Discussion and Correspondence

S.O.R. and membranes

By M. J. O'Carroll*

Some corrections and further comments are made to a paper by Lloyd and McCallion on the relation between the optimum S.O.R. parameter and the fundamental frequency of a membrane.

The description of the link between S.O.R. and membrane vibration as presented by Lloyd and McCallion (1968) is in parts misleading. In equation (5) and for the choice of Δt afterwards, $1/(4h^2)$ should correctly be $h^2/4$. Also in step 3 of the argument, μ_m does not represent a largest eigenvalue of L. In fact it represents minus the largest (negative) eigenvalue of the operator h^2L. That is also the smallest (positive) eigenvalue of $-h^2L$. To follow the notation of Wood (1967), as has been done elsewhere in Lloyd and McCallion's paper, this quantity should be μ_0 not μ_m. Their equation (6), in which η_m is the spectral radius of the Jacobi matrix, should have been

$$\eta_m \simeq 1 - 4\pi^2 f_1^2 \Delta t,$$

not $\eta_m = 1 - \pi^2 f_1^2$, where f_1 is the fundamental frequency of a membrane stretched on the given domain. As the finite difference net becomes finer, $\eta_m \to 1$ so the Jacobi, and the S.O.R., processes become slower.

Demonstration

There are two simple ways in which the relation between the Jacobi spectral radius, which determines the S.O.R. parameter, and the membrane frequency may be demonstrated.

(i) In the usual notation the Jacobi matrix is

$$J = L + U = I - A = I + \kappa D \tag{1}$$

where L is lower triangular and U upper triangular, κ is a positive constant and D is a matrix representing the discretised Laplace operator. For a square mesh of side h, $\kappa = h^2/4$. The eigenvalues of J and D are related by

$$\lambda_J = 1 + \kappa \lambda_D \tag{2}$$

and the spectral radius of J is

$$\rho_J = 1 - \kappa\alpha'$$
$$= 1 - \kappa 4\pi^2 f_1'^2.$$

i.e.

$$\rho_J \simeq 1 - \kappa 4\pi^2 f_1^2, \tag{3}$$

where α' is the smallest eigenvalue of $-D$ and f_1' is the fundamental frequency of a discrete system with operator D.

(ii) The Jacobi scheme is an iteration $U^{(n+1)} = JU^{(n)}$ where U denotes the error vector. Thus the errors are diminished, asymptotically, by a factor ρ_J for each iteration.

On the other hand the scheme is

$$U^{(n+1)} - U^{(n)} = (J - I)U^{(n)} = \kappa D U^{(n)},$$

which is the discrete representation of the equation $\partial U/\partial t = \nabla^2 U$, one iteration being given by a time increment $\Delta t = \kappa$. In the case of a square mesh of side h, again $\Delta t = h^2/4$. The standard separation of the time variable gives the eigenfunctions of the differential equation, the most slowly decaying having the factor $e^{-\alpha t}$, where α is the smallest eigenvalue of $-\nabla^2$ on the given domain with homogeneous boundary conditions. Now $\alpha = 4\pi^2 f_1^2$. Thus for one iteration the error is, asymptotically and approximately for a small mesh side, multiplied by a factor

$$e^{-\alpha\Delta t} \simeq 1 - \alpha\Delta t$$
$$\simeq 1 - \kappa 4\pi^2 f_1^2,$$

which then is approximately equal to ρ_J, agreeing with (3).

Elliptic equations

Linear and quasilinear elliptic equations may be treated by the S.O.R. method, by an iteration either of a Poisson-like equation or of a linear equation, the right-hand side or the coefficients being determined from the previous iterate. In the latter case the membrane analogy is not immediate.

For the Laplace equation approximated on a rectangular grid of sides h, k, the coefficient κ in (1) is $h^2k^2/2(h^2 + k^2)$. For the elliptic operator $a^2\partial^2/\partial x^2 + \partial^2/\partial y^2$, we have

$$\kappa = h^2k^2/2(h^2 + a^2k^2). \tag{4}$$

Under the transformation $x = ax'$, this operator becomes the Laplacian. Equation (2) now applies with λ_D representing an eigenvalue of the discretised elliptic operator in the x, y plane. That is approximately the Laplace eigenvalue in the x', y plane. By stretching the original domain in the x direction by a factor $1/a$ we obtain a domain for which the membrane frequency is related to ρ_J by equation (3).

For example, a compressible flow problem predominantly flowing in the x direction, with Mach number approximately M, may be calculated in terms of the velocity potential by this sort of S.O.R. with the optimum parameter estimated with the use of (3) and (4), where f_1 refers to the domain stretched x-wise by a factor $(1 - M^2)^{-1/2}$.

References

LLOYD, T., and MCCALLION, H. (1968). Bounds for the optimum over-relaxation factor for the S.O.R. solution of Laplace type equations over irregular regions, *Computer Journal*, Vol. 11, No. 3, p. 329.

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* *Department of Mathematics, Rugby College of Engineering Technology*

T. Lloyd and H. McCallion reply

We thank Mr. O'Carroll for bringing our attention to errors in our presentation of the paper Lloyd and McCallion (1968). We accept his criticism as valid but we would hasten to add that none of these errors affects the numerical results quoted, or in fact any of the paper except Sections 2(3) and 2(4). We solved the problem initially by the methods given in an appendix to Lloyd and McCallion (1967), and it was upon these results that figures and conclusions of the paper were based. The material to be presented did not, in our opinion, warrant the task of printing a full version of the analysis and so we decided to write it up as a short paper. We had, therefore, to base it upon a paper in the literature, our 1967 paper was not readily available at that time and so we decided to convert to the notation and terminology of Wood (1967), a paper which had come to our notice subsequent to the work of the paper. It was as a result of our mechanistic rather than thoughtful approach to this conversion of notation and terminology that the errors were introduced. We apologise for any inconvenience experienced.

T. LLOYD
H. MCCALLION

Reference

LLOYD, T., and MCCALLION, H. (1967). Recent Developments in Fluid Film Lubrication Theory. Lubrication and Wear: Fundamentals and Application to Design, *Proc. Instn. Mech. Engrs*, 1967–68, **182** (Pt. 3A), pp. 36–50.

Examinations by computer

By P. A. Ongley*

Although the literature on examinations by computer is not as yet voluminous, use of the multiple-choice paper associated with marking by computer is increasing. Computer examining involves two problems—marking by machine instead of by hand, and dictation of the pattern of the paper.

The computer will not only mark multiple-choice questionnaires and comment on the progress of each student but will also, by summarising class results question by question, provide information on the ability of individual questions to discriminate among more and less able students. It is claimed that the labour of marking is supplied by the machine rather than the examiner, and that the multiple-choice question can be marked more accurately than the essay. The case for the computer is made by Groves (1968), by Hinckley and Lagowski (1966) and by Smith, Schor and Donohue (1965).

With regard to accuracy, the essay-question examination can, with care, be made far more accurate than is generally supposed. Further, errors tend to cancel out, and at least in sessional and final examinations there are several papers. With regard to the labour of marking, the extra effort of setting is not inconsiderable; neither is the cost in computer time.

At this stage it may be wise to try to define examinations. An examination is an attempt to find out what knowledge the student has absorbed, how he can apply this, and what is his general understanding of his subject. The examination tells the student how he is progressing; it tells the teacher not only the student's grasp of his subject but also how successful the teaching has been and where any weaknesses lie. Feedback to both teacher and student is extremely important.

The multiple-choice question emphasises memory work. The handling of data, a very good exercise which cannot adequately be assessed by the multiple-choice question, approximates to what is done by a scientist or technologist when at work.

Insofar as feedback to the teacher is concerned, this is far greater with hand rather than computer marking.

The computer can mark accurately multiple-choice questions; it can be shown how individual students are progressing and in which areas any weaknesses lie. The imperfections lie in the restriction to multiple-choice questions and in impersonality. Some common drawbacks may, with care, be overcome; others are unavoidable. Some difficulties are avoidable, e.g.:

1. It can be difficult to obtain possible wrong answers that are not so different from the correct answer as to be obviously wrong, and so cut down the effective choice. Consider the formula of aluminium chloride is ACl, $AlCl_2$, $AlCl_3$, $AlBr_3$, AlCl, Al_2Cl_6. (Underline the right formula.) The choice ranges from exclusion of the ludicrous ACl and $AlBr_3$ to the sophisticated distinction between $AlCl_3$ and Al_2Cl_6.
2. The position of the right answer must vary from question to question. In particular it is known that students tend to shy clear of either the first or the last possibility. Obviously, unless the examiner is careful, the difficulty of choice may be very uneven and the odds much less than they seem.
3. Guesswork must be penalized. The correct score is approximately

$$S = R - \frac{W}{n-1}$$

where S = score
R = number right
W = number wrong
n = number of choices.

Some disadvantages, however, are unavoidable, e.g.:

1. In a foreign language examination the student cannot be asked to translate to or from the language. The only questions possible are of the types:
   (*a*) 'Mensa' means book, table, football, boy, school (underline the right word).
   (*b*) 'Give it to me' in French is 'Donnez (le, la, leur, les) moi' (underline the right word).
2. Not only must the answer be restricted to one of several possibilities; the correct form itself must be suggested. To recognize the correct answer among a number of

* *The University of Aston in Birmingham*