The runs up and down test

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The 'runs up and down' test has been shown to have certain advantages for testing a sequence of pseudo-random numbers for 'randomness'. The test is discussed in some detail: certain results, that have in the past been proved asymptotically, are derived exactly.

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1. Introduction

The need for long sequences of random numbers for use in Monte Carlo and simulation investigations is today usually met by the use of 'pseudo-random' sequences, i.e. sequences generated in a deterministic manner whose members display 'random-like' properties. In practice every sequence is not checked for such properties, but many sequences from a particular generator are checked before that generator is used.

The most frequently used generators are of the form

$$x_{i+1} = kx_i + c \, mod(m), \; i = 0, 1, 2, \ldots \qquad (1)$$

where i, k, c and x_0 are integers less than the integer m; the sequence $\{x_i/m|i = 1, 2, \ldots\}$ is then considered to be effectively a random sequence from the uniform distribution in (0, 1). A generator of this form is called multiplicative congruential if $c = 0$ and mixed congruential if $c \neq 0$. A suitable choice of k, c and m ensures that the members of the sequence are not repeated until a sufficient number have been generated, and also that the members pass certain statistical tests. Many papers discuss the choice of k, c and m (e.g. Hull and Dobell (1964), Kuehn (1961), Rotenberg (1960)). Downham and Roberts (1967) showed that if a sequence generated by this type of generator failed any of the tests usually applied, then almost always it failed the so-called 'runs up and down' test, which may therefore be said, in this sense, to be 'sensitive' for generators of the multiplicative and mixed congruential type.

Miller and Prentice (1968) have reopened investigations into additive congruential generators of the form

$$x_{i+1} = x_{i-s} + x_{i-t} \, mod(m), \quad i = max(s, t), max(s, t) + 1, \ldots \qquad (2)$$

where i, s, t, m and the x's are integers. Their results suggest that the 'runs up and down' test is also 'sensitive' for this type of generator.

The various expressions for the lengths of runs are derived in this paper for sequences, in which no two members are the same. For sequences generated by relation (1), the choice of k, c and m is intended to ensure that no number is repeated in a sequence of sufficient length. Thus, the expressions for run lengths are valid for all such sequences. Two members in a sequence generated by relation (2) may be the same, and, even worse, may be consecutive. If m is much larger than the required length of a sequence of random numbers, then few members are likely to be the same; only a small proportion of members taking the same value are likely to be consecutive. Thus, for sequences of random numbers to be used in simulation investigations, the expressions derived in this paper will usually be satisfactory. However, for the very long sequences that are sometimes required for Monte Carlo studies, these expressions are not necessarily valid, although they do hold for sequences generated by relation (2) provided that repeated numbers are omitted.

2. The 'runs up and down' test

Consider a sequence $\{x_i|i = 1, 2, \ldots, n\}$ generated by a pseudo-random number generator. A subsequence

$$x_{i-1}, x_i, \ldots, x_{i+r}, x_{i+r+1} \quad (2 \leqslant i \leqslant n - r - 1)$$

of $(r + 3)$ consecutive numbers is said to form an inside run 'up' of length r if

$$x_{i-1} > x_i < x_{i+1} < \ldots < x_{i+r} > x_{i+r+1}$$

Between the '$>$' signs there are r '$<$' signs. An end up 'up' of length r is defined either by

$$x_1 < x_2 < \ldots < x_{r+1} > x_{r+2} \quad (1 \leqslant {}_r < n - 2)$$

or

$$x_{n-r-1} > x_{n-r} < x_{n-r+1} < \ldots < x_n \; (1 \leqslant r \leqslant n - 2)$$

A complete run 'up' is defined by

$$x_1 < x_2 < \ldots < x_n,$$

and since there are $n - 1$ '$<$' signs such a run is of length $n - 1$. Runs 'down' are defined similarly.

For example, consider the sequence

$$S = \{22, 37, 81, 14, 42, 35, 20, 6, 19|n = 9\};$$

reading from left to right, there is a run up of length 2, a run down of length 1, a run up of length 1, a run down of length 3 and finally a run up of length 1.

The 'runs up and down' test is based on a comparison of the expected and actual numbers of runs of various lengths. Some relevant results, which have hitherto been derived only by asymptotic arguments, or approximately, can be derived easily by regarding runs as 'special arrangements' of monotone sequences. Consider a sequence

$$\{u_i|i = 1, 2, \ldots, m: u_i > u_j <=\geqslant i > j\} \qquad (3)$$

i.e. a sequence of m unequal quantities in ascending order of magnitude. A sequence

$$\{x_i|i = 1, 2, \ldots, m\},$$

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obtained by permuting the sequence (3), is said to be a 'special arrangement' of $u_1, u_2, \ldots, u_m$ if

$$x_1 > x_2 < x_3 < \ldots < x_{m-1} > x_m.$$

It is easily seen that the number of special arrangements:

with $(x_2 = u_1$ and $x_{m-1} = u_m) = (m-2)(m-3)$;
with $(x_1 = u_m$ and $x_2 = u_1) = (m-3)$;
with $(x_m = u_1$ and $x_{m-1} = u_m) = (m-3)$;
and
with $(x_m = u_1$ and $x_1 = u_m) = 1$.

Hence, the total number of special arrangements $= m^2 - 3m + 1$. If each arrangement is equally likely the probability of any particular arrangement is $1/m!$, so that the probability of a special arrangement is $(m^2 - 3m + 1)/m!$. In a pseudo-random sequence of length n there are $(n - r - 2)$ subsequences $x_{i-1}, x_i, \ldots, x_{n-r}$. Since an appropriate choice of constants will ensure a cycle long enough to rule out repeated numbers in a sequence of any required length, the elements of any such subsequences can be assumed all different, and the 'special arrangements' argument applies. Thus if

$$x_{i-1} > x_i < x_{i+1} < \ldots < x_{i+r} > x_{i+r+1} \quad (4)$$

the subsequence is a 'special arrangement' of $(r+3)$ numbers: but (4) is the condition defining an inside run up of length r. Hence the expected number of inside runs up of length r is equal to

$$\frac{((r+3)^2 - 3(r+3) + 1)(n-r-2)}{(r+3)!}$$
$$= \frac{(r^2+3r+1)(n-r-2)}{(r+3)!}, \quad r \leqslant n-2 \quad (5)$$

and, by symmetry, this expression is also the expected number of inside runs down.

By considering arrangements of the form

$$x_1 < x_2 < \ldots < x_{m-1} > x_m,$$

it can easily be verified that the expected number of end runs of length r, for $r < n-1$, is $4\dfrac{(r+1)}{(r+2)!}$.

As there are $n!$ possible arrangements of n numbers, the expected number of complete runs is $\dfrac{2}{n!}$.

Collecting these results, the expected number of runs of length r, $E(r)$, is given by,

$$E(r) = \begin{cases} 2 \times \dfrac{(r^2+3r+1)(n-r-2)}{(r+3)!} + 4 \times \dfrac{(r+1)}{(r+2)!} \\ = 2\left\{\dfrac{(r^2+3r+1)n - (r^3+3r^2-r-4)}{(r+3)!}\right\}, \\ \qquad r < n-1 \\ \dfrac{2}{n!} \qquad r = n-1. \end{cases} \quad (6)$$

The expected number of runs of length r or greater,

$$E'(r) = \sum_{\alpha=r}^{n-1} E(\alpha).$$

Now for any $(a_{-1}, a_0, a_1, a_2, a_3)$ such that $\sum_{i=-1}^{3} a_i = 0$ define

$$S_r(a_{-1}, a_0, a_1, a_2, a_3) = \sum_{\alpha=r}^{n-2} \sum_{i=-1}^{3} \frac{a_i}{(\alpha+i)!}, \quad r \geqslant 1$$
$$= \frac{a_{-1}}{(r-1)!} + \frac{a_{-1}+a_0}{r!} + \frac{a_{-1}+a_0+a_1}{(r+1)!}$$
$$+ \frac{a_{-1}+a_0+a_1+a_2}{(r+2)!} + \frac{a_0+a_1+a_2+a_3}{(n-2)!}$$
$$+ \frac{a_1+a_2+a_3}{(n-1)!} + \frac{a_2+a_3}{n!} + \frac{a_3}{(n+1)!} \quad (7)$$

Consider now $E(\alpha)$, for $\alpha < n-1$. The two terms in the numerator of (6) can be written as

$$\alpha^2 + 3\alpha + 1 = (\alpha+3)(\alpha+2) - 2(\alpha+3) + 1$$

and

$$\alpha^3 + 3\alpha^2 - \alpha - 4 = (\alpha+3)(\alpha+2)(\alpha+1) - 3(\alpha+3)(\alpha+2) + 3(\alpha+3) - 1,$$

whence

$$E'(r) = 2 \times \left\{nS_r(0, 0, 1, -2, 1) - S_r(0, 1, -3, 3, -1) + \frac{1}{n!}\right\}$$
$$= 2 \times \left[n \times \left\{\frac{1}{(r+1)!} - \frac{1}{(r+2)!} - \frac{1}{n!} + \frac{1}{(n+1)!}\right\} - \frac{1}{r!} + \frac{2}{(r+1)!} - \frac{1}{(r+2)!} + \frac{1}{(n-1)!} - \frac{2}{n!} + \frac{1}{(n+1)!} + \frac{1}{n!}\right]$$
$$= 2 \times \left\{\frac{(r+1)n - (r^2+r-1)}{(r+2)!}\right\} \quad 1 \leqslant r \leqslant n-1. \quad (8)$$

Herrman (1961) derived this result for large n.

3. Further discussion

Certain other asymptotic results have been found for sequences of numbers, but they could have been derived directly, in a manner similar to the derivation of $E'(r)$.

The expected number of runs $= E'(1) = \dfrac{2n-1}{3}$, a result derived independently by Bienayme (1874) and by Andre (1884).

$$\text{The mean length of a run} = \left(\frac{3}{2n-1}\right) \sum_{r=1}^{n-1} rE(r)$$
$$= \frac{6}{2n-1}\left[\sum_{r=1}^{n-2}\left\{\frac{nr(r^2+3r+1) - r(r^3+3r^2+r-4)}{(r+3)!}\right\} + \frac{n-1}{n!}\right],$$
$$= \frac{6}{2n-1}\left\{nS_1(0, 1, -3, 5, -3) - S_1(1, -3, 6, -7, 3) + \frac{n-1}{n!}\right\}$$

using (7) from Section 2,

$$= \frac{3(n-1)}{2n-1} \tag{9}$$

$$\left[\text{Alternatively, } \sum_{r=1}^{n-1} rE(r) = \sum_{r=1}^{n-1} E'(r)\right.$$

$$= 2\left\{nS_1(0, 0, 1, -1, 0) - S_1(0, 1, -2, 1, 0) + \frac{1}{n!}\right\}$$

$$\left. = n - 1.\right]$$

This is in agreement with Kermack and McKendrick (1938), who showed that for an infinite sequence the mean length of a run is 3/2. (They, in fact, defined the length of a run by the number of terms, so that they derived a mean length of 5/2.)

If one considers the totality of runs in the $n!$ different permutations of any sequence, then the proportion of runs of length $r < n - 1$, $P_r(n)$ say, is equal to

$$\frac{6}{2n-1}\left\{\frac{n(r^2+3r+1) - (r^3+3r^2-r-4)}{(r+3)!}\right\} \tag{10}$$

whence

$$\lambda_r = \lim_{n \to \infty} P_r(n) = \frac{3(r^2+3r+1)}{(r+3)!} \tag{11}$$

in agreement with Fisher (1926) and Kermack and McKendrick (1938). Levene and Wolfowitz (1944) warned against considering λ_p as 'the probability of a run of length r' and, in fact, showed that this phrase has no meaning.

To illustrate these results consider a sequence of length 4. As the inequality signs, rather than the sizes of the differences between consecutive numbers, are considered in proving these results, it is sufficient to consider the different arrangements of the integers $1, 2, 3, \ldots, n$. Let M_i = the total number of runs of length i, amongst the different possible arrangements of the integers $1, 2, \ldots, n$. With $n = 4$, $M_1 = 42$, $M_2 = 12$, $M_3 = 2$ and the total number of arrangements = 24.

$$\text{Thus, } E(1) = \frac{42}{24},\ E(2) = \frac{12}{24} \text{ and } E(3) = \frac{2}{24}.$$

$$P_1(4) = \frac{42}{56} \text{ and } P_2(4) = \frac{12}{56}.$$

The expected number of runs per arrangement

$$= \frac{M_1 + M_2 + M_3}{4!} = \frac{7}{3}.$$

The mean length of a run

$$= (42 + 2 \times 12 + 3 \times 2)/56 = \frac{9}{7}.$$

These results are consistent with the expression (6), (8), (9) and (10).

4. Statistical tests

A sequence of numbers is generated and $N_1, N_2, \ldots$ evaluated, where N_r is the number of runs of length r in the sequence. Levene and Wolfowitz (1944) have shown that for n large the statistic

$$\sum_{\alpha}(N_\alpha - E(\alpha))^2/E(\alpha)$$

is distributed as χ^2. Cochran (1954) suggested that such approximations are only satisfactory when 80% of the expected values are greater than 5. As α increases $E(\alpha)$ rapidly becomes less than 5, so that

$$\left(\sum_{\alpha=r}^{n-1} N_\alpha - E'(\alpha)\right)^2/E'(\alpha)$$

must be calculated. Hence,

$$\sum_{\alpha=1}^{r-1} \frac{(N_\alpha - E(\alpha))^2}{E(\alpha)} + \frac{\left(\sum_{\alpha=r}^{n-1} N_\alpha - E'(r)\right)^2}{E'(r)} \tag{12}$$

is distributed as χ^2 on $r - 1$ degrees of freedom, since there is one constraint—viz.

$$\sum_{\alpha} \alpha E(\alpha) = \sum_{\alpha} \alpha N_\alpha = n - 1.$$

For this test a sequence need be considered only as a series of inequality signs, e.g. the sequence 5, defined in Section 2, may be reduced to <<><>>><.

Consider a sequence of $n = 500$ numbers with $N_1 = 180$, $N_2 = 90$, $N_3 = 30$, $N_4 = 8$, $N_5 = 2$, $N_6 = 0$ and $N_7 = 1$. (As a check it can be easily verified that $\sum_i iN_i = 499$.)

From (6) $E(1) = 208{\cdot}4$, $E(2) = 91{\cdot}4$, $E(3) = 26{\cdot}3$, $E(4) = 5{\cdot}71$ and $E(5) = 1{\cdot}01$. Hence, in expression (12), r may be taken as 5 and from (8) $E'(5) = 1{\cdot}18$.

$$\chi_4^2 = (28{\cdot}4)^2/208{\cdot}4 + (1{\cdot}4)^2/91{\cdot}4 + (3{\cdot}7)^2/26{\cdot}3 + (2{\cdot}29)^2/5{\cdot}71 + (1{\cdot}82)^2/1{\cdot}18$$

$$= 8{\cdot}13.$$

Hence, there is not enough evidence to reject, at the 5% significance level, the hypothesis that such a sequence of runs could have occurred randomly. (In practice if a generator is unsatisfactory one usually gets many long runs, and a huge χ^2-value.)

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Book Review

Error Correcting Codes, Edited by Henry B. Mann, 1969, 228 pages. (John Wiley & Sons Ltd., Price 75s.)

The reader must first be warned that a more informative title for this book would have been '*Recent Advances* in Error Correcting Codes': being neither introductory nor encyclopaedic, it is not to be regarded as a manual or textbook. For example the first chapter is a Historical Survey by Mrs. F. J. MacWilliams which is delightful reading for anyone knowledgeable in the subject, but the lack of any bibliography makes it tantalising to the outsider; but of course it was addressed to experts, since the book is a record of the proceedings of a symposium organised by the Mathematics Research Centre of the U.S. Army at the University of Wisconsin. The computer user may be astonished that whereas he is accustomed to having one parity bit and 31 information bits in a word of 32 bits, a typical code discussed in this book provides 6 bits of information in each block of 32 bits; but the key words in the sponsorship of the symposium are *mathematics* and *U.S. army*. It is particularly for space vehicles that these elaborate codes have been designed, because the premium on weight and power is so high that almost any degree of complexity in the ground station can be tolerated if it allows some reduction in radio transmitter power on the space vehicle. Whether such tactics will ever be economic for terrestrial communication is another matter. But the mathematical interest is great and involves very varied topics. For example, the code discussed in Chapter 2 is presented graphically as a rectangular matrix of black and white squares: this is particularly appropriate since its construction is based on the Hadamard matrix which was first described in connection with the design of tessellated pavements.* The 'Fast Fourier Transform', which is a product of computer programming, has proved invaluable in the de-coding of certain types of code. The construction of error correcting codes may be based on combinatorial algebra or on topology. If some British mathematicians can be persuaded to read this book, and if as a result they become interested in the mathematical problems of error correcting codes, this may help to bring Britain forward in a subject in which we lag sadly behind the space-inspired Americans and Russians.

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*SYLVESTER, J. J., 'Tnoughts on inverse orthogonal matrices, simultaneous sign successions, and tessellated pavements in two or more colours, with applications to Newton's rule, ornamental tile-work and the theory or numbers.'
Phil.Mag. (ser.4), **34**, 461–475 (1867).