An algorithm for solving nonlinear programming problems subject to nonlinear inequality constraints

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An algorithm of the penalty function class which solves linear or non-linear optimisation problems subject to equality and/or inequality constraints is described. The 'penalty term' consists of exponential summands like exp $[T\cdot g(x)]$ where $T < 0$ and $g(x) \geqslant 0$ is a constraint. Computational experience is discussed. Convergence is proved and is typically first-order. The algorithm has found considerable application in its ability to distinguish readily between feasible and non-feasible (i.e. no-domain) problems.

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Many algorithms have been proposed to solve nonlinear programming problems (Zoutendijk, 1960; Fiacco and McCormick, 1963; Rosen, 1961). A recent review of some such methods has been published by Fiacco (1967). We present here an original variant of a method referred to as a 'sequential unconstrained minimisation technique' (SUMT) by some writers. It is a member of a class of methods using the concept of penalty functions. Development and applications of this technique have been actively pursued since mid-1966 with the result that at present, 'production computer decks' are being used routinely on nonlinear problems involving up to seventeen state variables subject to sixty-seven inequality constraints.

The ability of the algorithm to solve nonlinear programming problems is not unique. What is often important in our applications is the ability of the present method to sense a non-feasible problem in an unambiguous manner. A non-feasible problem is one for which no point of the space of independent variables can satisfy all constraints simultaneously. Henceforth, non-feasible problems will be referred to as 'no-domain problems'. Necessary and sufficient conditions characterising the no-domain case are not known at the present time. Three partial characterisations are available however and are proved in the appropriate section.

Numerical results are given, Lagrange multipliers for the equivalent equality constrained problem are identified and order of convergence is computed.

Description of the algorithm

No distinction is made between scalar and vector quantities since it is always clear from the context what is meant.

The general problem to be solved is Problem G:

$$\text{minimize } f(x);\ x \in E^N$$
$$\text{subject to } g_i(x) \geqslant 0;\ i = 1, \ldots, m.$$

Equality constraints are approximated by two opposing inequality constraints. That is, the equality $h(x) = 0$ is approximated by requiring

$$-\epsilon \leqslant h(x) \leqslant \epsilon$$

where $\epsilon > 0$ is an acceptable tolerance. Henceforth, no further discussion of equality constraints will be made.

The method of solution is as follows. Define an auxiliary function, $F_n(x)$ by

$$F_n(x) = f(x) + \sum_{i=1}^{m} \exp\,[T_{in}\cdot g_i(x)]$$

where the sequences $\{T_{in}\}$; $i = 1, \ldots, m$ each have the properties $0 > T_{in} > T_{i,n+1}$ and $\lim_{n\to\infty} T_{in} = -\infty$ for $i = 1, \ldots, m$.

If Problem G has a solution and if f and all g_i satisfy certain reasonable hypotheses, each $F_n(x)$ will have a unique minimum at x_n^* where

$$\min_x F_n(x) = F_n(x_n^*)$$

and $g_i(x_n^*) \geqslant 0$; $i = 1, \ldots, m$, for n sufficiently large.

It is shown in the next section that

$$\lim_{n\to\infty} [\min F_n(x)] = \inf_x f(x)$$

over the domain of x which satisfy all $g_i(x) \geqslant 0$. That is, the sequence of minima $\{F_n(x_n^*)\}$ converges to the solution of Problem G.

As a practical matter, x_1^* is found first and serves as the starting point for the search for x_2^* and so on.

It is instructive to examine the problem: $f(x) = x$, $g(x) = x - 1 \geqslant 0$, $T_n = -n$. It can be readily verified that

$$x_n^* = 1 + (\log n)/n$$

and $\quad F_n(x_n^*) = x_n^* + 1/n \quad$ for all n.

The algorithm can be applied equally well to find the maximum of $f(x)$ subject to $g_i(x) \geqslant 0$; $i = 1, \ldots, m$ by defining the auxiliary function to be

$$F_n(x) = -f(x) + \sum_{i=1}^{m} \exp [T_{in} \cdot g_i(x)].$$

Convergence of the algorithm

The proof of convergence of the algorithm follows the format used by Fiacco and McCormick (1963). We first define

$$D_i = \{x | g_i(x) \geqslant 0\};\ i = 1, \ldots, m.$$

The set $D = \bigcap_{i=1}^{m} D_i$ is called the domain of the problem—sometimes called the set of feasible points. To avoid trivial cases, it will be assumed that each $g_i(x)$ is a meaningful function in the sense that D_i is non-empty. We assume that the following hypotheses underlie all subsequent claims.

1. D° is the interior of D and $D^\circ \neq \phi$
2. f and all g_i are in $C^{(1)}$
3. the set $S_R = \{x | f(x) \geqslant R\} \cap D$ is bounded for every R
4. $\inf_D f(x) = f_0 \geqslant 0$.

Certain inequalities become easier to handle if $f_0 \geqslant 0$. If $f_0 < 0$ but bounded, there is no loss in adding a positive constant to it.

Let $\Sigma(n, x)$ denote $\sum_{i=1}^{m} \exp [T_{in} . g_i(x)]$.

Lemma 1:

(i) $F_n(x)$ is bounded below in D for each n and
(ii) one local minimum of $F_n(x)$ exists at $x_n^* \in D^\circ$ for each sufficiently large n.

Proof:

Claim (i) follows from the obvious fact that

$$\Sigma(n, x) > 0 \quad \text{for all } n \text{ and } x \text{ so that}$$
$$0 \leqslant f_0 = \inf_D f(x) \leqslant f(x) + \Sigma(n, x) = F_n(x).$$

To prove (ii), choose $x_0 \in D^\circ$ so that $f(x_0)$ is sufficiently close to f_0. This may be done by choosing η so that $0 < \eta < 1$ and x_0 so that $0 < f(x_0) - f_0 < \eta$. By the divergent property of the sequences $\{T_{in}\}$; $i = 1, \ldots, m$ there is an n_0 such that for any $n > n_0,$ $F_n(x_0) < 1 + f_0$.

Define a set S by

$$S(x_0, n) = \{x | F_n(x) \leqslant F_n(x_0)\} \cap D \text{ for } n > n_0.$$

It is then easy to show that

- S is closed
- $S \neq \phi$ (since $x_0 \in S$)
- S contains no boundary points so $S \subset D^\circ$
- if $y \in D - S$, $F_n(y) > F_n(x_0)$.

To finish the proof of (ii), note that $S(x_{0,}\ n)$ is closed and bounded by virtue of hypothesis (3). By hypothesis (2), F_n is continuous on S, a compact set, so F_n has a minimum at some point $x_n^* \in S(x_0,n)$ for each $n > n_0$. Since $F_n(y) > F_n(x_0) \geqslant F_n(x_n^*)$ on $D - S$, x_n^* is a local minimum of F_n in D°.

This result is stated in slightly different terms in

Theorem 1:

Subject to hypotheses (1) through (4), for sufficiently large n, there exists at least one local minimum of $F_n(x)$ (at x_n^*). For any such minimum, $x_n^* \in D^\circ$ and $\nabla F_n(x_n^*) = 0$.

Proof:

Lemma 1 proves everything except $\nabla F_n(x_n^*) = 0$ which follows as a necessary condition for an interior minimum.

In order to prove convergence of $\{F_n(x_n^*)\}$ to f_0, an additional hypothesis is required:

5. The minimum of $F_n(x)$ is unique.

Lemma 2 is required before proving the final results.

Lemma 2:

$F_{n+1}(x_{n+1}^*) \leqslant F_n(x_n^*)$ for n sufficiently large.

Proof:

Choose n so large that x_n^* and x_{n+1}^* are both in D°. Then

$$\Sigma(n + 1, x_n^*) \leqslant \Sigma(n, x_n^*)$$

and

$$F_n(x_n^*) = f(x_n^*) + \Sigma(n, x_n^*) \geqslant f(x_n^*) + \Sigma(n + 1, x_n^*).$$

Since a unique minimum exists by hypothesis (5),

$$f(x_n^*) + \Sigma(n + 1, x_n^*) \geqslant f(x_{n+1}^*) + \Sigma(n + 1, x_{n+1}^*) = F_{n+1}(x_{n+1}^*)$$

and hence $F_n(x_n^*) \geqslant F_{n+1}(x_{n+1}^*)$.

Theorem 2, proving convergence, follows.

Theorem 2:

Hypotheses (1) through (5) imply that

$$\lim_{n \to \infty} \{\min F_n(x)\} = f_0.$$

Proof:

For any $\epsilon > 0$, choose $x^* \in D^\circ$ so that $f(x^*) < f_0 + \epsilon/2$. Next choose n_0 so that the following hold for every $n > n_0$:

- $\underset{i}{\text{maximum}} \{\exp [T_{in} \cdot g_i(x^*)]\} < \epsilon/2m$
- minimum $F_n(x)$ exists
- $F_{n+1}(x_{n+1}^*) \leqslant F_n(x_n^*)$

By hypothesis (5), F_n attains its minimum at a unique point x_n^*. Then $\Sigma(n_0, x^*) < \epsilon/2$ and $f(x^*) < f_0 + \epsilon/2$ so that

$$F_n(x_n^*) \leqslant F_{n_0}(x_{n_0}^*) \leqslant F_{n_0}(x^*) < f_0 + \epsilon.$$

Since $F_n(x_n^*) \geqslant f(x_n^*) \geqslant f_0 > f_0 - \epsilon$, it follows that $|F_n(x_n^*) - f_0| < \epsilon$ for all $n > n_0$ and the theorem is proved.

Corollary:

Let x^* be the unique point for which $f(x^*) = f_0$. Then $\{x_n^*\} \to x^*$.

Proof:

From Theorem 2,

$0 < f(x_n^*) - f_0 < F_n(x_n^*) - f_0 < \epsilon$ for n sufficiently large. Thus, $\{f(x_n^*)\} \to f_0$. For any positive number η, let S_η denote the neighbourhood of x^* defined by $S_\eta = \{x | |x_i - x_i^*| < \eta;\ i = 1, 2, \ldots, N\}$ and suppose that $\{x_n^*\}$ does not converge to x^*. Then there is a $\delta > 0$ such that for some $\bar{n} > n_0$ (where n_0 is arbitrarily large), $x_{\bar{n}}^*$ is not in S_δ. Let $\bar{f} = \min_{D - S_\delta} f(x)$. Then $f(x_{\bar{n}}^*) \geqslant \bar{f}$ and $f(x_{\bar{n}}^*) - f_0 > \bar{f} - f_0 > 0$ since x^* is unique.

This contradicts the convergence of $\{f(x_n^*)\}$ to f_0. Thus $\{x_n^*\} \to x^*$.

In concluding the section on convergence, it should be remarked that the algorithm has been observed to converge under much weaker conditions than those needed in the convergence proof. There is ample experimental evidence that the domain D need not be convex and that the $-g_i$ and f need not be either convex or in $C^{(1)}$

Lagrange multipliers and order of convergence

Associated with problem G is a saddle value problem S. Problem S is:

Find vectors $\bar{x} \geqslant 0$ and $\bar{\mu} \geqslant 0$ such that

$$F(x, \bar{\mu}) \geqslant F(\bar{x}, \bar{\mu}) \geqslant F(\bar{x}, \mu)$$

for all $x \geqslant 0$ and $\mu \geqslant 0$ where

$$F(x, \mu) = f(x) - \sum_{i=1}^{m} \mu_i g_i(x).$$

The μ_i are called Lagrange multipliers (LM's). From the Kuhn-Tucker Equivalence Theorem (Kuhn and Tucker, 1951) and the definition of a saddle point it follows that x^* solves problem G only if there exists $\mu_i^* \geqslant 0$ such that

$$\nabla f(x^*) = \sum_{i=1}^{m} \mu_i^* \nabla g_i(x^*) \qquad (1)^*$$

and

$$\sum_{i=1}^{m} \mu_i^* . g_i(x^*) = 0 \qquad (2)$$

where $g_i(x^*) \geqslant 0$ for all i.

These results are subject to appropriate hypotheses and the 'Kuhn-Tucker constraint qualification' all of which are implied by the hypotheses invoked in Theorem 2. Equation (2) implies that if for some i, $g_i(x^*) > 0$, then $\mu_i^* = 0$ and if $g_i(x^*) = 0$, $\mu_i^* \geqslant 0$.

To apply this result to the present case, fix n and note that $\nabla F_n(x_n^*) = 0$. It follows that

$$0 = \nabla f(x_n^*) + \sum_{i=1}^{m} T_{in} . \nabla g_i(x_n^*) . \exp\,[T_{in} . g_i(x_n^*)]. \qquad (3)$$

Let λ_{in} denote $T_{in} . \exp\,[T_{in} . g_i(x_n^*)]$. Then (3) may be rewritten in the form

$$\nabla f(x_n^*) = -\sum_{i=1}^{m} \lambda_{in} . \nabla g_i(x_n^*). \qquad (4)$$

By the corollary of Theorem 2, $\lim_{n\to\infty} x_n^* = x^*$, and since $f \in C^{(1)}$, $\lim_{n\to\infty} \nabla f(x_n^*) = \nabla f(x^*)$. Therefore, $\lim_{n\to\infty} \sum_{i=1}^{m} \lambda_{in} . \nabla g_i(x_n^*)$ exists and is equal to

$$\sum_{i=1}^{m} \lim_{n\to\infty} \lambda_{in} . \nabla g_i(x_n^*).$$

* It is assumed further that $x^* > 0$.

Since each $g_i \in C^{(1)}$ also, $\lim_{n\to\infty} \nabla g_i(x_n^*) = \nabla g_i(x^*)$ and it then follows that for each i, $\lim_{n\to\infty} \lambda_{in}$ exists making it possible to define $\lambda_i = \lim_{n\to\infty} \lambda_{in}$. Using this λ_i and passing to the limit in (4), equation (4) becomes

$$\nabla f(x^*) = -\sum_{i=1}^{m} \lambda_i . \nabla g_i(x^*). \qquad (5)$$

From equations (1) and (5),

$$\sum_{i=1}^{m} (\mu_i^* + \lambda_i) . \nabla g_i(x^*) = 0. \qquad (6)$$

The vectors $\nabla g_i(x^*)$ are assumed linearly independent and then

$$\mu_i^* = -\lambda_i;\ i = 1, \ldots, m. \qquad (7)$$

Thus, the LM's are given by

$$\mu^* = -\lim_{n\to\infty} T_{in} . \exp\,[T_{in} . g_i(x_n^*)]. \qquad (8)$$

From (8) it is clear that $\mu_i^* = 0$ if the g_i is not binding. Let p denote the number of binding constraints.

Exact LM's calculated from a tightly converged solution of the corresponding equality constraint problem are compared in **Table 1** with LM's derived from (8) for a problem in which $m = 17$, $p = 4$, $N = 7$. The difference $\mu_i^* - \mu_i$ could have been reduced by continuing the sequential solution to larger T_{in} (equation (8)).

The availability of LM's provides a means of computing order of convergence. If the sequence $\{x_n\}$ converges to α and if there exists real numbers ρ and $C \neq 0$ such that

$$\lim_{n\to\infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^\rho} = C,$$

then ρ is called the order of convergence of $\{x_n\}$ and C is called the asymptotic error constant (Traub, 1964).

Table 1

Theoretical (μ_i^*) versus computed Lagrange multipliers

i	μ_i^*	μ_i
1	$4{\cdot}482 \times 10^{-6}$	$4{\cdot}544 \times 10^{-6}$
2	$4{\cdot}583 \times 10^{-6}$	$4{\cdot}643 \times 10^{-6}$
3	$2{\cdot}364 \times 10^{-6}$	$2{\cdot}453 \times 10^{-6}$
4	$5{\cdot}322$	$5{\cdot}344$

To find ρ and C for the sequence $\{F_n(x_n^*)\}$, we rewrite $F_n(x)$ in the form

$$F_n(x) = f(x) + \sum_{NBC} (n, x) + \sum_{BC} (n, x)$$

where NBC denotes the sum over all non-binding constraints while BC denotes the sum over all binding constraints. Note that the concept of order of convergence, as defined above, does not apply to the case of no binding constraints. Accordingly, we suppose that $\sum_{BC}$ contains at least one summand. It is customary in most applications to choose $T_{in} = K^n . T_{i0}$ for $n \geqslant 1$ and all i where $K > 1$. For such choices of the penalty factors,

$$F_n(x_n^*) - f(x_n^*) = -\frac{1}{K^n}\left[\sum_{NBC} \frac{\lambda_{in}}{T_{i0}} + \sum_{BC} \frac{\lambda_{in}}{T_{i0}}\right].$$

It has been noted earlier that

$$\lim_{n\to\infty} \sum_{NBC} \frac{\lambda_{in}}{T_{i0}} = 0 \text{ and } \lim_{n\to\infty} \sum_{BC} \frac{\lambda_{in}}{T_{i0}} = \sum_{BC} \frac{\lambda_i}{T_{i0}}.$$

Using these relations, the result

$$\lim_{n\to\infty} \frac{F_{n+1}(x_{n+1}^*) - f(x_{n+1}^*)}{F_n(x_n^*) - f(x_n^*)} = \frac{1}{K}$$

follows. This implies that the order of convergence is 1 and that the asymptotic error constant is $1/K$. Numerous computational verifications have confirmed this result.

It is tempting to believe that large K's will produce a tightly converged solution 'more quickly' than smaller K's will. For large K's, fewer members of the sequence $\{F_n(x_n^*)\}$ are required to attain a given small neighbourhood of x^*. However, it is almost invariably true that more actual computation time is required to find successive members of the sequence $\{x_n^*\}$ for large K than for small K. There appears then to be an optimal choice of K for each problem in the sense that such a K will enable one to 'converge' a problem with the least amount of computation.

The case of no-domain

To motivate the discussion of the no-domain case consider the problem in one variable—viz. find min $f(x)$ subject to $x \leqslant a$ and $b \leqslant x$ where $a < b$. Restated, this is equivalent to the problem: find $\min_x f(x)$

$$\text{subject to } g_1(x) = a - x \geqslant 0$$
$$\text{and} \quad g_2(x) = x - b \geqslant 0.$$

Obviously this is a no-domain problem. Suppose that it is not known *a priori* that this is a no-domain problem and the auxiliary function is defined in the usual way by

$$F_n(x) = f(x) + \exp[T_{1n}.g_1(x)] + \exp[T_{2n}.g_2(x)].$$

Whether the original problem has a domain or not, $F_n(x)$ is defined wherever $f(x)$ and $g_i(x)$ are defined—a crucial property. To provide a concrete situation, take $T_{1n} = T_{2n} = -n$, $f(x) = x$, $a = 1$ and $b = 2$. For these choices, F_n becomes

$$F_n(x) = x + \exp[-n(1-x)] + \exp[-n(x-2)].$$

For large n, $x_n^* = 3/2$ asymptotically, and

$$F_n(x_n^*) = 3/2 + 2.\exp[n/2].$$

Moreover, $F_n(x_n^*)$ rises exponentially with n to arbitrarily large positive values. This behaviour of the algorithm carries over to problems in higher dimensions and provides the basis for deciding in practice whether or not a problem is a no-domain problem.

As suggested in the example, if the problem has no domain, it appears that there are points x_n^*, for sufficiently large n, at which $F_n(x_n^*)$ is a minimum and $\lim_{n\to\infty} F_n(x_n^*) = +\infty$. Necessary and sufficient conditions characterising the no-domain case are not known at the present time. Three partial characterisations are proved in Theorems 3, 4 and 5 which follow.

Theorem 3:

If for all n sufficiently large (i) F_n has a minimum at x_n^* and (ii) if $\lim_{n\to\infty} F_n(x_n^*) = +\infty$, then no point of D is in any compact subset of E^N.

Proof:

Let S denote any compact subset of E^N. Then f is bounded on S so let $C(S) = \sup_S f(x)$. Choose any number $M > C(S) + m$. By hypotheses, there exists $n_0(M)$ so that for all $n > n_0(M)$,

$$f(x) + \Sigma(n, x) > f(x_n^*) + \Sigma(n, x_n^*) = F_n(x_n^*) > M > C(S) + m.$$

Then

$$C(S) + \Sigma(n, x) > f(x) + \Sigma(n, x) > C(S) + m$$

and it follows that $\Sigma(n, x) > m$ for any $x \in S$ and for all $n > n_0(M)$. If $g_i(x) \geqslant 0$ for all i, then $T_{in}.g_i(x) \leqslant 0$ and $\exp[T_{in}.g_i(x)] \leqslant 1$ from which $\Sigma(n, x) \leqslant m$ for all n. Therefore, $\Sigma(n, x) > m$ implies $g_i(x) < 0$ for some i and all $x \in S$. Hence by definition of no-domain, S contains no domain points.

A variation of theorem 3 follows.

Theorem 4:

If for all n sufficiently large, (i) F_n has a unique minimum at x_n^* and (ii) $\lim_{n\to\infty} F_n(x_n^*) = +\infty$, then $D = \phi$.

Proof:

Suppose $x_0 \in D$. Then $\Sigma(n, x_0) \leqslant m$.
Since x_n^* is the minimum of F_n,

$F_n(x_n^*) \leqslant F_n(x_0)$ or

$F_n(x_n^*) \leqslant f(x_0) + \Sigma(n, x_0) \leqslant f(x_0) + m$, a bound independent of n.

Therefore $\{F_n(x_n^*)\}$ is bounded contradicting (ii). Thus $D = \phi$.

Theorem 5:

If for all n sufficiently large, (i) F_n has a minimum, x_n^*, (ii) if $D = \phi$ and (iii) if f is bounded below by $M \geqslant 0$ for all $x \in E^N$, then

$$\lim_{n\to\infty} F_n(x_n^*) = +\infty.$$

Proof:

By (ii) there exists an $\epsilon < 0$ such that for some $i = k$,

$$\lim_{n\to\infty} g_k(x_n^*) \leqslant \epsilon < 0.$$

Then

$$\begin{aligned}\lim_{n\to\infty} F(x_n^*) &= \lim_{n\to\infty} [f(x_n^*) + \Sigma(n, x_n^*)] \\ &\geqslant \lim_{n\to\infty} [M + \Sigma(n, x_n^*)] \\ &\geqslant \lim_{n\to\infty} [M + \Sigma'(n, x_n^*) + \exp[T_{kn}.g_k(x_n^*)]] \\ &\geqslant \lim_{n\to\infty} [M + \Sigma'(n, x_n^*) + \exp[T_{kn}.\epsilon]] \\ &\geqslant \lim_{n\to\infty} [\exp[T_{kn}.\epsilon]] = +\infty\end{aligned}$$

Table 2

No-domain example in six variables

FUNCTION NAME	n 1	2	3	4	5
x_1^*	0·2842	0·2831	0·2827	0·2824	0·2820
x_2^*	0·7083	0·7013	0·6957	0·6925	0·6915
x_3^*	0·3261	0·3237	0·3243	0·3254	0·3258
x_4^*	0·1533	0·1495	0·1478	0·1471	0·1466
x_5^*	0·3056	0·2933	0·2847	0·2802	0·2785
x_6^*	0·0601	0·0495	0·0443	0·0419	0·0411
$f(x_n^*)$	3·260	3·230	3·205	3·192	3·187
$F_n(x_n^*)$	17·4	47·4	481·0	5·86E + 4	8·1E + 8

where $\Sigma'(n, x_n^*) = \Sigma(n, x_n^*) - \exp[T_{kn} \cdot g_k(x_n^*)]$.

Theorem 4 is the result most often applied in practice to detect no-domain cases. For complex problems it is rarely possible to guarantee a unique minimum for F_n. Moreover, observations on any finite set of $F_n(x_n^*)$ values however large, cannot guarantee that $\lim_{n\to\infty} F_n(x_n^*) = +\infty$.

These objections notwithstanding, much computational experience has shown that (at least for applications with engineering significance) if the sequence $\{F_n(x_n^*)\}$ of minima increases rapidly for $n = 4, 5, 6$, etc. when $T_{in} = K^n . T_{i0}(K = 2, 3)$, the problem at hand is indeed a no-domain problem. A typical example of this behaviour is shown in **Table 2** for a problem in which $N = 6$, $m = 19$ and $K = 2$.

It is worth noting that these properties could be used to test a family of inequality constraints for feasibility by simply applying the algorithm to Problem G with $f(x) \equiv 0$. A similar suggestion was apparently made by Motzkin (1952) for the case of linear $g_i(x)$.

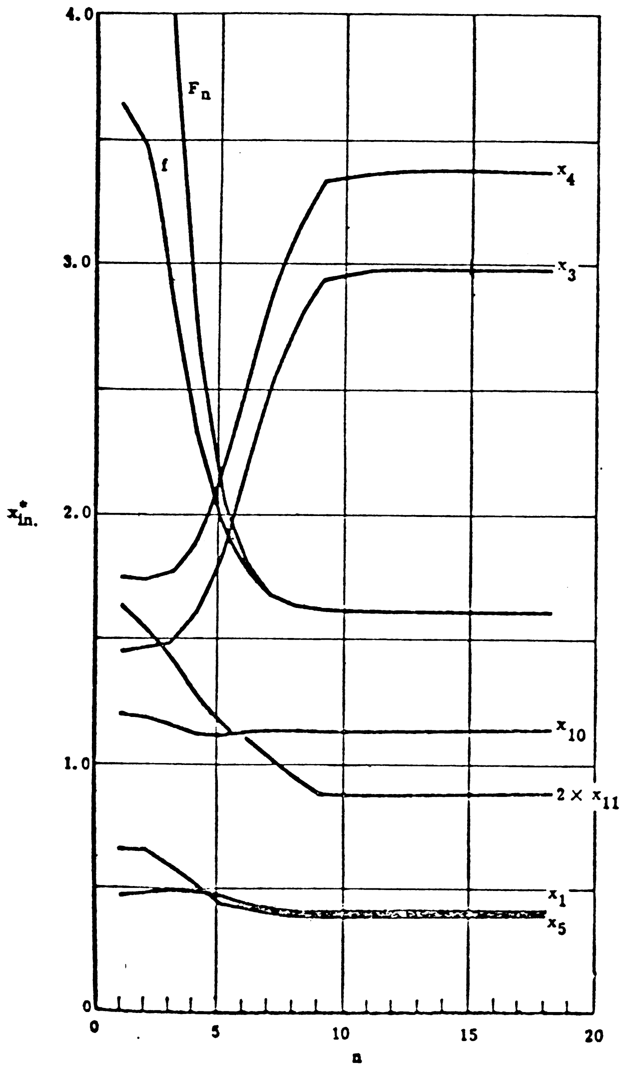

Fig. 1. Plots of x_{in}^* versus *n* for various *i*

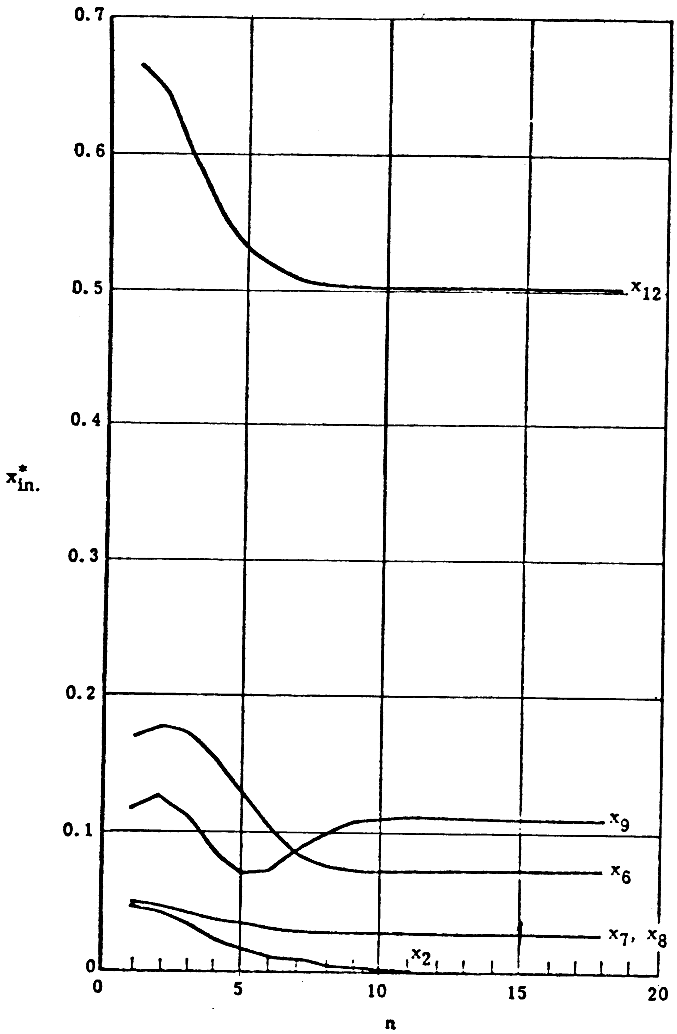

Fig. 2. Plots of x_{in}^* versus *n* for various *i*

Convergence behaviour in a twelve variable problem

The method described herein has been applied to many physically meaningful problems. As such, the objective functions and constraints are often quite complex and frequently contain functions involving absolute values and logical 'if' statements. Such structures have partial derivatives which experience finite jumps at points or along arcs in E^N.

These problems are readily handled by the algorithm provided an appropriate method for locating the minimum of $F_n(x)$ is used. The well-known Fletcher-Powell-Davidon (Fletcher and Powell, 1963; Davidon, 1959) method (FPD), which requires continuous first partial derivatives, has been used in the vast majority of our applications and when used, the occurrence of jumps in the partial derivatives cannot be tolerated. Various schemes are available to smooth such discontinuities. Our work employs a variant of the arctangent function to do this. An example of how a large problem converges is shown in **Fig. 1** and **Fig. 2** and in **Table 3**. For this example, $N = 12$, $m = 67$, $p = 10$ and $T_{in} = 2^{n/2}.T_{i0}$; $i = 1, \ldots, 67$. The FPD method was used to minimize F_n and the problem was computed on the IBM 360/44 computer in extended (16 decimal) precision. Computing time was 86·4 minutes. The detailed structure of these convergence paths (Figs. 1 and 2) depends on many factors—i.e. the starting point for finding x_1^*, the choice of T-sequences, etc. These data were extracted from a file of production runs and have no intrinsic significance here except to illustrate the convergence behaviour of a large problem. Table 3 lists all the data plotted in Figs. 1 and 2. Table 3 also shows the behaviour of the μ_i^* for all ten binding constraints (all other μ_i^* are rapidly going to zero).

From $n = 9$ onward, the μ_i^* are 'practically converged'. The noise observed in these values can be attributed directly to the limited precision with which the x_n^* were determined. For each n, x_n^* was said to be converged when $\max_i |\partial F_n(x)/\partial x_i| \leqslant 0{\cdot}1$. Note the extreme range of the μ_i^*:—from $\cdot 5 \times 10^{-6}$ to $4{\cdot}4$.

There is nothing in the theory of convergence which requires anything special of the T-sequences except that

$$0 > T_{in} > T_{i,n+1} \text{ for all } i \text{ and } n \geqslant 0$$

and

$$\lim_{n\to\infty} T_{in} = -\infty.$$

In most applications to date, T-sequences have been defined by

$$T_{in} = K^n.T_{i0};\; i = 1, \ldots, m$$

and $T_{i0} < 0$. There has never been a clear need to use different K's for different constraints. Other methods of defining T-sequences have been found useful. The most important of these is the so-called 'constant $T - g$ product method'. An almost arbitrary set of T_{i0} are

Table 3

Typical convergence behaviour of a 12-variable problem

n	x_{1n}^*	x_{2n}^*	x_{3n}^*	x_{4n}^*	x_{5n}^*	x_{6n}^*	x_{7n}^*	x_{8n}^*	x_{9n}^*	x_{10n}^*	x_{11n}^*	x_{12n}^*
1	0·467037	0·048137	1·44619	1·74703	0·656308	0·170542	0·044916	0·044165	0·118909	1·19516	0·807825	0·663840
2	0·484049	0·043871	1·45467	1·73773	0·652150	0·178412	0·045089	0·045301	0·126551	1·19210	0·769976	0·643841
3	0·491110	0·034472	1·49282	1·76906	0·589870	0·174744	0·041826	0·042565	0·111113	1·16016	0·709137	0·585041
4	0·485855	0·024135	1·60593	1·88570	0·507164	0·157993	0·037528	0·038361	0·087592	1·12097	0·638072	0·554408
5	0·467993	0·017088	1·85306	2·15201	0·444748	0·130200	0·034250	0·034968	0·071656	1·11395	0·584225	0·528923
6	0·441001	0·013752	2·19656	2·52285	0·415242	0·130534	0·031642	0·032084	0·073730	1·13316	0·549769	0·513571
7	0·418902	0·008575	2·52650	2·88333	0·401892	0·086176	0·029644	0·029818	0·088600	1·13536	0·505188	0·505392
8	0·411850	0·004120	2·78323	3·16270	0·396575	0·077359	0·028597	0·028649	0·100913	1·13342	0·468666	0·501809
9	0·411247	0·000987	2·94279	3·33519	0·394679	0·073318	0·028097	0·028102	0·108798	1·13041	0·444024	0·500267
10	0·411179	0·000541	2·96992	3·36481	0·394488	0·072669	0·028044	0·028046	0·109915	1·13005	0·441481	0·500118
11	0·411133	0·000310	2·98184	3·37784	0·394408	0·072388	0·028022	0·028023	0·110492	1·12989	0·440275	0·500056
12	0·411106	0·000185	2·98743	3·38395	0·394373	0·072257	0·028011	0·028012	0·110803	1·12981	0·439671	0·500029
13	0·411092	0·000114	2·99022	3·38700	0·394356	0·072193	0·028006	0·028006	0·110978	1·12976	0·439356	0·500015
14	0·411087	0·000073	2·99170	3·38862	0·394347	0·072159	0·028004	0·028004	0·111080	1·12974	0·439184	0·500009
15	0·411078	0·000048	2·99248	3·38946	0·394343	0·072141	0·028002	0·028002	0·111141	1·12973	0·439085	0·500005
16	0·411073	0·000032	2·99291	3·38994	0·394340	0·072131	0·028001	0·028001	0·111181	1·12972	0·439030	0·500003
17	0·411074	0·000022	2·99321	3·39026	0·394339	0·072124	0·028001	0·028001	0·111204	1·12972	0·438994	0·500002
18	0·411070	0·000016	2·99336	3·39042	0·394338	0·072121	0·028001	0·028001	0·111221	1·12971	0·438971	0·500001

n	$F_n(x_n^*)$	$f(x_n^*)$	μ_{1n}^*	μ_{2n}^*	μ_{3n}^*	μ_{4n}^*	μ_{5n}^*	μ_{6n}^*	μ_{7n}^*	μ_{8n}^*	μ_{9n}^*	μ_{10n}^*
1	8·17558	3·65912	—	—	—	—	—	—	—	—	—	—
2	6·11999	3·49765	—	—	—	—	—	—	—	—	—	—
3	4·05644	2·90479	—	—	—	—	—	—	—	—	—	—
4	2·72959	2·32642	—	—	—	—	—	—	—	—	—	—
5	2·06421	1·95123	—	—	—	—	—	—	—	—	—	—
6	1·79601	1·76541	—	—	—	—	—	—	—	—	—	—
7	1·68248	1·67271	—	—	—	—	—	—	—	—	—	—
8	1·63341	1·63054	0·5214E−6	0·1726E−5	0·3329E−5	0·1909E−5	3·727	3·881	0·5007E−1	0·3428	2·118	0·5109
9	1·61165	1·61128	0·4902E−6	0·2606E−5	0·3000E−5	0·1871E−5	4·127	4·384	0·4722E−1	0·3077	2·174	0·5097
10	1·60941	1·60925	0·4865E−6	0·2722E−5	0·2973E−5	0·1873E−5	4·162	4·431	0·4707E−1	0·3037	2·180	0·5108
11	1·60846	1·60839	0·4851E−6	0·2732E−5	0·2949E−5	0·1865E−5	4·212	4·443	0·4711E−1	0·3016	2·187	0·5098
12	1·60802	1·60799	0·4844E−6	0·2730E−5	0·2943E−5	0·1866E−5	4·199	4·470	0·4668E−1	0·3012	2·185	0·5114
13	1·60781	1·60779	0·4839E−6	0·2730E−5	0·2940E−5	0·1866E−5	4·207	4·478	0·4668E−1	0·3007	2·185	0·5110
14	1·60770	1·60769	0·4907E−6	0·2724E−5	0·2948E−5	0·1864E−5	4·189	4·476	0·4651E−1	0·3034	2·172	0·5148
15	1·60764	1·60764	0·4902E−6	0·2723E−5	0·2937E−5	0·1860E−5	4·211	4·476	0·4554E−1	0·2996	2·184	0·5084
16	1·60761	1·60760	0·4866E−6	0·2707E−5	0·2937E−5	0·1870E−5	4·235	4·479	0·4661E−1	0·2965	2·203	0·5132
17	1·60759	1·60759	0·4872E−6	0·2728E−5	0·2928E−5	0·1872E−5	4·214	4·470	0·4569E−1	0·2972	2·191	0·5159
18	1·60758	1·60758	0·4823E−6	0·2700E−5	0·2950E−5	0·1871E−5	4·241	4·472	0·4673E−1	0·3005	2·182	0·5106

chosen in order to find x_1^*. Having done this, a new set of T's, $\{T_{i1}\}$ are defined by

$$T_{i1} = K'/|g_i(x_1^*)|; \; i = 1, \ldots, m$$

where $K' = -2$ (or -3, or -4 etc.). These are used to find x_2^* from which

$$T_{i2} = 2.K'/|g_i(x_2^*)|, \text{ or in general}$$

$$T_{in} = n.K'/|g_i(x_n^*)|; \; i = 1, \ldots, m; \; n \geqslant 1.$$

This scheme has no known theoretical justification but it appears to have a 'smoothing property' in the sense that convergence paths often appear to be smoother and as a result, less computation time is required to find x_n^* than if the 'simple method' ($T_{in} = K^n . T_{i0}$) is used. A worthwhile rule-of-thumb, when the simple method is being used, is to choose the T_{i0} so that the products $T_{i0}g_i(x)$ act like constraints all of which have approximately the same LM's. When this is done, $\partial f/\partial[T_{i0}g_i]_{x^*}$ all have the same value for $i = 1, \ldots, p$ and one can say that each $T_{i0}g_i(x)$ which binds, has the same influence on f.

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Book review

Theory of Problem Solving—An Approach to Artificial Intelligence, by R. B. Banerji, 1969; 189 pages. (*Elsevier Publishing Co. Ltd.*, £6.50.)

As Banerji points out in his preface, this book is an account of work at Case University rather than a general account of work to date in this area of Artificial Intelligence Research. The title is misleading since although a general definition of a 'problem' is given, wide enough to cover puzzles and 2-person games, nothing worth calling a theory of problem solving is developed.

The first half of the book deals with alternative formulations of the notion of a problem and the associated notions of strategy, winning strategy, etc. A number of 'theorems' are produced and duly proved but they are all rather obvious consequences of the definitions. The reader has to plough through a lot of formalism for scant reward. Two classes of games are then defined and studied: Nim-like games and Tic-tac-toe games (Tic-tac-toe = noughts and crosses). The latter include 3-dimensional Tic-tac-toe and Go-Moku. For Nim-like games some graph theoretic ideas are introduced. It transpires that the 'graph of the game' can sometimes be expressed as the sum of simpler graphs and that this helps one to find positions which enable a win to be forced. The technique is confined to Nim-like games. The notion of forcing positions is also explored for the Tic-tac-toe class of games and a method for discovering such positions is given. A Go-Moku program is referred to but not described.

The second half of the book deals with concept-formation, alias 'induction' or 'pattern recognition', especially as a tool for classifying situations in a game. It considers how to find a derived property as a Boolean combination of simpler ones so as to account for given examples. Some algorithms are given but there is no information about their effectiveness. A language for describing more elaborate concepts is descricribed, in fact first order logic with some primitives to handle pairs and strings, but no algorithms are given for this.

The style throughout is set-theoretic. Banerji rightly deprecates the looseness of much earlier writing in this field. Unfortunately the new-found precision only exposes the lack of any general theory of problem solving. This is mostly formalism rather than mathematics. There has indeed been useful and non-obvious work in the area of semi-enumerative search methods e.g. dynamic programming, branch and bound methods, the 'alpha-beta heuristic' for game playing and Samuel's work on learning in checkers. But although this book gives detailed study to two classes of games it does not come up with any new generally applicable technique, nor does it provide any really helpful framework for previously existing work. Since mathematical concepts as simple as a finite state automata or even a semi-group have produced interesting theories there is no *a priori* reason why there should not be a theory of problem solving. But we are still waiting for one.

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