The stability of the Du Fort-Frankel method for the diffusion equation with boundary conditions involving space derivatives

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It is shown that the Du Fort-Frankel method is unstable for the diffusion equation, if the usual central difference approximations are made to linear boundary conditions involving first order space derivatives. This is shown to be true even when the corresponding differential equation is stable. A modified boundary condition is presented which is proved to be stable provided the differential equation is stable.

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1. Introduction

The explicit formula, due to Du Fort and Frankel (1953), for the numerical solution of second order parabolic differential equations, has in recent years been applied to the numerical solution of non-linear equations (e.g. Fromm (1966)), with various boundary conditions. The purpose of this paper is to re-examine the stability of the method for the diffusion equation with linear boundary conditions involving first derivatives. It is shown that, if central differences are used in approximations to such boundary conditions, instability will occur unless the conditions are modified in a manner consistent with the Du Fort-Frankel method. Although these results are only obtained for the diffusion equation in one space dimension, they may be extended to some problems in higher dimensions using direct products of matrices and Afriat's Theorem for eigenvalues of partitioned matrices.

2. The problem and its partial discretization

The problem we shall consider is that of solving the equation

$$\frac{\partial\phi(x, t)}{\partial t} = \frac{\partial^2\phi(x, t)}{\partial x^2} \tag{1}$$

in the region $0 \leqslant x \leqslant 1$, $t \geqslant 0$ with initial condition

$$\phi(x, 0) = g(x) \quad 0 \leqslant x \leqslant 1 \tag{2}$$

and boundary conditions

$$\alpha_0\phi(0, t) - \beta_0 \frac{\partial\phi(0, t)}{\partial x} = \gamma_0(t), \tag{3a}$$

$$\alpha_1\phi(1, t) + \beta_1 \frac{\partial\phi(1, t)}{\partial x} = \gamma_1(t), \tag{3b}$$

where $\gamma_0(t)$ and $\gamma_1(t)$ are continuous and bounded as $t \to \infty$. We suppose first that (1) is replaced by a partial discretization. We divide the region $0 \leqslant x \leqslant 1$ into a set of $(n + 1)$ mesh points $x_j = jh$, where $h = 1/n$ is the mesh length. We replace (1) by the system of ordinary differential equations

$$\frac{d\mathbf{\Phi}(t)}{dt} = \frac{1}{h^2}\mathbf{M\Phi}(t) + \mathbf{b}(t) \tag{4}$$

where $\mathbf{\Phi}(t)^T = [\phi_0(t), \phi_1(t), \ldots \phi_n(t)]$. $\phi_j(t)$ denotes the solution of these ordinary differential equations along the line $x = jh$ for $t \geqslant 0$. $\mathbf{M}$ is an $(n + 1) \times (n + 1)$ matrix derived from the boundary conditions and the usual three point difference approximation to $\partial^2\phi/\partial x^2$. The elements of the vector $\mathbf{b}(t)$ depend entirely on the known terms in the boundary conditions. The initial conditions are $\phi_j(0) = g(jh)$ for $j = 0, 1, 2, \ldots, n$.

3. Stability of the partial discretization

We consider first the stability of (4). If $\lambda_0, \lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{M}$, we say that equations of the form (4) are *stable*, if $\max_r \mathscr{R}(\lambda_r) < 0$, or, if $\max_r \mathscr{R}(\lambda_r) = 0$ and every λ_r, with $\mathscr{R}(\lambda_r) = 0$ is a simple zero of the minimal polynomial of $\mathbf{M}$ (i.e. the Jordan normal form of $\mathbf{M}$ contains only submatrices of order 1×1 corresponding to such eigenvalues). Otherwise we say that (4) is *unstable*. In general, a perturbation of the initial value of the solution vector of (4) will produce a uniformly bounded error for the stable case and an unbounded error for the unstable case.

This may be proved using the results for Jordan matrices established by Varga (1962) (particularly Lemma 8.1).

If (3*a*) and (3*b*) are replaced by

$$\alpha_0\phi_0(t) - \beta_0 \frac{\phi_1(t) - \phi_{-1}(t)}{2h} = \gamma_0(t) \qquad (5a)$$

and

$$\alpha_1\phi_n(t) + \beta_1 \frac{\phi_{n+1}(t) - \phi_{n-1}(t)}{2h} = \gamma_1(t), \qquad (5b)$$

the matrix $\boldsymbol{M}$ is given by

$$\boldsymbol{M} = \begin{bmatrix} -2\left(1+\frac{\mu_0}{n}\right) & 2 & & & & \\ 1 & -2 & 1 & & 0 & \\ & 1 & -2 & 1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & 0 & & \cdot & \cdot & \cdot \\ & & & 1 & -2 & 1 \\ & & & & 2 & -2\left(1+\frac{\mu_1}{n}\right) \end{bmatrix}$$

where $\mu_0 = \alpha_0/\beta_0$ and $\mu_1 = \alpha_1/\beta_1$. Taking a particular case of the results of Campbell and Keast (1968), we deduce that the system (4) is stable for $\mu_0\mu_1 + (\mu_0 + \mu_1) \geqslant 0$ and $(\mu_0 + \mu_1) \geqslant 0$. Campbell and Keast also show that if these conditions are satisfied, the solutions of the parabolic equation (1) are bounded.

4. The eigenvalues of M

In the next section we shall need more information regarding the eigenvalues of $\boldsymbol{M}$. If $\boldsymbol{x}$ and λ are an eigenvector and eigenvalue of $\boldsymbol{M}$ with

$$\boldsymbol{Mx} = \lambda\boldsymbol{x},$$

then the components x_j of $\boldsymbol{x}$ satisfy the difference equations

$$x_{j+1} - (2+\lambda)x_j + x_{j-1} = 0, \qquad (6)$$

with boundary conditions

$$x_{-1} = x_1 - 2\frac{\mu_0}{n}x_0 \qquad (7a)$$

$$x_{n+1} = x_{n-1} - 2\frac{\mu_1}{n}x_n. \qquad (7b)$$

The general solution of (6) is

$$x_j = A.\cos j\theta + B.\sin j\theta \qquad (8)$$

where

$$\lambda = -4.\sin^2\frac{\theta}{2} \qquad (9)$$

unless $\lambda = 0$ or -4 when a term of the form $j(\pm 1)^j$ must be included in (8)). The general solution (8) satisfies (7*a*) and (7*b*) if

$$A\frac{\mu_0}{n} - B.\sin\theta = 0$$

and

$$A\left(\cos(n+1)\theta - \cos(n-1)\theta + 2\frac{\mu_1}{n}.\cos n\theta\right)$$
$$+ B\left(\sin(n+1)\theta - \sin(n-1)\theta + 2\frac{\mu_1}{n}.\sin n\theta\right) = 0.$$

We obtain a non-zero solution if the determinant of the coefficients of A and B in the above equations is zero, that is, after some manipulation, if

$$f(\theta) \equiv \left(\cos 2\theta + 2\frac{\mu_0\mu_1}{n^2} - 1\right).\sin n\theta$$
$$+ 2\left(\frac{\mu_0+\mu_1}{n}\right)\sin\theta\cos n\theta = 0. \qquad (10)$$

The $(n+1)$ eigenvalues of M are thus given by (9) where θ is given by the $(n+1)$ roots (excluding one root $\theta = 0$ and one root $\theta = \pi$) of (10).

We show that $f(\theta)$ has at least $n-1$ real zeros (apart from $\theta = 0$ and π). For $r = 1, 2, \ldots, (n-1)$,

$$f\left(\frac{r\pi}{n}\right) = \frac{2(\mu_0+\mu_1)}{n}(-1)^r\sin\left(\frac{r\pi}{n}\right)$$

which alternates in sign and, therefore, there are at least $(n-2)$ real zeros, at least one being in each of the intervals $[r\pi/n, (r+1)\pi/n]$, $r = 1, 2, \ldots (n-2)$. Also, as $f(\pi) = 0$ and

$$f'(\pi) = 2(-1)^n(\mu_0\mu_1 - \mu_0 - \mu_1)/n, \qquad (11)$$

there is a further real root in $((n-1)\pi/n, \pi)$ if $\mu_0\mu_1 - \mu_0 - \mu_1$ and $\mu_0 + \mu_1$ are of opposite signs. If $\mu_0\mu_1 - \mu_0 - \mu_1 = 0$, then $f'(\pi) = 0$ and there is a root $\theta = \pi$, when $\lambda = -4$ is an eigenvalue with corresponding eigenvector components $x_j = (\mu_0 j - n)(-1)^j$.

We now seek roots of the form $\theta = \pi + i\psi$. Let

$$G(\psi) \equiv -i(-1)^n f(\pi + i\psi)$$
$$= \left(\operatorname{ch} 2\psi + 2\frac{\mu_0\mu_1}{n^2} - 1\right).\operatorname{sh} n\psi$$
$$-2\left(\frac{\mu_0+\mu_1}{n}\right)\operatorname{sh}\psi\operatorname{ch} n\psi.$$

$$G(\psi) \sim \tfrac{1}{4}e^{(n+2)\psi} \text{ as } \psi \to \infty$$
$$>0,$$
$$G(0) = 0$$

and

$$G'(0) = \frac{2}{n}(\mu_0\mu_1 - \mu_0 - \mu_1)$$

and thus there is at least one root of the form $\theta = \pi + i\psi$ if $\mu_0\mu_1 - \mu_0 - \mu_1 < 0$. So far, for $\mu_0\mu_1 - \mu_0 - \mu_1 < 0$ we have located at least n roots for $\mu_0 + \mu_1 > 0$ and $(n-1)$ roots for $\mu_0 + \mu_1 < 0$.

If $\mu_0\mu_1 - \mu_0 - \mu_1 > 0$ and $\mu_0 + \mu_1 > 0$, which implies $\mu_0 > 0$ and $\mu_1 > 0$, let

$$\psi^* = \operatorname{sh}^{-1}(\sqrt{\mu_0\mu_1}/n)$$

when

$$G(\psi^*) = 4.\frac{\mu_0\mu_1}{n^2}.\operatorname{sh} n\psi^*$$
$$-2\left(\frac{\mu_0+\mu_1}{n^2}\right)\sqrt{\mu_0\mu_1}.\operatorname{ch} n\psi^*$$
$$< \frac{2}{n^2}(2\mu_0\mu_1 - (\mu_0+\mu_1)\sqrt{\mu_0\mu_1}).\operatorname{ch} n\psi^*$$
$$= \frac{-2}{n^2}\sqrt{\mu_0\mu_1}(\sqrt{\mu_0} - \sqrt{\mu_1})^2.\operatorname{ch} n\psi^*$$
$$\leqslant 0.$$

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Thus in this case there are two roots of the form $\theta = \pi + i\psi$, as $G(0+) > 0$ and $G(\infty) > 0$.

We can similarly consider the behaviour of $f(\theta)$ at $\theta = 0$ and seek roots of the form $\theta = i\psi$. All $(n+1)$ roots can thus be located and a summary of results is shown in **Table 1.** If a root is of the form $\theta = i\psi$, the corresponding eigenvalue satisfies

$$\lambda = 4.\text{sh}^2 \frac{\psi}{2} > 0 \qquad (12)$$

and (4) is unstable. If a root is $\theta = \pi + i\psi$ then

$$\lambda = -4.\text{ch}^2 \frac{\psi}{2} < 0 \qquad (13)$$

which does not make (4) unstable. If θ is real the corresponding eigenvalue is clearly non-positive. There are no repeated zeros of the minimal polynomial of $\boldsymbol{M}$, as $\boldsymbol{M}$ is similar to a symmetric matrix as was shown by Campbell and Keast, and, therefore, Table 1 is consistent with their results.

For λ given by (12), the corresponding eigenvector has components of the form

$$x_j = C.\text{sh}\, j\psi$$

where C is arbitrary. For λ given by (13) the components are

$$x_j = C.\text{ch}\, j\psi$$

The eigenvectors are therefore hyperbolic in form and do not represent waves.

Table 1

Number of roots of (10) of the form $\theta = i\psi$ and $\theta = \pi + i\psi$. The remaining roots are real

$\mu_0 + \mu_1$	$\mu_0\mu_1 + \mu_0 + \mu_1$	$\mu_0\mu_1 - \mu_0 - \mu_1$	$\theta = i\psi$	$\theta = \pi + i\psi$
$\geqslant 0$	$\geqslant 0$	$\geqslant 0$	0	2
		<0	0	1
	<0	<0	1	1
<0	$\geqslant 0$	$\geqslant 0$	2	0
	<0	$\geqslant 0$	1	0
		<0	1	1

5. Stability of the Du Fort-Frankel method

For a time increment δt, the Du Fort-Frankel formula for integrating (4) is

$$\frac{1}{2.\delta t}(\boldsymbol{\Phi}_{s+1} - \boldsymbol{\Phi}_{s-1}) \qquad (14)$$
$$= \frac{1}{h^2}[(\boldsymbol{M} + 2\boldsymbol{I})\boldsymbol{\Phi}_s - \boldsymbol{\Phi}_{s+1} - \boldsymbol{\Phi}_{s-1}] + \boldsymbol{b}(s.\delta t)$$

where $\boldsymbol{\Phi}_s^T = [\phi_{0,s}, \phi_{1,s}, \ldots, \phi_{n,s}]$ and $\phi_{j,s}$ is the solution of the difference equation at the point $(jh, s.\delta t)$. We rewrite (14) in the single step form

$$\begin{bmatrix} \boldsymbol{\Phi}_{s+1} \\ \boldsymbol{\Phi}_s \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} \boldsymbol{\Phi}_s \\ \boldsymbol{\Phi}_{s-1} \end{bmatrix} + \begin{bmatrix} 2.\delta t.(1+2\alpha)^{-1}\boldsymbol{b}(s.\delta t) \\ \boldsymbol{O} \end{bmatrix} \qquad (15)$$

where $\boldsymbol{A}$ is the $2(n+1) \times 2(n+1)$ matrix

$$\boldsymbol{A} = \begin{bmatrix} 2(1+2\alpha)^{-1}\alpha(\boldsymbol{M}+2\boldsymbol{I}) & (1+2\alpha)^{-1}(1-2\alpha)\boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{O} \end{bmatrix} \qquad (16)$$

and $\alpha = \delta t/h^2 (>0)$. We shall say that the recurrence relation (15) is *stable*, if all eigenvalues of $\boldsymbol{A}$ lie in or on the unit circle and eigenvalues on the unit circle are simple zeros of the minimal polynomial of $\boldsymbol{A}$. Otherwise we shall say that (15) is *unstable*. As for the partial discretization, in general a perturbation of the solution vector in (15) will produce uniformly bounded errors for a stable relation and unbounded errors for an unstable relation. (See e.g. Varga (1962).)

Theorem 1

The Du Fort-Frankel method (14) is stable, if and only if the eigenvalues of $\boldsymbol{M}$ lie in or on the ellipse E in the complex plane, defined by

$$\left[\frac{\mathscr{R}(\lambda+2)}{2}\right]^2 + [\alpha.\mathscr{I}(\lambda)]^2 = 1, \qquad (17)$$

and any eigenvalues on E are only simple zeros of the minimal polynomial of $\boldsymbol{M}$.

Proof:

The eigenvalues of the matrix $\boldsymbol{A}$ of (16) are the $2(n+1)$ roots of the $(n+1)$ equations

$$(1+2\alpha)\nu^2 - 2\alpha(\lambda_r + 2)\nu - (1-2\alpha) = 0$$
$$r = 0, 1, 2 \ldots, (n+1), \qquad (18)$$

where λ_r, $r = 0, 1 \ldots, (n+1)$ are the eigenvalues of $\boldsymbol{M}$.

We consider the transformation

$$2\alpha(\lambda+2) = (1+2\alpha)\nu + \frac{(2\alpha-1)}{\nu} \qquad (19)$$

from the complex ν-plane to the λ-plane. The closed region between the circle $|\nu| = |2\alpha - 1|/(2\alpha+1)$ and $|\nu| = 1$ is mapped on to the closed interior of E (**Fig. 1**). The circle $|\nu| = \sqrt{[|2\alpha-1|/(2\alpha+1)]}$ is mapped into a cut of length $2\sqrt{|4\alpha^2 - 1|}/\alpha$ which is along the real axis a for $2\alpha - 1 \geqslant 0$ and along the line $\mathscr{R}(\lambda) = -2$ for $2\alpha - 1 < 0$.

The inverse correspondence is such that to each point in or on E in the λ-plane, there corresponds two points in or on the unit circle in the ν-plane. To a point outside E, there corresponds one point inside $|\nu| = |1-2\alpha|/(1+2\alpha)$ and one point outside the unit circle.

The final part of the theorem is proved by considering the eigenvectors of $\boldsymbol{M}$ and $\boldsymbol{A}$. If an eigenvalue $\tilde{\lambda}$ is on E and is of multiplicity p, there are m corresponding Jordan submatrices in the Jordan normal form of $\boldsymbol{M}$, if the space spanned by the eigenvectors corresponding to $\tilde{\lambda}$ is of dimension m. The two roots of (18) for $\lambda_r = \tilde{\lambda}$ are distinct and, therefore, $\boldsymbol{A}$ has a corresponding eigenvalue $\tilde{\nu}$ on the unit circle and of multiplicity p. As the eigenvectors of $\boldsymbol{A}$ are of the form $\begin{bmatrix} \nu\boldsymbol{x} \\ \boldsymbol{x} \end{bmatrix}$, where $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{M}$, the eigenvectors of $\boldsymbol{A}$ corresponding to $\tilde{\nu}$ span a space of dimension m. Thus there are m

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Jordan submatrices corresponding to $\tilde{\nu}$ in the Jordan normal form of $\boldsymbol{A}$. Hence all the Jordan submatrices corresponding to $\tilde{\nu}$ are 1×1 (i.e. $\tilde{\nu}$ is a simple zero of the minimal polynomial) if and only if $m = p$.

6. The effect of boundary conditions on the Du Fort-Frankel method

For the boundary conditions (5*a*) and (5*b*), we have shown that either $\boldsymbol{M}$ has a positive eigenvalue when the system (4) is unstable, or $\boldsymbol{M}$ has an eigenvalue of the form $-4.\mathrm{ch}^2 \tfrac{1}{2}\psi$ where ψ is real. In both cases, $\boldsymbol{M}$ has an eigenvalue outside E and we conclude that the Du Fort-Frankel method is unstable, even when (4) is stable. This instability may be considered as due to the effect of the 'hyperbolic' form of the corresponding eigenvectors.

Du Fort and Frankel show that their method is stable if the boundary conditions, applied to the solution of (6), are of the form

$$x_1 - x_0 = c_0 x_0 \qquad (20a)$$

$$x_{n-1} - x_n = c_1 x_n \qquad (20b)$$

where c_0 and c_1 are positive constants. Such conditions may be obtained by replacing (3*a*), for example, by either

$$\alpha_0\phi_0(t) - \beta_0\left(\frac{\phi_1(t) - \phi_0(t)}{h}\right) = \gamma_0(t) \qquad (21)$$

or

$$\frac{\alpha_0}{2}(\phi_{-1/2}(t) + \phi_{1/2}(t)) - \frac{\beta_0}{h}(\phi_{1/2}(t) - \phi_{-1/2}(t)) = \gamma_0(t). \quad \ldots\ldots (22)$$

For the latter condition, the grid is taken at points $(i + \tfrac{1}{2})h$ so that the boundaries bisect a mesh interval. In replacing (3*a*) by (21), the truncation error introduced is $O(h)$ and, as $c_i = \alpha_i h/\beta_i$, $i = 0, 1$, we require $\alpha_i/\beta_i > 0$ if c_0 and c_1 are to be positive. This restriction may however be relaxed by completing a stability analysis for (20*a*) and (20*b*), similar to that of Sections 3 and 4. In view of its large truncation error we shall discard (21).

The truncation error for (19) is $O(h^2)$ and to ensure that c_0 and c_1 are positive, we require $\alpha_i h/(\beta_i - \alpha_i h/2) > 0$ but again this condition can be relaxed. Unless we have $h < 2|\beta_i/\alpha_i|$ for $i = 0$ and 1, (22) will not be satisfactory. For example, if $h = 2\beta_0/\alpha_0$, (19) reduces to the Dirichlet condition $\phi_{-1/2} = \gamma_0(t)/\alpha_0$. As this restriction on h may be severe, we shall consider instead a modification of conditions (5*a*) and (5*b*).

7. Modified boundary condition

We shall assume throughout this section that $\mu_0\mu_1 + \mu_0 + \mu_1 \geqslant 0$ and $\mu_0 + \mu_1 \geqslant 0$ when (4) is stable.

We investigate the use of the difference approximation

$$\frac{\alpha_0}{2}(\phi_{0,\,s+1} + \phi_{0,\,s-1}) - \frac{\beta_0}{2h}(\phi_{1,\,s} - \phi_{-1,\,s}) = \gamma_0(s.\delta t) \quad \ldots\ldots (23a)$$

$$\frac{\alpha_1}{2}(\phi_{n,s+1} + \phi_{n,s-1}) + \frac{\beta_1}{2h}(\phi_{n+1,s} - \phi_{n-1,s}) = \gamma_1(s.\delta t) \quad \ldots\ldots (23b)$$

with the Du Fort-Frankel method. The truncation error for these boundary conditions is $0(\delta t^2) + 0(h^2)$. The resulting method may be written in the form

$$\begin{bmatrix}\boldsymbol{\Phi}_{s+1}\\ \boldsymbol{\Phi}_s\end{bmatrix} = \boldsymbol{B}\begin{bmatrix}\boldsymbol{\Phi}_s\\ \boldsymbol{\Phi}_{s-1}\end{bmatrix} + \boldsymbol{c}_s, \qquad (24)$$

where $\boldsymbol{B}$ is the $2(n+1) \times 2(n+1)$ matrix

$$\boldsymbol{B} = \begin{bmatrix}\boldsymbol{N} & \boldsymbol{D}\\ \boldsymbol{I} & \boldsymbol{O}\end{bmatrix},$$

$\boldsymbol{N}$ is the $(n+1) \times (n+1)$ tridiagonal matrix

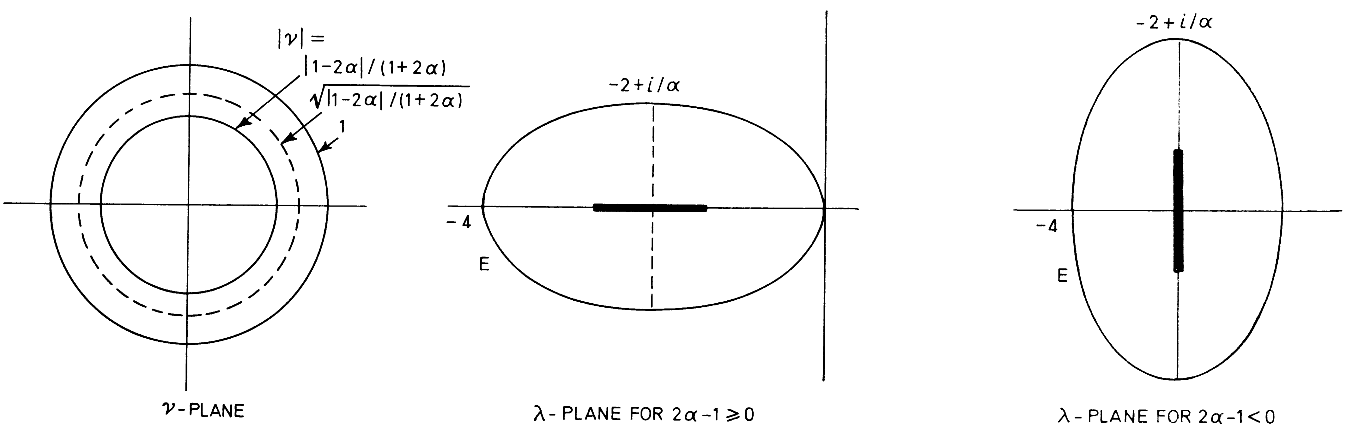

Fig. 1. The transformation (19)

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$$
N = \begin{bmatrix}
0 & \dfrac{4\alpha}{1 + 2\alpha\left(1 + \dfrac{\mu_0}{n}\right)} & 0 & & & \\
\left(\dfrac{2a}{1 + 2\alpha}\right) & 0 & \left(\dfrac{2\alpha}{1 + 2\alpha}\right) & & & \\
& \cdot & \cdot & \cdot & & \\
& & \cdot & \cdot & \cdot & \\
& & \cdot & \cdot & \cdot & \\
& & \left(\dfrac{2\alpha}{1 + 2\alpha}\right) & 0 & \left(\dfrac{2\alpha}{1 + 2\alpha}\right) \\
& 0 & & \dfrac{4\alpha}{1 + 2\alpha\left(1 + \dfrac{\mu_1}{n}\right)} & 0
\end{bmatrix}
$$

and D is the $(n + 1) \times (n + 1)$ diagonal matrix

$$
D = \begin{bmatrix}
\dfrac{1 - 2\alpha\left(1 + \dfrac{\mu_0}{n}\right)}{1 + 2\alpha\left(1 + \dfrac{\mu_0}{n}\right)} & & & & 0 \\
& \dfrac{1 - 2\alpha}{1 + 2\alpha} & & & \\
& & \cdot & & \\
& & \cdot & & \\
& & \cdot & \dfrac{1 - 2\alpha}{1 + 2\alpha} & \\
0 & & & & \dfrac{1 - 2\alpha\left(1 + \dfrac{\mu_1}{n}\right)}{1 + 2\alpha\left(1 + \dfrac{\mu_1}{n}\right)}
\end{bmatrix}.
$$

c_s is a suitable vector. All elements of N, D and c_s are finite as $\mu_0 \geqslant -1$ and $\mu_1 \geqslant -1$.

If ν is an eigenvalue of B with corresponding partitioned eigenvector $\begin{bmatrix} y \\ x \end{bmatrix}$, then clearly,

$$y = \nu x$$

and, if $y^T = [y_0 y_1 \ldots y_n]$ and $x^T = [x_0 x_1 \ldots x_n]$,

$$\left(\frac{2\alpha}{1 + 2\alpha}\right) y_{i-1} + \left(\frac{2\alpha}{1 + 2\alpha}\right) y_{i-1} + \left(\frac{1 - 2\alpha}{1 + 2\alpha}\right) x_i = \nu y_i$$

$$i = 1, 2, \ldots, (n - 1).$$

Thus

$$2\alpha\nu x_{i-1} + [1 - 2\alpha - \nu^2(1 + 2\alpha)]x_i + 2\alpha\nu x_{i+1} = 0$$

$$i = 1, 2, \ldots, (n - 1). \qquad (25)$$

For the first and last elements we obtain,

$$4\alpha\nu x_1 + \left\{1 - 2\alpha\left(1 + \frac{\mu_0}{n}\right) - \nu^2\left[1 + 2\alpha\left(1 + \frac{\mu_0}{n}\right)\right]\right\} x_0 = 0 \qquad (26a)$$

and

$$4\alpha\nu x_{n-1} + \left\{1 - 2\alpha\left(1 + \frac{\mu_1}{n}\right) - \nu^2\left[1 + 2\alpha\left(1 + \frac{\mu_1}{n}\right)\right]\right\} x_n = 0. \qquad (26b)$$

For $\nu \neq 0$ we can rewrite (25) as

$$2\alpha x_{i-1} + \left[\frac{1 - 2\alpha}{\nu} - \nu(1 + 2\alpha)\right] x_i + 2\alpha x_{i+1} = 0$$

$$\ldots\ldots \qquad (27)$$

and on substituting (25) in (26*a*) and (26*b*), we obtain

$$x_{-1} = x_1 - \frac{\mu_0}{n}\left(\nu + \frac{1}{\nu}\right) x_0 \qquad (28a)$$

$$x_{n+1} = x_{n-1} - \frac{\mu_1}{n}\left(\nu + \frac{1}{\nu}\right) x_n. \qquad (28b)$$

Now for $\mu_0 \geqslant 0$, $\mu_1 \geqslant 0$, $1 - 2\alpha > 0$, $1 - 2\alpha(1 + \mu_0/n) \geqslant 0$ and $1 - 2\alpha(1 + \mu_1/n) \geqslant 0$, the Gershgorin discs of B are

$$|z| \leqslant \frac{1 + 2\alpha(1 - \mu_j/n)}{1 + 2\alpha(1 + \mu_j/n)} \quad j = 0 \text{ and } 1$$

$$|z| \leqslant 1.$$

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Hence by the extension to Gershgorin's Theorem for irreducible matrices (see e.g. Varga (1962)), all the eigenvalues of $\boldsymbol{B}$ lie inside the unit circle. The eigenvalues of $\boldsymbol{B}$ are continuous functions of α, μ_0 and μ_1 and we show that $\boldsymbol{B}$ cannot have an eigenvalue on the unit circle unless $\mu_0\mu_1 + \mu_0 + \mu_1 = 0$. We thus conclude that all eigenvalues must lie inside the unit circle if $\mu_0\mu_1 + \mu_0 + \mu_1 > 0$ and $\mu_0 + \mu_1 > 0$.

If ν is an eigenvalue of $\boldsymbol{B}$ and it lies on the unit circle, $(\nu + 1/\nu)$ is real and the difference equation (27), subject to boundary conditions (28*a*) and (28*b*), is the same as (6) subject to (7*a*) and (7*b*), with μ_i replaced by $\mu_i(\nu + 1/\nu)/2$ for $i = 0$ and 1 and λ related to ν by (19). In Section 3, it was shown that for any real μ_0 and μ_1, all solutions of (6) must yield real values of λ and thus in this case λ must be real. Hence from (19) and **Fig. 1,** we can see that either $\lambda = 0$ and $\nu = +1$, or $\lambda = -4$ and $\nu = -1$. $\lambda = 0$ gives a non-trivial solution of (6), subject to (7*a*) and (7*b*), if and only if

$$\mu_0\mu_1 + (\mu_0 + \mu_1) = 0, \qquad (29)$$

which for (25) subject to (26*a*) and (26*b*) becomes

$$\frac{1}{2}\left(\nu + \frac{1}{\nu}\right)\cdot\left(\mu_0\mu_1\cdot\frac{1}{2}\left(\nu + \frac{1}{\nu}\right) + \mu_0 + \mu_1\right) = 0$$

For $\nu = 1$ this reduces to (29). Similarly $\lambda = -4$ is a solution of (6) subject to (7*a*) and (7*b*), if and only if

$$\mu_0\mu_1 - (\mu_0 + \mu_1) = 0$$

and, for (25) subject to (26*a*) and (26*b*) with $\nu = -1$, this again reduces to (29). Thus we have shown for all positive values of α, that unless (29) is satisfied, $\boldsymbol{B}$ cannot have an eigenvalue on the unit circle. If (29) is satisfied, $\boldsymbol{B}$ will have simple eigenvalues at $\nu = \pm 1$. We conclude that the Du Fort-Frankel method using (23*a*) and (23*b*) is stable when (1) is stable.

As in the main Du Fort-Frankel approximation, a term of the form $(\delta t)^2 . \delta^2\phi/\delta t^2$ has effectively been added to the boundary conditions. This has altered the difference equations sufficiently to ensure that the 'hyperbolic' eigenvector is damped.

8. Numerical example

We consider as an example the case $\alpha_0 = \beta_0 = \alpha_1 = \beta_1 = 1$, $\gamma_0(t) = 0$, $\gamma_1(t) = 2e^{-t}.\cos 1\cdot 0$ when the exact solution of (1) is

$$\psi(x, t) = e^{-t}(\cos x + \sin x).$$

Results for $n = 19$ and $\alpha = 0\cdot 25$ are presented in **Table 2** for the Du Fort-Frankel method using (5*a*) and (5*b*) (with $\phi_j(s.\delta t)$ replaced by $\phi_{j,\,s}$) and (23*a*) and (23*b*). E_s is the maximum absolute error, i.e.

$$E_s = \max_j |\phi_{j,\,s} - \phi(jh, s.\delta t)|$$

Table 2

Numerical results for problem of Section 8

s	BOUNDARY CONDITIONS	
	(5*a*) and (5*b*) $E_s \times 10^6$	(23*a*) and (23*b*) $E_s \times 10^6$
500	137	137
1,000	142	141
2,000	111	107
3,000	88	68
4,000	131	41
5,000	426	23
6,000	1,798	13
7,000	7,907	7
8,000	34,960	4

The results are consistent with the theory presented in this paper.

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