Chebyshev solution of differential, integral and integro-differential equations

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This paper describes a new method for the numerical solution of linear integral equations of Fredholm type and of Volterra type. The method has been extended to the linear integro-differential equations and ordinary differential equations. It can also be applied to non-linear problems. In each case numerical examples are treated and the method compares quite favourably with other known methods.

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1. Introduction

El-gendi (1964) considered the use of finite Fourier series for approximating the integral

$$\int_{-1}^{x} f(t)dt, \ (-1 \leqslant x \leqslant 1) \tag{1}$$

at the points

$$x_m = -\cos\frac{m\pi}{N}, \ m = 0, 1, \ldots, N \tag{2}$$

We know that if the function values at the points (2) are known, the Chebyshev coefficients for this function can be directly computed. So the previous procedure gives a finite Chebyshev expansion for the integral.

The purpose of this paper is to develop another method based on the Clenshaw and Curtis quadrature scheme. This method is used to solve linear integral equations, integro-differential equations and ordinary differential equations. Like the previous method the new procedure will provide the solution in terms of a finite Chebyshev expansion.

2. The Clenshaw and Curtis quadrature scheme

We assume that the function $f(x)$ is defined and 'well-behaved' in $(-1 \leqslant x \leqslant 1)$. Clenshaw and Curtis (1960) give the following procedure for the numerical integration of $f(x)$, based on the approximation

$$f(x) = \sum_{r=0}^{N}{}'' a_r T_r(x) \tag{3}$$

where
$$a_r = \frac{2}{N}\sum_{j=0}^{N}{}'' f(x_j) T_r(x_j) \tag{4}$$

and
$$x_j = \cos\left(\frac{j\pi}{N}\right), \ j = 0, 1, \ldots, N. \tag{5}$$

Here $T_r(x)$ is the r-th Chebyshev polynomial. A summation symbol with double primes denotes a sum with first and last terms halved.

Formulae for both the definite and indefinite integral are derived from the relations

$$\int_{-1}^{x} T_n(t)dt = \begin{cases} \dfrac{T_{n+1}(x)}{2(n+1)} - \dfrac{T_{n-1}(x)}{2(n-1)} + \dfrac{(-1)^{n+1}}{n^2-1} & \text{if } n \geqslant 2 \\ \frac{1}{4}\{T_2(x) - 1\} & \text{if } n = 1 \\ T_1(x) + 1 & \text{if } n = 0 \end{cases} \tag{6}$$

The indefinite integral $\int_{-1}^{x} f(t)dt$ is approximated by

$$\int_{-1}^{x} f(t)dt = \sum_{j=0}^{N}{}'' a_j \int_{-1}^{x} T_j(t)dt = \sum_{r=0}^{N+1} C_r T_r(x) \tag{7}$$

where

$$\left.\begin{aligned} C_0 &= \sum_{\substack{j=0\\ j\neq 1}}^{N}{}'' \frac{(-1)^{j+1}a_j}{j^2-1} - \tfrac{1}{4}a_1 \\ C_k &= \frac{a_{k-1} - a_{k+1}}{2k}, \ k = 1, 2, \ldots, N-2 \\ C_{N-1} &= \frac{a_{N-2} - \frac{1}{2}a_N}{2(N-1)} \\ C_N &= \frac{a_{N-1}}{2N} \\ C_{N+1} &= \frac{\frac{1}{2}a_N}{2(N+1)} \end{aligned}\right\} \tag{8}$$

If we insert the expressions for C_r in (7) and use (4), then after certain arrangements we can define the elements of the matrix B defined in the relation

$$\left[\int_{-1}^{x} f(t)dt\right] = B[f] \tag{9}$$

where B is a square matrix of order $(N+1)$; the elements of the column matrix $[f]$ are given by $f_j = f\left(-\cos\frac{j\pi}{N}\right)$ $j = 0, 1, \ldots, N$. The right-hand side of (9) gives approximations for the integral at the points (2). In other words, we here evaluate the integral $\int_{-1}^{x} f(t)dt$, at the points

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$x_m = -\cos\frac{m\pi}{N}$, $m = 0, 1, \ldots, N$, rather than evaluate the Chebyshev coefficients of the integral. The two approaches are equivalent in the sense that if we know the values of the integral at x_m its Chebyshev coefficients can be directly evaluated from a formula similar to (4).

The main advantage of using the suggested approach is that for a certain value of N the elements of the matrix B can be evaluated once and for all. Economisation in computation will be achieved if, for example, the matrix B is stored, for different values of N, on cards. For $N = 4$ the elements of the matrix B are given in **Table 1**. We may notice that $\sum_{j=0}^{N} b_{ij} = \left(1 - \cos\frac{i\pi}{N}\right)$ $i = 0, 1, \ldots, N$; and this can be used for checking purposes.

3. Matrix approximations

From relation (9) we can deduce the following approximations which will be of importance in treating different problems.

(i)
$$\int_{-1}^{1} f(x)dx = \sum_{s=0}^{N} b_{Ns} f_s \tag{10}$$

where b_{Ns}, $s = 0, 1, \ldots, N$ are the elements of the last row of the matrix B. It is easy to verify that for N even we have

$$b_{Ns} = \frac{4}{N}\sum_{j=0}^{N/2}{}'' \frac{1}{1-4j^2}\cos\frac{2j\pi s}{N}, \quad s = 1, 2, \ldots, N-1$$

$$b_{N0} = b_{NN} = \frac{1}{N^2 - 1}.$$

(ii) For $-1 \leqslant x \leqslant 1$,

$$\left[\int_{-1}^{1} k(x, s)y(s)ds\right] = C[y] \tag{11}$$

where the right-hand side defines the operator $\int_{-1}^{1} k(x, s)$ $y(s)ds$ at the points (2). The elements of the matrix C are defined as

$$C_{ij} = b_{Nj}k_{ij},\ k_{ij} = k\left(-\cos\frac{i\pi}{N}, -\cos\frac{j\pi}{N}\right),$$
$$i, j = 0, 1, \ldots, N$$

and the elements of the column $[y]$ are $y_j = y\left(-\cos\frac{j\pi}{N}\right)$, $j = 0, 1, \ldots, N$.

Similarly

(iii)
$$\left[\int_{-1}^{x} k(x, s)y(s)ds\right] = D[y] \tag{12}$$

where $d_{ij} = b_{ij}k_{ij}$.

(iv)
$$\left[\int_{-1}^{x}\int_{-1}^{x} k(x, s)y(s)dsdx\right] = E[y] \tag{13}$$

where $E = BD$.

(v) As a special case of (13) we have

$$\left[\int_{-1}^{x}\int_{-1}^{x} y(s)dsdx\right] = F[y] \tag{14}$$

where $F = B^2$.

We have also

(vi)
$$\int_{-1}^{1}\int_{-1}^{x} f(t)dtdx = \sum_{i=0}^{N} s_i f_i \tag{15}$$

where s_i; $i = 0, 1, \ldots, N$ are the last row of the matrix F.

(vii)
$$\left[\int_{-1}^{x}\int_{-1}^{1} k(x, s)y(s)dsdx\right] = G[y] \tag{16}$$

where $G = BC$ and C is defined in (11).

Finally, when the range is $(0 \leqslant x \leqslant 1)$ we have the approximation

$$\left[\int_{0}^{x} f(x)dx\right] = A[f] \tag{17}$$

where $A = \frac{1}{2}B$ and the right-hand side defines the integral at the points

$$x_i = \tfrac{1}{2}\left(1 - \cos\frac{i\pi}{N}\right),\ i = 0, 1, \ldots, N \tag{18}$$

The elements of the column $[f]$ are also defined at these points. We notice that the interval has been normalised using the transformation $x = \frac{1}{2}(1 + t)$, $-1 \leqslant t \leqslant 1$. The Chebyshev expansion of the integral will be in this case in terms of

$$T_r^*(x) = T_r(2x - 1). \tag{19}$$

The operators from (11) to (16), can be similarly defined by replacing the matrix B by A and the points (2) by (18).

4. Numerical integration

To illustrate the approximations (9) and (17) we consider the following.

Table 1

The matrix B, $N = 4$

b_{ij}, ij	0	1	2	3	4
0	0	0	0	0	0
1	0·1194036	0·1900634	−0·0242641	0·0132867	−0·0055964
2	0·0333333	0·6202201	0·4000000	−0·0868867	0·0333333
3	0·0722631	0·5200466	0·8242641	0·3432699	−0·0527369
4	0·0666667	0·5333333	0·8000000	0·5333333	0·0666667

Example 1.

To evaluate $f(x)$ defined as

$$f(x) = e^{-1} + \int_{-1}^{x} e^t dt, \quad -1 \leqslant x \leqslant 1 \tag{20}$$

and $$g(x) = 1 + \int_{0}^{x} e^t dt, \quad 0 \leqslant x \leqslant 1. \tag{21}$$

If we use (9) we have the representation

$$[f] = [e^{-1}] + B[e^x] \tag*{(20)$'$}$$

In (20)$'$ the elements of $[e^{-1}]$ are all equal to e^{-1} and those of $[e^x]$ are equal to e^{x_i}, where x_i are defined in (2). The right-hand side of (20)$'$ gives $f(x_i)$ for $i = 0, 1, \ldots, N$. If we use (17) we have

$$[g] = [1] + A[e^x] \tag*{(21)$'$}$$

where the elements of the column [1] are all equal to one and those of $[e^x]$ are equal to e^{x_i} and of $[g]$ are $g(x_i)$ where x_i are defined in (18).

Table 2 shows the results if we take $N = 4$, so we can use the matrix defined in **Table 1**. The Chebyshev coefficients of $f(x)$ are computed so they can be compared with the exact values.

Table 2

Numerical example 1

i	CHEBYSHEV COEFFICIENTS OF $f(x)$		VALUES OF $g(x_i)$, $x_i = \frac{1}{2}(1 - \cos i\pi/4)$	
	THE METHOD	EXACT	THE METHOD	EXACT
0	2·5322	2·5321	1·0	1·0
1	1·1303	1·1303	1·157711	1·157713
2	0·2714	0·2715	1·648729	1·648721
3	0·0449	0·0443	2·347973	2·347975
4	0·0057	0·0055	2·718281	2·718282

5. Fredholm integral equations

We now consider the equation

$$y(x) - \lambda \int_{-1}^{1} k(x, s) y(s) ds = f(x) \tag{22}$$

where $-1 \leqslant x,\ s \leqslant 1$. This equation is known as a Fredholm integral equation of the second kind where λ is a given parameter, $f(x)$ is a given function and $y(x)$ is the function to be found. We shall consider non-singular integral equations, i.e. the kernel $k(x, s)$ is continuous and bounded.

Recalling relation (11), we may represent (22) in the form

$$(I - \lambda C)[y] = [f] \tag{23}$$

where I is the unit matrix and the elements of $[f]$ are $f_i = f(x_i)$ and x_i is defined in (2). The system (23) is to be solved for the solution at the points (2) and the Chebyshev coefficients can be evaluated directly if they are required. For the sake of comparison we now mention briefly another method which provides the solution as a Chebyshev expansion.

6. Elliott's method

For the numerical solution of (22), Elliott (1963) suggested for $y(x)$ the Chebyshev expansion,

$$y(x) = \tfrac{1}{2}a_0 + \sum_{r=1}^{N} a_r T_r(x). \tag{24}$$

In order to determine the $(N + 1)$ coefficients a_r, equation (22) is to be replaced by the $(N + 1)$ equations

$$y(x_i) - \lambda \int_{-1}^{1} k(x_i, s) y(s) ds = f(x_i) \tag{25}$$

for the $(N + 1)$ points $x_i = \cos\frac{i\pi}{N}$, $i = 0, 1, \ldots, N$.

For each x_i the kernel $k(x_i, s)$ is approximated by a polynomial of degree M in the form

$$k(x_i, s) = \sum_{r=0}^{M}{}'' \, b_r(x_i) T_r(s) \tag{26}$$

where $$b_r(x_i) = \frac{2}{M} \sum_{s=0}^{M}{}'' \, k\left(x_i, \cos\frac{\pi s}{M}\right) \cos\left(\frac{\pi rs}{M}\right)$$

The expansions (24) and (26) are then inserted in (25) and after some algebraic manipulations we have a system of $(N + 1)$ equations in the coefficients $a_0, a_1, \ldots, a_N$.

Compared with the suggested method the method of Elliott contains much work.

Example 2.

As an example we take Love's equation, also treated by Fox and Goodwin (1953), by Elliott (1963) and by El-gendi (1964). The equation is

$$y(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{y(s)}{1 + (x - s)^2} ds = 1. \tag{27}$$

Here the solution is symmetric about $x = 0$ and for $N = 8$ we get the results shown in **Table 3**. For the sake of comparison the Chebyshev coefficients are computed and the method of Elliott has been used taking $N = M = 8$. The results in the last column of this table are obtained by Elliott's method if we take $N = 20$ and $M = 16$. For this example the suggested method gives better accuracy than that of El-gendi (1964).

Table 3

Numerical example 2

THE CHEBYSHEV COEFFICIENTS IN $y(x) = \frac{1}{2}a_0 + \sum_{r=1}^{n} a_{2r} T_{2r}(x) (N = 2n)$			
a_{2r}	THE METHOD $N = 8$	METHOD OF ELLIOTT $N = M = 8$	METHOD OF ELLIOTT $N = 20,\ M = 16$
a_0	1·4151850	1·4151850	1·4151850
a_2	493851	493850	493851
a_4	— 10481	— 10483	— 10475
a_6	— 2310	— 2308	— 2327
a_8	195	195	200

7. Volterra integral equations

We now consider the equation of Volterra type

$$y(x) - \int_{-1}^{x} k(x, s)y(s)ds = f(x) \qquad (28)$$

where $-1 \leqslant x,\ s \leqslant 1$. Recalling approximation (12), the matrix representation of (28) is given by

$$(I - D)[y] = [f] \qquad (29)$$

The solution of this system gives approximations for $y(x_i)$, $i = 0, 1, \ldots, N$, from which the Chebyshev coefficients can be evaluated.

Example 3.

We consider Volterra's equation

$$y(x) = x + \int_0^x (s - x)(y(s)ds, 0 \leqslant x \leqslant 1 \qquad (30)$$

with exact solution $y = \sin x$. The same example has been solved by Jain and Sharma (1967) using Lobatto quadrature. The results obtained by the suggested method with $N = 4$ are shown in **Table 4**.

Table 4

Numerical example 3

$x_i = \frac{1}{2}(1 - \cos i\pi/4)$	THE METHOD	EXACT
x_0	0	0
x_1	0·145921	0·145924
x_2	0·479438	0·479426
x_3	0·753615	0·753621
x_4	0·841468	0·841471

8. Integro-differential equations

We now consider the class of integro-differential equations defined as

$$y'(x) + p(x)y(x) = q(x) + \int_{-1}^{x} k(x, t)y(t)dt \qquad (31)$$

with the initial condition

$$y(-1) = \eta \text{ and } -1 \leqslant x \leqslant 1 \qquad (32)$$

Intergrating (31) we get

$$y(x) + \int_{-1}^{x} p(t)y(t)dt = h(x) + \int_{-1}^{x}\int_{-1}^{x} k(x, t)y(t)dtdx \qquad (33)$$

where $$h(x) = \eta + \int_{-1}^{x} q(t)dt$$

and the initial condition is satisfied. This preliminary integration often has a very good effect on accuracy. Fox (1962) uses the method of 'prior integration' to get better accuracy with Lanczos' method.

Now, using the matrices B and E defined in (9) and (13) respectively we get

$$(I + S)[y] = [h] \qquad (34)$$

where $$S = BP - E$$

and P is a diagonal matrix whose diagonal is given by $p_i = p\left(-\cos\frac{i\pi}{N}\right)$, $i = 0, 1, \ldots, N$. The elements of $[h]$ are given by $h\left(-\cos\frac{i\pi}{N}\right)$ and if we solve the system (34) we get the solution of equation (31) under the condition (32) at the points $-\cos\frac{i\pi}{N}$, $i = 0, 1, \ldots, N$.

We now consider two examples which have also been solved by J. Day (1967).

Example 4.

Our first example is the integro-differential equation

$$y' = 1 - \int_0^x y(s)ds,\ y(0) = 0,\ 0 \leqslant x \leqslant 1 \qquad (35)$$

which has the exact solution $y(x) = \sin x$.

Integrating this equation we get

$$y(x) = x - \int_0^x\int_0^x y(s)dsdx. \qquad (36)$$

The matrix representation will be

$$(I + F)[y] = [x] \qquad (37)$$

where $[x]$ is a column matrix whose elements are $x_i = \frac{1}{2}\left(1 - \cos\frac{i\pi}{N}\right)$, $i = 0, 1, \ldots, N$ and $F = A^2$ where A is defined in (17). Taking $N = 4$ we have the results shown in **Table 5**.

Table 5

Numerical example 4

$x_i = \frac{1}{2}(1 - \cos i\pi/4)$	THE METHOD	EXACT	CHEBYSHEV COEFFICIENTS OF THE METHOD
x_0	0·0	0·0	0·8998513
x_1	0·1459244	0·1459237	0·4252206
x_2	0·4794224	0·4794255	− 293436
x_3	0·7536206	0·7536208	− 44855
x_4	0·8414703	0·8414710	1531

Example 5.

Our second example is given by

$$y'(x) + y(x) = 1 + 2x + \int_0^x x(1 + 2x)e^{s(x-s)}y(s)ds \qquad (38)$$

with $y(0) = 1$ and $0 \leqslant x \leqslant 1$. The exact solution is $y(x) = e^{x^2}$. The integrated form of this equation is

$$y(x) + \int_0^x y(s)ds = 1 + x + x^2 + \int_0^x\int_0^x x(1 + 2x)e^{s(x-s)}y(s)dsdx \qquad (39)$$

Replacing the integral operators in (39) by the appropriate matrices and taking $N = 4$ and 8 the errors found are shown in **Table 6**. By error we mean, error = |true value—approximate value|. Since equation (38) is inherently unstable, the errors are rather large near the end of the range and there has not been much gain in accuracy in going from $N = 4$ to $N = 8$.

Table 6

Numerical example 5

$x_i = \frac{1}{2}(1 - \cos i\pi/4)$	$N = 4$	$N = 8$
x_0	0	0
x_1	$2\cdot05 \times 10^{-4}$	$9\cdot9 \times 10^{-6}$
x_2	$3\cdot35 \times 10^{-3}$	$1\cdot09 \times 10^{-3}$
x_3	$1\cdot69 \times 10^{-2}$	$6\cdot31 \times 10^{-3}$
x_4	$2\cdot37 \times 10^{-2}$	$9\cdot98 \times 10^{-3}$

9. Other linear problems

The suggested method can be applied to other problems which we now mention briefly.

(i) For Fredholm integral equations, with Kernels which, though continuous, possess discontinuities in their first derivatives, i.e. for the problem

$$y(x) - \lambda \int_{-1}^{1} k(x, s)y(s)ds = f(x) \tag{40}$$

where $k(x, s) = \begin{cases} k_1(x, s) \text{ for } -1 \leqslant s \leqslant x \\ k_2(x, s) \text{ for } x \leqslant s \leqslant 1 \end{cases}$

equation (40) is replaced by

$$y(x) - \lambda \left\{ \int_{-1}^{x} k_1(x, s)y(s)ds + \left(\int_{-1}^{1} - \int_{-1}^{x} \right) k_2(x, s)y(s)ds \right\} = f(x). \tag{41}$$

We here apply the matrix representations (11) and (12).

(ii) Other types of integro-differential equations can also be considered. For instance, if we have

$$y'(x) + p(x)y = q(x) + \int_{-1}^{1} k(x, s)y(s)ds$$

with $$y(-1) = \eta, \ -1 \leqslant x \leqslant 1 \tag{42}$$

the integrated form of this equation will be

$$y(x) + \int_{-1}^{x} p(t)y(t)dt = h(x) + \int_{-1}^{x} \int_{-1}^{1} k(x, s)y(s)dsdx$$

$$h(x) = \eta + \int_{-1}^{x} q(t)dt \tag{43}$$

We can here use the approximations (9) and (15).

(iii) For the second-order differential equation

$$y''(x) + p(x)y = q(x), \ -1 \leqslant x \leqslant 1 \tag{44}$$

with the boundary conditions

$$y(-1) = A, \ y(+1) = B$$

the integrated form is given by

$$y(x) + \int_{-1}^{x} \int_{-1}^{x} p(t)y(t)dtdx - \left(\frac{1+x}{2}\right) \int_{-1}^{1} \int_{-1}^{x} p(t)y(t)dtdx = g(x) \tag{45}$$

where

$$g(x) = \tfrac{1}{2}(B + A) + \tfrac{1}{2}(B - A)x + \int_{-1}^{x} \int_{-1}^{x} q(t)dtdx - \left(\frac{1+x}{2}\right) \int_{-1}^{1} \int_{-1}^{x} q(t)dtdx$$

We can here apply the approximations (14) and (15).

10. Non-linear problems

The method can also be applied to non-linear problems after using a linearisation process. For instance, we consider the two-point boundary problem

$$\frac{dy_1}{dx} = \frac{y_1^2}{y_2}, \ \frac{dy_2}{dx} = \frac{y_2^2}{y_1} \tag{46}$$

with the conditions

$$y_1(0) = 1, \ y_1(1) = e \tag{47}$$

and which has the exact solution $y_1(x) = y_2(x) = e^x$. This example has also been treated by Sylvester and Meyer (1965). Now, if $z_1(x)$ and $z_2(x)$ are approximate or assumed values of the solution where $z_1(x)$ satisfies the conditions (47), the linearised equations may be written in the form

$$\left.\begin{aligned} \frac{d\epsilon_1}{dx} - \frac{2z_1}{z_2}\epsilon_1 + \frac{z_1^2}{z_2^2}\epsilon_2 &= -\frac{dz_1}{dx} + \frac{z_1^2}{z_2} \\ \frac{d\epsilon_2}{dx} + \frac{z_2^2}{z_1^2}\epsilon_1 - \frac{2z_2}{z_1}\epsilon_2 &= -\frac{dz_2}{dx} + \frac{z_2^2}{z_1} \end{aligned}\right\} \tag{48}$$

with $\epsilon_1(0) = \epsilon_1(1) = 0$

where $\epsilon_i(x) = y_i(x) - z_i(x), \ (i = 1, 2)$.

In our method we proceed to integrate (48) to get

$$\epsilon_1(x) - \int_0^x \frac{2z_1}{z_2}\epsilon_1 dx + \int_0^x \frac{z_1^2}{z_2^2}\epsilon_2 dx = 1 - z_1(x) + \int_0^x \frac{z_1^2}{z_2} dx$$

$$\epsilon_2(x) + \int_0^x \frac{z_2^2}{z_1^2}\epsilon_1 dx - \int_0^x \frac{2z_2}{z_1}\epsilon_2 dx - C = -z_2(x) + \int_0^x \frac{z_2^2}{z_1} dx$$

$$\epsilon_1(1) = 0, \ 0 \leqslant x \leqslant 1$$

where C is constant of integration.

If we use the matrix approximation (17), the problem is reduced to a system of linear equations to be solved by standard methods. A new trial solution is obtained by adding the correction $\epsilon_i(x)$ to the previous $z_i(x)$. The process is repeated until a certain convergence criterion, for example $\max_x |\epsilon_i(x)| < \delta$, where δ is a given small positive quantity, is satisfied. In this example, we have started with the initial guess $z_1(x) = (e - 1)x + 1, \ z_2(x) = 2$.

If we take $N = 4$ and $\delta = 10^{-5}$ we obtain, at the fourth iteration, the results shown in **Table 7**.

11. Conclusion

In the various problems considered in this paper the suggested method computes $y(x_i)$ rather than the Chebyshev coefficients. However, the Chebyshev coefficients can be simply computed and the solution is expressed in the form

$$y(x) = \sum_{r=0}^{N}{}' a_r T_r(x)$$

We may need to solve for two or more values of N, comparing coefficients and thereby deciding when to stop by inspection. For the different problems considered the computations will be reduced if the matrix B defined in (9) is available for different values of N.

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Table 7

Numerical example 6

$x_i = \frac{1}{2}(1 - \cos i\pi/N)$	$y_1(x_i)$	$y_2(x_i)$	EXACT VALUES e^{x_i}
x_0	1	1	1
x_1	1·157711	1·157711	1·157713
x_2	1·648731	1·648733	1·648721
x_3	2·347974	2·347979	2·347975
x_4	2·718282	2·718290	2·718282

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Notice to Members of The British Computer Society

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The recent notice about the circulation of *The Computer Journal* has given rise to some unfortunate misunderstandings which the Society is anxious to correct.

First, no one is to be deprived of his right to receive the *Journal*; second, the *Journal* is not to cease publication.

It is known that some members do not wish to receive the *Journal* and it is felt that the cost of printing and posting unwanted copies of the *Journal* should be saved and applied to the benefit of members in other ways.

It was recognised that the right to receive the *Journal* must continue but it was felt that those who really wanted it should be asked to make a decision and, if they decided that they needed it, should be asked to request it by returning the card.

In this way, The British Computer Society would be sure that the *Journal* goes only to those with a real need and use for it and that no money is wasted.

As a result of the unfortunate misunderstandings which have occurred the question is being reconsidered and further announcement will be included with the Annual Report.