It has been shown by Booton, *et al.* (1953) that the evaluation of the integral depends on the solution of a set of linear algebraic equations. Briefly, the proof entails separating the $c(s)$ and $d(s)$ polynomials into the sum of two fractions $a(s)/d(s)$ and $b(s)/d(-s)$ and the integral becomes

$$I_n = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{a(s)}{d(s)} ds$$

A change of variable $s = -s'$ is then made and the final solution, by an application of the initial value theorem, is found to be

$$I_n = a_{n-1}/d_n \tag{4.1}$$

Therefore the integral evaluation requires the calculation of only one coefficient, a_{n-1}.

In practice, the numerator and denominator coefficients $c(s)$ and $d(s)$ are used to form the elements of two matrices $[C]$ and $[D]$. The $[C]$ matrix is set up by summing products of the c_k coefficients

$$[C] = \begin{bmatrix} C_0 \\ C_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ C_{n-2} \end{bmatrix} \tag{4.2}$$

where

$$\left.\begin{aligned} 2C_m &= \sum_{k=0}^{m} (-1)^k c_k c_{m-k} \text{ for } 0 \leqslant m \leqslant n-1 \\ &\text{and} \\ 2C_m &= \sum_{k=m-n+1}^{n-1} (-1)^k c_k c_{m-k} \text{ for } n \leqslant m \leqslant 2n-2 \end{aligned}\right\} \tag{4.3}$$

The $[D]$ matrix is formed directly from the d_k coefficients but differs for n odd or even.
For n odd

$$[D] = \begin{bmatrix} d_0 & 0 & 0 & \cdot & \cdot & 0 \\ d_2 & d_1 & d_0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{n-1} & d_{n-2} & d_{n-3} & \cdot & \cdot & d_0 \\ 0 & d_n & d_{n-1} & \cdot & \cdot & d_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & d_n & d_{n-1} \end{bmatrix}$$

and for n even

$$[D] = \begin{bmatrix} d_0 & 0 & 0 & \cdot & \cdot & 0 \\ d_2 & d_1 & d_0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_n & d_{n-1} & d_{n-2} & \cdot & \cdot & d_1 \\ 0 & 0 & d_n & \cdot & \cdot & d_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & d_n & d_{n-1} \end{bmatrix} \tag{4.4}$$

I_n is now obtained by solving the matrix equation

$$[D] \times [A] = [C] \tag{4.5}$$

where

$$[A] = \begin{bmatrix} a_0 \\ -a_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ (-1)^{n-1}a_{n-1} \end{bmatrix} \tag{4.6}$$

In fact it is only necessary to compute a_{n-1} and the integral is then given by (4.1).

5. Implementation

The method described above has been programmed in FORTRAN for the ICT 1909 computer in eight segments, Epton and Gough (1967). Initially, the MASTER segment reads the process parameters and other constant data before calling SUBROUTINE GENCOEFS. This routine computes the numerator and denominator coefficients of (1.2) by multiplying out the factors given in (3.6). In order to do this, GENCOEFS calls three further SUBROUTINES PADE, POLYADD and POLYMULT which generate the time delay approximations given by (3.3) and (3.4), and add and multiply polynomials in s. In addition PADE uses REAL FUNCTION IFACT which generates the factorials for the Padé coefficients using (3.2).

Control is then transferred to SUBROUTINE DEFINT which is a general procedure designed to evaluate the standard integral using relations (4.1) to (4.6). The solution is carried out by augmenting the $[D]$ matrix with the $[C]$ matrix and using Gaussian elimination. Round-off errors are minimised by choosing the largest element of the appropriate column as a pivot. Termination of the program occurs if $[D]$ is found to be singular, since this implies that unrealistic data has been used.

After one evaluation of the integral, the MASTER segment increments one of the control parameters and repeats the above procedure.

Finally, when the required number of evaluations have been computed (or when the value of the integral becomes negative or infinite, indicating the onset of instability) the answers are normalised and a GRAPHPLOT routine is called to present the results in graphical form.

Where possible, the program has been tested by comparison with results obtained using the standard tables. For $n = 30$ the program occupies 8000 store locations but this requirement may easily be reduced by altering the DIMENSION statement. For $n \leqslant 10$, typical running times for computing normalised curves are less than 60 seconds.

Fig. 2 shows a typical set of normalised controllability curves plotted as a function of controller gains for varying transmission line lengths. The curves show clearly the manner in which the controller gain must be reduced in order to avoid instability.

In conclusion, it may be said that the successful

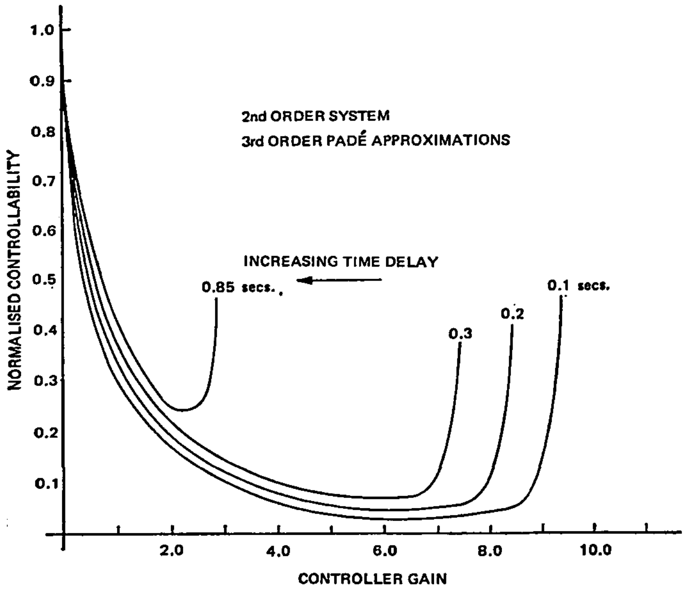

Fig. 2. Normalised controllability as a function of controller gain for various time delays

implementation of this method has shown many possibilities for the rapid design and optimal operation of feedback control schemes. The program development continues and its use in an on-line computer control scheme is being prepared. It is envisaged that an Argus 400 process control computer involving about 30 feedback loops will be used to monitor variations in process parameters. The computer will then interrupt the ICT 1909 computer via a high speed data link so that the higher-level computer can compute the optimal control settings.

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