Discrete, nonlinear string vibrations*

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This paper develops a new approach, computer oriented in the sense that both the model and the equations of motion are discrete, to the problems arising from the vibrations of an elastic string.

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1. Introduction

Problems related to the vibrations of an elastic string have been studied for many years by mathematicians, physicists and engineers (see, e.g. all the quoted references and the references contained therein). For such problems, we will develop in this paper a new approach which is completely computer oriented, in the sense that both the model and the equations of motion are discrete. Thereby, we will be able to study nonlinear motion of a vibrating string by means only of arithmetic processes.

2. The discrete string

A discrete string is one which is composed of a finite number of particles. It will be treated mathematically as an ordered set of $n+2$ circular, homogeneous particles C_k, $k = 0, 1, 2, \ldots, n, n+1$, with respective centres (x_k, y_k), as shown typically in **Fig. 1**. Of course,

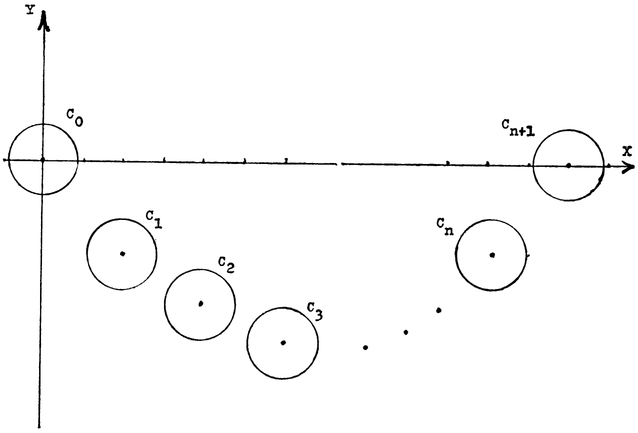

Fig. 1

by the molecular theory of matter, real strings are discrete strings for which n is relatively large.

Our problem will be that of describing the return of a discrete string to a position of equilibrium from an arbitrary position of tension when n is relatively small. The resulting motion can be considered as an approximation to that of a real string, the improvement of which is dependent largely upon one's computer capability. It will be assumed throughout that C_0 and C_{n+1} are fixed while $C_1, C_2, \ldots, C_n$ are free to move, and that

$$x_0 = y_0 = y_{n+1} = 0. \tag{2.1}$$

3. Velocity and acceleration of a particle

To facilitate dealing with the motion of a discrete vibrating string, it will be convenient to develop first the concepts of velocity and acceleration for a particle which moves in a fixed direction. Throughout, the location of the particle will be identified with the location of the centre of the particle.

For $\Delta t > 0$, let $t_k = k\Delta t$, $k = 0, 1, \ldots, q-1, q$. At time t_k let a particle which is in motion along an S axis have its centre at s_k. We wish to define the velocity v_k and acceleration a_k of the particle at each time t_k, $k = 1, 2, \ldots, q$. For this purpose, consider first the interval $t_0 \leqslant t \leqslant t_1$. Suppose, as shown in **Fig. 2**, one

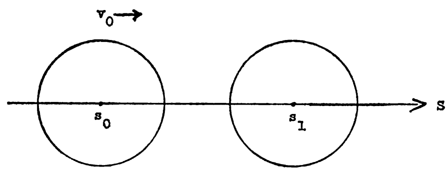

Fig. 2

knows v_0 in addition to s_0 and s_1. For example, when a particle's motion begins from a position of rest, one would know that $v_0 = 0$. Let us try to define $v_1 = v(t_1)$ in a fashion that will use *all* the given data. Such is the case when one defines v_1 implicitly by the smoothing formula

$$\frac{s_1 - s_0}{\Delta t} = \frac{v_0 + v_1}{2}. \tag{3.1}$$

Note that if one were to define v_1 by the backward difference formula

$$v_1 = \frac{s_1 - s_0}{\Delta t}, \tag{3.2}$$

then the knowledge of v_0 would have been neglected. On the other hand, if one were to define velocity by a

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forward difference formula, it would follow that

$$v_0 = \frac{s_1 - s_0}{\Delta t} \qquad (3.3)$$

and $v_0 = 0$ would imply that $s_0 = s_1$. However, it is undersirable physically to imply that a particle whose initial velocity is zero cannot move during the first time interval of length Δt.

The above considerations then motivate the general definition

$$\frac{s_k - s_{k-1}}{\Delta t} = \frac{v_{k-1} + v_k}{2}, \quad k = 1, 2, \ldots, q, \qquad (3.4)$$

for velocity v_k.

With regard to acceleration $a_k = a(t_k)$, $k = 1, 2, \ldots, q$, one rarely knows a_0, so that one is forced to define a_1 by the backward difference

$$a_1 = \frac{v_1 - v_0}{\Delta t},$$

from which we are motivated to define a_k by

$$a_k = \frac{v_k - v_{k-1}}{\Delta t}, \quad k = 1, 2, \ldots, q. \qquad (3.5)$$

From (3.4) and (3.5), it follows readily (Greenspan, 1968a) that

$$v_1 = \frac{2}{\Delta t}[s_1 - s_0] - v_0 \qquad (3.6a)$$

$$v_k = \frac{2}{\Delta t}[s_k + (-1)^k s_0 + 2\sum_{j=1}^{k-1}(-1)^j s_{k-j}] + (-1)^k v_0, \quad k \geqslant 2 \qquad (3.6b)$$

$$a_1 = \frac{2}{(\Delta t)^2}[s_1 - s_0 - v_0 \Delta t] \qquad (3.7a)$$

$$a_2 = \frac{2}{(\Delta t)^2}[s_2 - 3s_1 + 2s_0 + v_0 \Delta t] \qquad (3.7b)$$

$$a_k = \frac{2}{(\Delta t)^2}\{s_k - 3s_{k-1} + 2(-1)^k s_0 + 4\sum_{j=2}^{k-1}[(-1)^j s_{k-j}] + (-1)^k v_0 \Delta t\}, \quad k \geqslant 3. \qquad (3.7c)$$

4. The law of motion

In terms of the definitions of Section 3, the motion of a particle is assumed to be governed by a generalised Newton's equation of the form

$$F(t_{k-1}) = ma(t_k); \quad k = 1, 2, \ldots, q. \qquad (4.1)$$

The form of (4.1) has particular computational value since the left side will be a function only of $s_0, s_1, \ldots, s_{q-1}$, while the right side will be a linear combination of $s_0, s_1, \ldots, s_{q-1}, s_q$. For then (4.1) can be solved explicitly for s_q, and the particle's position can be generated easily on a digital computer for a large number of time steps. From the resulting recursion formula for s_q, one has immediately the existence and uniqueness of each s_q for given s_0 and v_0, provided only that F is always defined.

The validity of the conservation laws for (4.1), which is in part related to the stability of the numerical procedure to be followed, follows readily as in Greenspan (1968a).

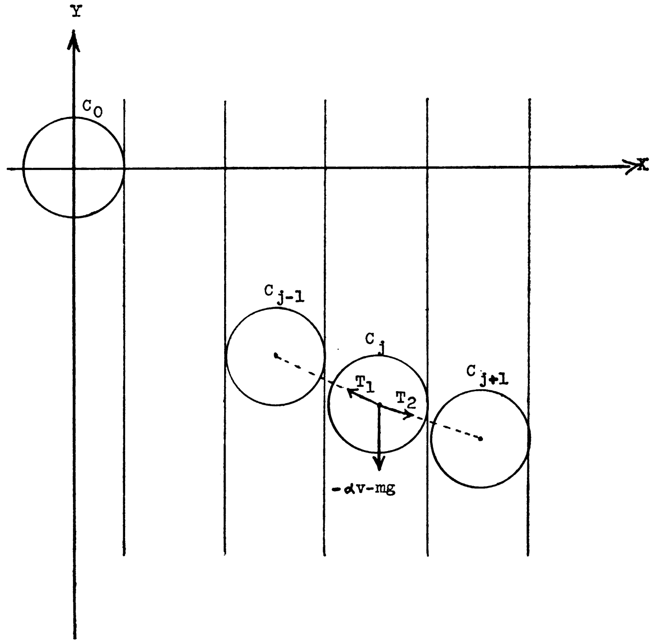


Fig. 3

5. Equations of string vibration

We will proceed under the popular assumption that each particle of a discrete string can move in the vertical direction only. The string is then said to exhibit transverse vibrations.

Let $x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1}$ and $x_i - x_{i-1} = \Delta x$, $i = 1, 2, \ldots, n+1$. At time t_k, $k = 0, 1, 2, \ldots, q$, measured in seconds, let C_j, as shown in **Fig. 3**, be a typical particle in motion. In order to incorporate the time dependence of the centres of C_{j-1}, C_j and C_{j+1}, let the respective centres of these particles at time t_k be $(x_{j-1}, y_{j-1,k})$, $(x_j, y_{j,k})$, $(x_{j+1}, y_{j+1,k})$, where each coordinate is measured in feet.

In studying the motion of C_j, we will take into account only tensile, viscous, and gravitational forces. For this purpose, let T_1 be the tensile force between C_{j-1} and C_j, let T_2 be the tensile force between C_j and C_{j+1}, and let the viscosity vary with the velocity of the particle. Then (4.1) takes the particular form

$$|T_2|\frac{(y_{i+1,k-1} - y_{i,k-1})}{[(\Delta x)^2 + (y_{i+1,k-1} - y_{i,k-1})^2]^{1/2}} - |T_1|\frac{(y_{i,k-1} - y_{i-1,k-1})}{[(\Delta x)^2 + (y_{i,k-1} - y_{i-1,k-1})^2]^{1/2}} - \alpha v_{i,k-1} - mg = ma_{i,k}; \quad k = 1, 2, 3, \ldots, (5.1)$$

where $g \geqslant 0$, $\alpha \geqslant 0$ and m is the mass of C_j. By means of (3.6) and (3.7), it follows readily from (5.1) that

$$y_{i,1} = y_{i,0} + \left(1 - \frac{\alpha \Delta t}{2m}\right) v_{i,0}\Delta t + \frac{(\Delta t)^2}{2m} \times \left\{|T_2|\frac{(y_{i+1,0} - y_{i,0})}{[(\Delta x)^2 + (y_{i+1,0} - y_{i,0})^2]^{1/2}} - |T_1|\frac{(y_{i,0} - y_{i-1,0})}{[(\Delta x)^2 + (y_{i,0} - y_{i-1,0})^2]^{1/2}} - mg\right\}; \quad i = 1, 2, \ldots, ! \qquad (5\ 2a$$

$$y_{i,2} = \left(3 - \frac{\alpha \Delta t}{m}\right) y_{i,1} - \left(2 - \frac{\alpha \Delta t}{m}\right) y_{i,0} - \left(1 - \frac{\alpha \Delta t}{2m}\right) v_{i,0} \Delta t + \frac{(\Delta t)^2}{2m} \left\{ |T_2| \frac{(y_{i+1,1} - y_{i,1})}{[(\Delta x)^2 + (y_{i+1,1} - y_{i,1})^2]^{1/2}} - |T_1| \frac{(y_{i,1} - y_{i-1,1})}{[(\Delta x)^2 + (y_{i,1} - y_{i-1,1})^2]^{1/2}} - mg \right\}; \quad i = 1, 2, \ldots, n \qquad (5.2b)$$

$$y_{i,k} = \left(3 - \frac{\alpha \Delta t}{m}\right) y_{i,k-1} + (-1)^{k-1} \left(2 - \frac{\alpha \Delta t}{m}\right) y_{i,0} + 2\left(2 - \frac{\alpha \Delta t}{m}\right) \left(\sum_{j=2}^{k-1} [(-1)^{j-1} y_{i,k-j}]\right) + (-1)^{k+1} \left(1 - \frac{\alpha \Delta t}{2m}\right) v_{i,0} \Delta t + \frac{(\Delta t)^2}{2m} \left[|T_2| \frac{(y_{i+1,k-1} - y_{i,k-1})}{[(\Delta x)^2 + (y_{i+1,k-1} - y_{i,k-1})^2]^{1/2}} - |T_1| \frac{(y_{i,k-1} - y_{i-1,k-1})}{[(\Delta x)^2 + (y_{i,k-1} - y_{i-1,k-1})^2]^{1/2}} - mg \right]; \quad k \geqslant 3,\ i = 1, 2, \ldots, n \qquad (5.2c)$$

Before considering actual dynamical problems, it is worth noting that in practice it is of value to know the steady state, or terminal position, of a vibrating string. With this position available *a priori*, one can actually check a particular computation to see whether or not it is converging or diverging. The steady state can often be obtained by applying the generalized Newton's method (Greenspan, 1968b) to the algebraic system:

$$\frac{|T_2|(y_{i+1} - y_i)}{[(\Delta x)^2 + (y_{i+1} - y_i)^2]^{1/2}} - \frac{|T_1|(y_i - y_{i-1})}{[(\Delta x)^2 + (y_i - y_{i-1})^2]^{1/2}} = mg; \quad i = 1, 2, \ldots, n, \qquad (5.3)$$

which results easily from (5.1) after setting $a_{i,k} \equiv v_{i,k-1} \equiv 0$.

6. Examples

A large number of examples using (5.2*a*)–(5.2*c*) were run at the University of Wisconsin Computing Center. In this section we will discuss several which are both illustrative and of physical interest. In all cases the output is given graphically with 100 additional points interpolated linearly between each pair of consecutive particles and the strings are all of approximately the same weight.

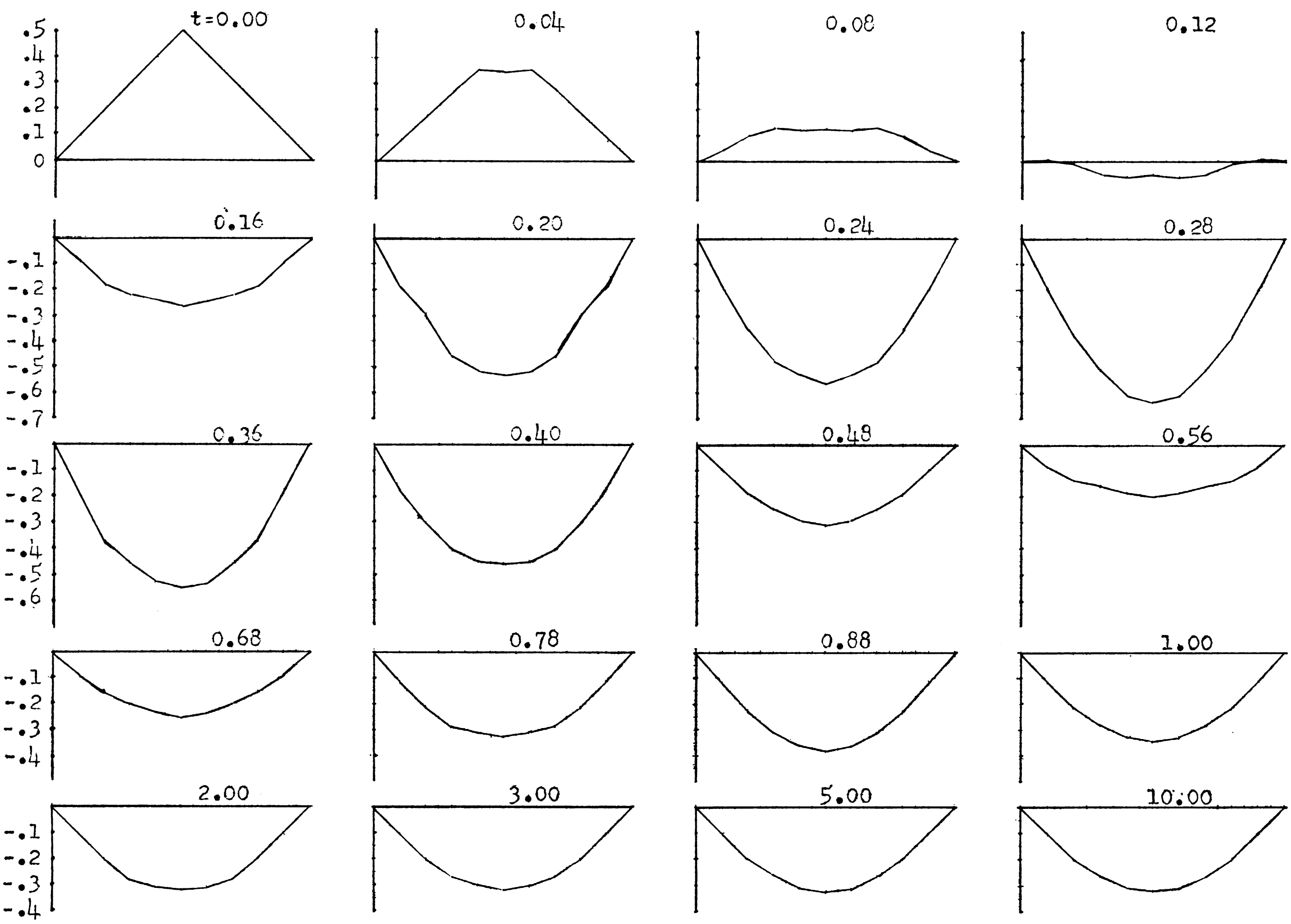


Fig. 4

Example 1: Consider an eleven particle string with $x_i = \frac{i}{10}$, $i = 0, 1, 2, \ldots, 10$; with

$$T_1 = T_0 \left[1 + \left|\frac{y_{i,k-1} - y_{i-1,k-1}}{\Delta x}\right| + \frac{\epsilon}{2}\left(\frac{y_{i,k-1} - y_{i-1,k-1}}{\Delta x}\right)^2\right] \quad (6.1)$$

$$T_2 = T_0 \left[1 + \left|\frac{y_{i+1,k-1} - y_{i,k-1}}{\Delta x}\right| + \frac{\epsilon}{2}\left(\frac{y_{i+1,k-1} - y_{i,k-1}}{\Delta x}\right)^2\right]; \quad (6.2)$$

and with $\alpha = 0\cdot6$, $m = 0\cdot1$, $T_0 = 9$, $\Delta t = 0\cdot002$, $\Delta x = 0\cdot1$, $n = 9$, $g = 32\cdot2$, $\epsilon = 0\cdot01$. The string is placed in a position of tension by bringing the centre particle to the point $(0\cdot5, 0\cdot5)$. The particles to the left of centre are positioned on the line $y = x$ and those to the right of centre on the line $y = -x + 1$. The resulting configuration is that shown for $t = 0\cdot00$ in **Fig. 4**. The string is released from its position of tension and its stable, strongly damped motion is shown typically from $t = 0\cdot00$ to $t = 10\cdot00$ in Fig. 4. At $t = 10$ the moving particles were oscillating no more than $2\cdot10^{-4}$ and were located at $(0\cdot1, -0\cdot1137)$, $(0\cdot2, -0\cdot2023)$, $(0\cdot3, -0\cdot2669)$, $(0\cdot4, -0\cdot3080)$, $(0\cdot5, -0\cdot3236)$, $(0\cdot6, -0\cdot3080)$, $(0\cdot7, -0\cdot2669)$, $(0\cdot8, -0\cdot2023)$, $(0\cdot9, -0\cdot1137)$. The steady state positions, found by a method described in the next example, are $(0\cdot1, -0\cdot1137)$, $(0\cdot2, -0\cdot2022)$, $(0\cdot3, -0\cdot2668)$, $(0\cdot4, -0\cdot3079)$, $(0\cdot5, -0\cdot3235)$, $(0\cdot6, -0\cdot3079)$, $(0\cdot7, -0\cdot2668)$, $(0\cdot8, -0\cdot2022)$, $(0\cdot9, -0\cdot1137)$. The total computing time consumed on the UNIVAC 1108 was under 14 seconds.

Example 2: Consider now a twenty-one point string with $x_i = \frac{i}{10}$, $i = 0, 1, 2, \ldots, 20$; with T_1 and T_2 defined by (6.1)–(6.2); and with $\alpha = 0\cdot15$, $m = 0\cdot05$, $T_0 = 12\cdot5$, $\Delta t = 0\cdot00025$, $\Delta x = 0\cdot1$, $n = 19$, $g = 32\cdot2$, $\epsilon = 0\cdot01$. The string is placed in a position of tension by bringing the centre particle to the point (1, 1). The particles to the left of centre are positioned on $y = x$ and those to the right of centre on $y = -x + 2$. The resulting configuration is that shown for $t = 0$ in **Fig. 5**. The string is released from its position of tension and its downward motion from $t = 0$ to $t = 0\cdot35$ is shown typically in Fig. 5, while its upward motion from $t = 0\cdot35$ to $t = 0\cdot69$ is shown typically in **Fig. 6**. The lower curve in **Fig. 7** is the string's position after six seconds, at which time its maximum oscillation is less than $0\cdot005$. The upper curve in Fig. 7 labelled S, is the steady state solution, which was obtained as follows. Substitution of the given parameters into (5·3) yields the system

$$(12\cdot5)\left[1 + \left|\frac{y_{i+1} - y_i}{0\cdot1}\right| + (0\cdot005) \times \left(\frac{y_{i+1} - y_i}{0\cdot1}\right)^2\right] \frac{(y_{i+1} - y_i)}{[(0\cdot01) - (y_{i+1} - y_i)^2]^{1/2}}$$

$$-(12\cdot5)\left[1 + \left|\frac{y_i - y_{i-1}}{0\cdot1}\right| + (0\cdot005)\times \left(\frac{y_i - y_{i-1}}{0\cdot1}\right)^2\right] \frac{(y_i - y_{i-1})}{[(0\cdot01) - (y_i - y_{i-1})^2]^{1/2}} = 1\cdot61;$$

$$i = 1, 2, \ldots, 19. \quad (6.3)$$

This system was replaced by an alternate system of ten

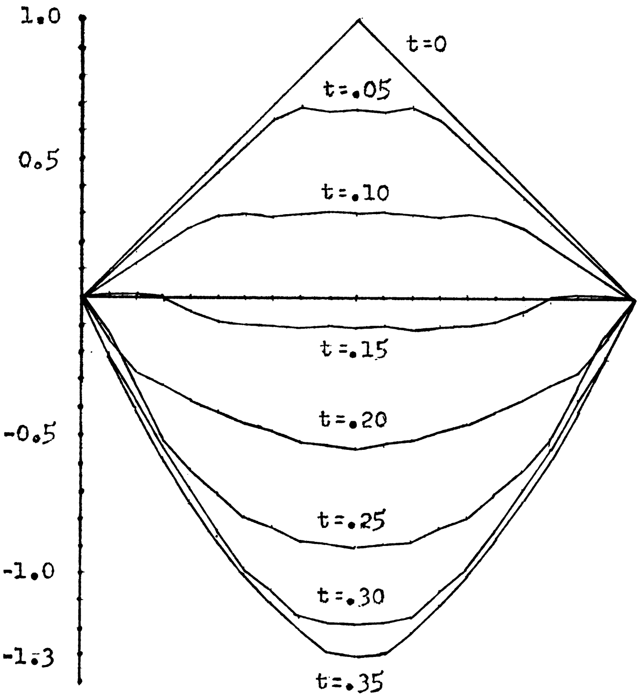


Fig. 5

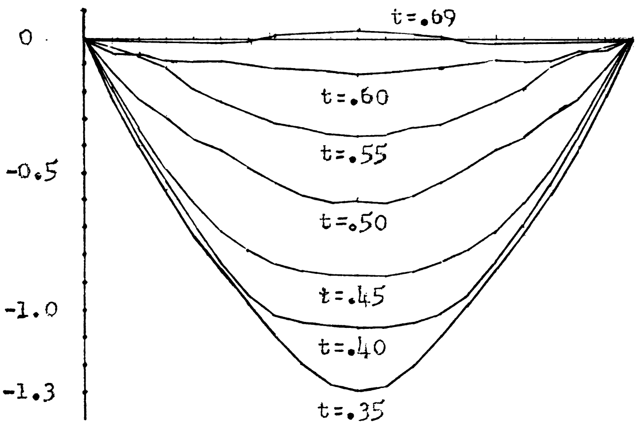


Fig. 6

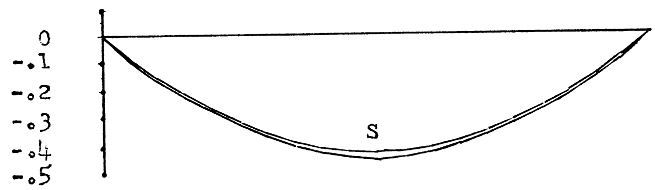
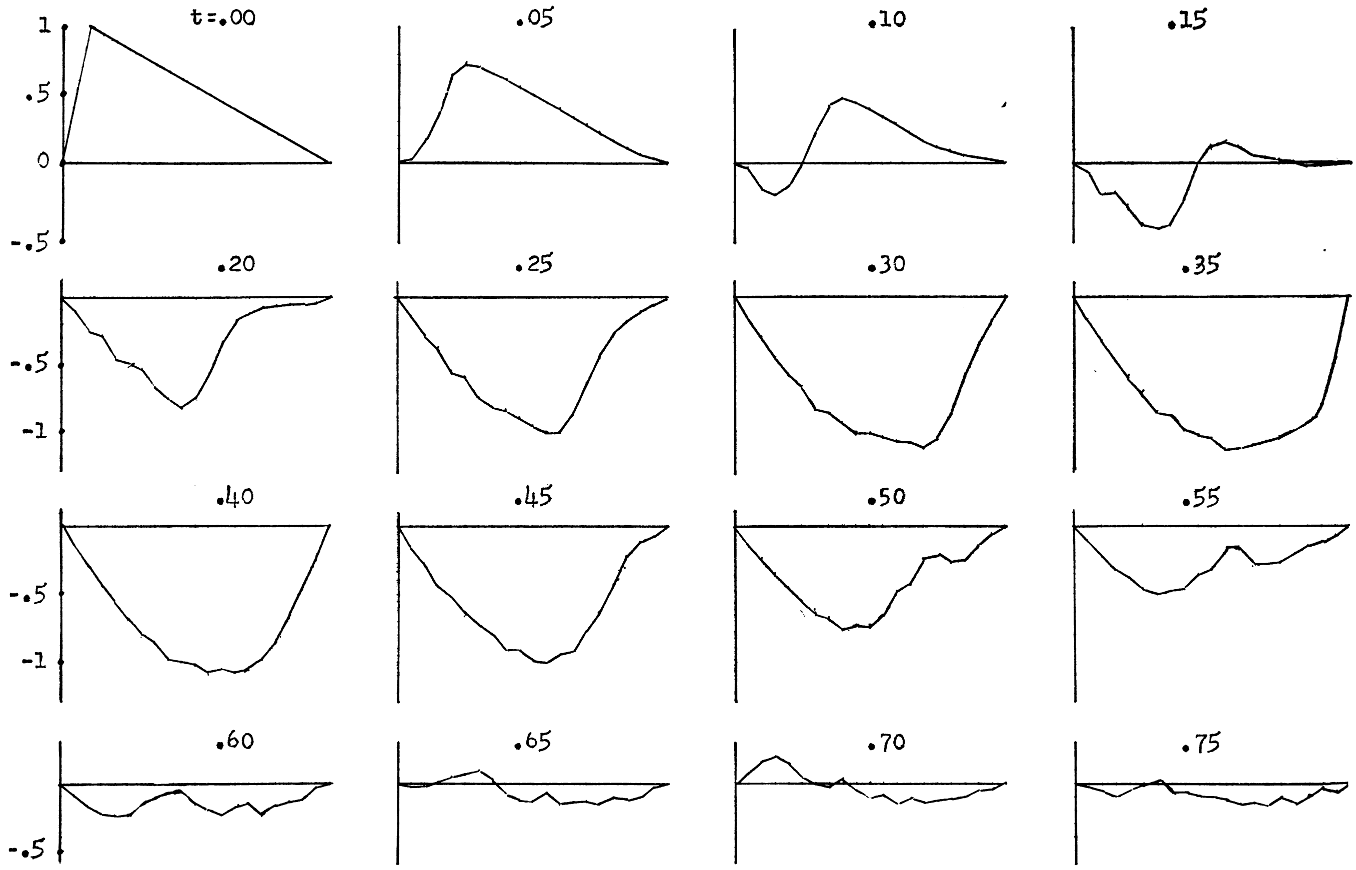


Fig. 7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0
-.5
-1

Fig. 8

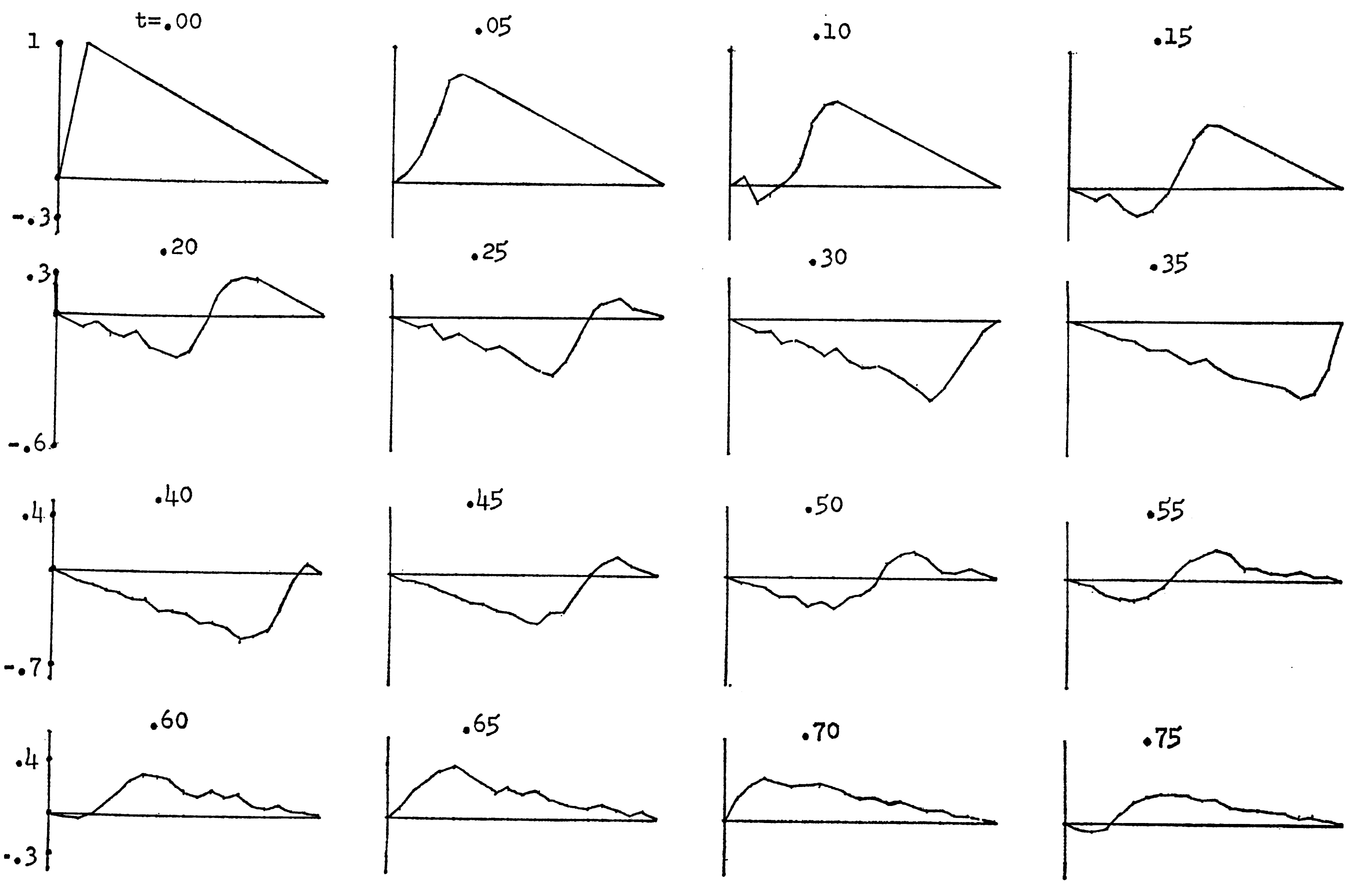
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Fig. 9

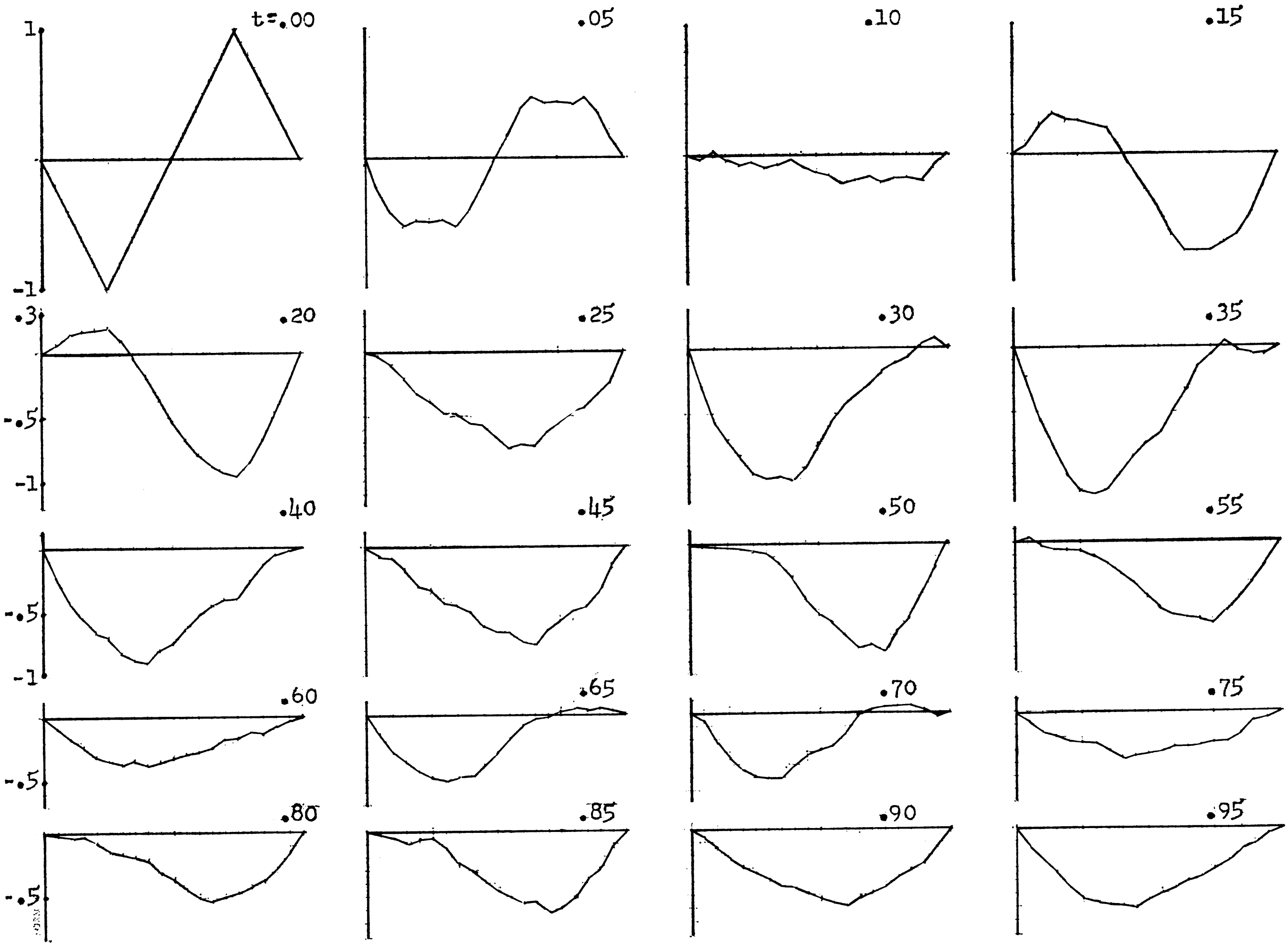
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Fig. 10

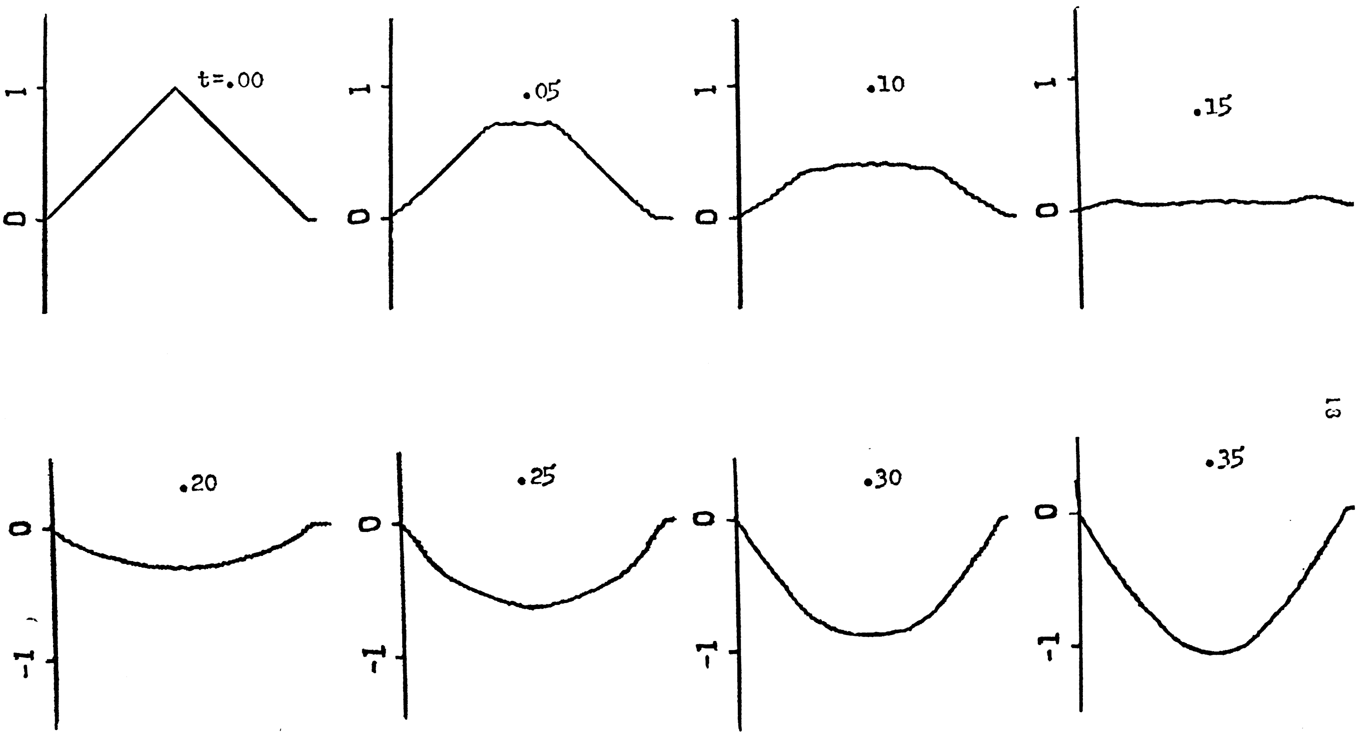
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Fig. 11

equations in $y_1, y_2, \ldots, y_{10}$ by applying the symmetry assumption $y_i = y_{20-i}$, $i = 0, 1, 2, \ldots, 9$ and the 'hanging chain' assumption $y_i > y_{i+1}$, $i = 0, 1, 2, \ldots, 9$ to (6·3). The alternate system was solved by the generalized Newton's method (Greenspan, 1968b) and $y_{11}, y_{12}, \ldots, y_{19}$ were then defined by symmetry. The resulting values of $y_1, y_2, \ldots, y_{19}$ were substituted finally into (6·3) and were verified to be the required solution. At the end of six seconds, the particles of the string were at most 0·0025 from steady state. The total UNIVAC 1108 computer time consumed for six seconds of vibrations and for determination of the steady state was under 50 seconds.

Example 3*:* The string in Example 2 was considered again but with a different initial position. The first particle was placed at (0·1, 0·5), the second particle at (0·2, 1), and the remaining particles on $y = -\frac{5}{9}(x - 2)$, as shown for $t = 0{\cdot}00$ in **Fig. 8**. The first 0·75 seconds of motion is shown typically in Fig. 8. Convergence to steady state S, shown in Fig. 7, was at a rate comparable to that of Example 2.

Example 4*:* The string in Example 3 was considered again but without gravity, that is, with $g = 0$. The first 0·75 seconds of motion is shown typically in **Fig. 9**. Convergence to the steady state solution $y_i = 0$, $i = 1, 2, \ldots, 19$ was at a rate comparable to that of Example 2.

Example 5*:* The string in Example 2 was considered again but with an initial position defined as follows. The fifth particle was set at (0·5, −1) and the fifteenth particle at (1·5, 1). The particles to the left of the fifth were set on $y = -2x$, those between the fifth and the fifteenth on $y = 2x - 2$, and those to the right of the fifteenth on $y = -2x + 4$. The resulting configuration is that shown for $t = 0{\cdot}00$ in **Fig. 10**, where the first 0·95 seconds of motion is shown. Convergence to steady state S shown in Fig. 7 was at a rate comparable to that of Example 2.

Example 6*:* Consider a forty-one particle string with $x_i = \frac{i}{20}$, $i = 0, 1, 2, \ldots, 40$; with T_1 and T_2 defined by (6·1)–(6·2); and with $\alpha = 0{\cdot}125$, $m = 0{\cdot}025$, $T_0 = 10{\cdot}0$, $\Delta t = 0{\cdot}00025$, $\Delta x = 0{\cdot}05$, $n = 39$, $g = 32{\cdot}2$, $\epsilon = 0{\cdot}01$. The string is placed in a position of tension by bringing the centre particle to (1, 1), the particles to the left of centre to $y = x$, and the particles to the right of centre to $y = -x + 2$, as shown for $t = 0{\cdot}00$ in **Fig. 11**. The need for greater accuracy than that required of Examples 1–5 led to use of the CDC 3600 for the computation. The volume of the output became so excessive that it was graphed directly, without printout, by a Calcomp 570 digital plotter. The first 0·35 seconds of motion is shown typically in Fig. 11 as the string executes its initial movement downwards. A full four seconds of vibrations were graphed and showed a convergence to steady state similar to that of Examples 1–5. The entire computing time was under ten minutes.

7. Remarks

The intuition used in constructing the examples of Section 6 can be outlined as follows. A variety of initial conditions and parameters are inserted into (5.2*a*)–(5.2*c*) and the computer is programmed to give 5–10 seconds of vibration. If no convergent cases result, then Δt and ϵ are decreased while α and m are increased. When a convergent case results, others can be constructed with a steady state closer to the horizontal by decreasing m a small amount, while still others with larger oscillations can be constructed by decreasing α a small amount. If a decrease in α or Δx results in divergence, then Δt must also be decreased to retain the convergent behaviour.

Other convergent examples, not seriously different in behaviour from those of Section 6, were obtained with $\epsilon = 0{\cdot}1$, $\epsilon = 0{\cdot}001$, and with T defined by raising the bracketed terms in (6.1) and (6.2) to the powers $\frac{1}{2}$ and $\frac{3}{2}$. Studies of light strings led to such rapid convergence that the graphical output was relatively uninteresting. Nevertheless, it became apparent quickly that the number and variety of interesting parameter choices and initial positions of tension was so vast that no attempt could be made at present to study them all. Initial studies of a 201 particle string with transverse oscillations only and of a 21 particle string which allowed also for longitudenal motion resulted in no significant results due to a shortage of available computing time.

Finally, it should be noted that (6.1) and (6.2) follow from the simple assumptions of Fermi, Pasta and Ulam (1955). More complex formulas, which allow for each particle to have a relatively large radius, can also be developed and will be similar to those of Carrier (1945).

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