Analysis of numerical iterative methods for solving integral and integrodifferential equations

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An expository account is given of the application of the contraction mapping theorem to the solution of certain integral and integrodifferential equations iteratively. An analysis is given also of the errors incurred by using polynomial approximations for the iterates.

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1. Introduction

Recently in this journal Wolfe (1969) proposed certain iterative methods for obtaining the numerical solution of four classes of integral and integrodifferential equations. The first two of these are the integral equation

$$y(x) = f(x) + \int_a^b g(x, t; y(t))dt \tag{1}$$

and the integrodifferential equation

$$y'(x) = h(x, y(x)) + \int_a^b g(x, t; y(t))dt; \; y(a) = c. \tag{2}$$

Following Noble (1964), I will refer to these as equations of Fredholm type, although as a rule this term is more strictly reserved for the narrower class of linear equations, which satisfy the Fredholm theory. That is, equations in which $g(x, t; y(t)) \equiv k(x, t).y(t)$, say. The third and fourth classes considered are the Volterra type of equations obtained from (1) and (2) by replacing the upper limit of integration b by x. In each case, the solution $y(x)$ is required on the finite interval $[a, b]$.

Equation (2) may be rewritten in the form

$$y(x) = c + \int_a^x h(u, y(u))du + \int_a^x \int_a^b g(u, t; y(t))dt\,du. \tag{3}$$

The corresponding Volterra equation may be rewritten similarly, so that all four types of equation are seen to be particular examples of functional equations of the form

$$y = Ty. \tag{4}$$

For certain operators T, called contraction mappings, it is well known that the equation (4) may be solved iteratively by choosing an initial iterate $y_0 (= y_0(x))$ and calculating recursively

$$y_{n+1} = Ty_n, \quad n = 0, 1, 2, \ldots \tag{5}$$

A full discussion of contraction mappings is given in Collatz (1966), Goldstein (1967) and Saaty (1967). An admirable account of the long history of iterative methods applied to integral equations is given by Wouk (1964), where there is also a very extensive bibliography.

As a gentler introduction to these ideas, one may well wish first of all to become familiar with the notion of contraction mappings as applied to the solution of systems of nonlinear equations. See, for example, the excellent account of this in Henrici (1964).

The present paper aims to do two things. The first aim, on the theoretical side, is to give an elementary exposition of the contraction mapping theorem applied to the solution of the four types of integral and integrodifferential equations listed above. The need for this is perhaps greater in the case of the integrodifferential equations, which are not so commonly discussed in the literature as the integral equations. Secondly, in practice it is usually not possible to calculate *exactly* the theoretical sequences $\{y_n\}$ which are generated by (5); due to the need to evaluate integrals, we must work with some computed sequences, say $\{y_n^*\}$. The last section of the paper analyses the behaviour of certain computed sequences $\{y_n^*\}$.

2. Convergence of the iterative methods

For the integral equation (1), the iterative process defined by (5) is

$$y_{n+1}(x) = f(x) + \int_a^b g(x, t; y_n(t))dt, \quad n = 0, 1, \ldots \tag{6}$$

Let us assume that $f(x)$ and $g(x, t; y(t))$ are continuous functions of x, for $a \leqslant x \leqslant b$, and that $g(x, t; y(t))$ also satisfies a Lipschitz condition of the form

$$|g(x, t; y(t)) - g(x, t; z(t))| \leqslant L(x, t).|y(t) - z(t)| \tag{7}$$

for $a \leqslant x, t \leqslant b$ and all functions $y(t)$ and $z(t)$ belonging to some set S. Initially, let S be the set of all continuous functions defined on $[a, b]$. In what follows, $||y||$ is used to denote $\sup_{a \leqslant x \leqslant b} |y(x)|$.

From (6) we then have

$$||y_{n+1} - y_n|| \leqslant G||y_n - y_{n-1}||, \tag{8}$$

where

$$G = \sup_{a \leqslant x \leqslant b} \int_a^b L(x, t)dt. \tag{9}$$

We may treat the corresponding Volterra integral equation similarly. For the Volterra equation the inequality (8) will hold, where G is given by (9), except that the upper limit of integration b must be replaced by x.

For both types of integral equations, if the appropriate $G < 1$, the sequence $\{y_n(x)\}$ of continuous functions will be a Cauchy sequence which will therefore converge uniformly to some continuous limit function, say $Y(x)$. It is easily verified that $Y(x)$ is a solution of the given equation. To illustrate this for the integral equation (1) we may write

$$Y(x) - f(x) - \int_a^b g(x, t;\ Y(t))dt = Y(x) - y_{n+1}(x)$$
$$+ \int_a^b [g(x, t;\ y_n(t)) - g(x, t;\ Y(t))]dt.$$

Therefore

$$\sup_{a \leqslant x \leqslant b} |Y(x) - f(x) - \int_a^b g(x, t;\ Y(t))dt|$$
$$\leqslant ||Y - y_{n+1}|| + G||y_n - Y||. \quad (10)$$

Since both terms on the right side of this inequality may be made arbitrarily small by choosing n sufficiently large, it follows that $Y(x)$ is a solution of (1). It may be verified also that the solution of (1) is unique. For suppose (1) has two distinct solutions $Y(x)$ and $Z(x)$. Then

$$Y(x) - Z(x) = \int_a^b [g(x, t;\ Y(t)) - g(x, t;\ Z(t))]dt.$$

Therefore, from (7) and (9),

$$||Y - Z|| \leqslant G||Y - Z||,$$

which provides a contradiction.

So far, we have imposed no restriction on the functions which are produced by the iterative process and for which the Lipschitz condition (7) holds. Similar results hold for restricted $y(x)$. In particular, for the integral equation of Fredholm type we may state the following result.

Theorem 1. Given two functions $p(x) \leqslant q(x)$ on $[a, b]$, let S denote the set of continuous functions $y(x)$ such that

$$p(x) \leqslant y(x) \leqslant q(x), \quad \text{for} \quad a \leqslant x \leqslant b.$$

Suppose that $y(x) \in S$ implies that $y^*(x) \in S$ also, where

$$y^*(x) = f(x) + \int_a^b g(x, t;\ y(t))dt.$$

Also, let us suppose that

$$|g(x, t;\ y(t)) - g(x, t;\ z(t))| \leqslant L(x, t)|y(t) - z(t)|$$

for $a \leqslant x, t \leqslant b$ and all functions $y(x)$ and $z(x) \in S$, that f and g are continuous in x, and that

$$G = \sup_{a \leqslant x \leqslant b} \int_a^b L(x, t)dt < 1.$$

Then, for any $y_0(x) \in S$, the sequence $\{y_n\}$ defined by the iterative method

$$y_{n+1}(x) = f(x) + \int_a^b g(x, t;\ y_n(t))dt, \quad n = 0, 1, \ldots,$$

converges uniformly to the unique and continuous solution of the Fredholm equation which satisfies the inequalities $p(x) \leqslant Y(x) \leqslant q(x)$.

This result is a special case of the Banach theorem or contraction mapping theorem of functional analysis. See, for example, Goldstein (1967).

A further dividend from the contraction mapping theorem is an estimate of the error at any stage, $Y(x) - y_n(x)$. By repeated use of (8), we have

$$||y_{n+1} - y_n|| \leqslant G^n||y_1 - y_0||.$$

Therefore for any $m > n \geqslant 0$

$$||y_{m+n} - y_n|| \leqslant ||y_{m+n} - y_{m+n-1}|| + \ldots + ||y_{n+1} - y_n||$$
$$\leqslant (G^{m+n-1} + \ldots + G^n).||y_1 - y_0||.$$

Thus

$$||y_{m+n} - y_n|| < G^n||y_1 - y_0||/(1 - G). \quad (11)$$

We may now write

$$||Y - y_n|| \leqslant ||Y - y_{m+n}|| + ||y_{m+n} - y_n||. \quad (12)$$

Letting $m \to \infty$, the first term on the right of (12) tends to zero. We then obtain from (12) and (11) that

$$||Y - y_n|| \leqslant G^n||y_1 - y_0||/(1 - G). \quad (13)$$

To provide an *a priori* bound for $||Y - y_n||$ without requiring advance knowledge of the iterate $y_1(x)$, we may replace (13) by

$$||Y - y_n|| \leqslant G^n||q - p||/(1 - G),$$

where $p(x)$ and $q(x)$ are the functions referred to above.

The integrodifferential equations may be treated in the same way. For instance, the iterative process (5) applied to the integrodifferential equation (2) is

$$y_{n+1}(x) = c + \int_a^x h(u, y_n(u))du$$
$$+ \int_a^x \int_a^b g(u, t;\ y_n(t))dt\, du. \quad (14)$$

We will assume that the function h satisfies a Lipschitz condition

$$|h(x, y(x)) - h(x, z(x))| \leqslant M(x).|y(x) - z(x)|, \quad (15)$$

for all $y(x)$ and $z(x) \in S$. For the iterative process (14), if $y \in S \Rightarrow Ty \in S$, it follows that successive iterates satisfy the inequality (8), where

$$G = \int_a^b M(u)du + \int_a^b \int_a^b L(u, t)dt\, du. \quad (16)$$

In the case of the Volterra integrodifferential equation, we can see that successive iterates satisfy the inequality (8). This time, G is given by

$$G = \int_a^b M(u)du + \int_a^b \int_a^u L(u, t)dt\, du. \quad (17)$$

Thus, for all four types of integral and integrodifferential equations considered here, we can write down a theorem, as for the Fredholm integral equation above. In what follows, S again denotes the set of continuous functions $y(x)$ such that $p(x) \leqslant y(x) \leqslant q(x)$ for $a \leqslant x \leqslant b$.

Theorem 2. If $y \in S \Rightarrow Ty \in S$, if the relevant Lipschitz conditions (7) and (15) hold on S and the appropriate $G < 1$, then for any $y_0 \in S$ the iterative scheme $y_{n+1} = Ty_n$ converges uniformly to the unique solution Y of $y = Ty$ on S. Further, for any $n > 0$,

$$||Y - y_n|| \leqslant G^n ||q - p||/(1 - G).$$

3. Numerical examples

To illustrate these results, I will consider three equations which have previously been considered by other writers, including Wolfe (1969).

Example 1. First let us examine the following Fredholm integral equation (a particular case of Love's equation), which has been solved previously by Fox and Goodwin (1953) and by Elliott (1963):

$$y(x) = 1 + \frac{1}{\pi}\int_{-1}^{1} \frac{y(t)dt}{1 + (t - x)^2}. \qquad (18)$$

In this example,

$$L(x, t) = \frac{1}{\pi}\frac{1}{1 + (t - x)^2}$$

and from (9)

$$G = \sup_{-1 \leqslant x \leqslant 1} \frac{1}{\pi}\int_{-1}^{1} \frac{dt}{1 + (t - x)^2}.$$

Thus

$$G = \frac{1}{\pi}\tan^{-1} 2 < 1.$$

Here, as for all linear equations, we may take S unrestricted, as the set of all continuous functions defined on $[-1, 1]$. For this equation, the theoretical iterative process will converge with any choice of initial iterate $y_0(x)$.

Example 2. The Volterra integrodifferential equation

$$y'(x) = 1 + 2x - y(x) + \int_0^x x(1 + 2x)e^{t(x-t)} y(t)dt \quad (19)$$

has been considered by several writers, including Pouzet (1960) and Day (1967). In this case

$$L(x, t) = x(1 + 2x)e^{t(x-t)},$$
$$M(x) = 1$$

and we may, as in the previous example, take S unrestricted. It may be pointed out that there is, of course, an essential difference between Fredholm and Volterra equations. For an equation of Fredholm type, the interval on which the iterates $y_n(x)$ have to be computed must be at least that of the range of integration $[a, b]$. With the Volterra equation, which is an initial value problem, we may seek to compute a numerical solution iteratively on an interval, say $[a, a + \delta]$, for any choice of $\delta > 0$. Therefore, for Volterra equations, we can ensure that $G < 1$ by choosing δ sufficiently small. Hence, certainly for linear Volterra equations, we can always find an interval $[a, a + \delta]$ on which the iterative process will converge to the solution. (For nonlinear equations we have also to ensure that if $y \in S$, $Ty \in S$.)

For (19) then, on the interval $[0, \delta]$, we have from (17) that

$$G = \int_0^\delta du + \int_0^\delta \int_0^u u(1 + 2u)e^{t(u-t)}dt\, du.$$

On $[0, \frac{1}{2}]$, for example, we see that $G < 1$ and so convergence follows. On $[0, 1]$, our sufficient criterion for convergence, $G < 1$, is not satisfied.

Example 3. The Volterra integral equation

$$y(x) = 1 - x + \int_0^x (xe^{t(x-2t)} + e^{-2t^2})[y(t)]^2 dt \quad (20)$$

has been solved by Laudet and Oules (1960) and Day (1966). For this problem we must first find an appropriate set S. It may be noted that, for $0 \leqslant x \leqslant 1$, $0 \leqslant y(x) \leqslant e^{x^2}$ implies that $0 \leqslant y^*(x) \leqslant e^{x^2}$, where $y^* = Ty$. Thus, for S, we may take the set of continuous functions $y(x)$ such that $0 \leqslant y(x) \leqslant e^{x^2}$. (In verifying the right hand inequality, we happen to hit upon the solution, $y(x) = e^{x^2}$.) Therefore, on S, the integrand in (20) satisfies a Lipschitz condition (7), with

$$L(x, t) = 2(xe^{t(x-t)} + e^{-t^2}).$$

On the interval $[0, \delta]$,

$$G = \sup_{0 \leqslant x \leqslant \delta} \int_0^x 2(xe^{t(x-t)} + e^{-t^2})dt,$$

which gives

$$G = \int_0^\delta 2(\delta e^{t(\delta-t)} + e^{-t^2})dt.$$

We may check that for $\delta = \frac{1}{3}$, $G < 1$. The iterative process will converge if we compute the iterates on the interval $[0, \frac{1}{3}]$, beginning with any continuous $y_0(x)$ satisfying $0 \leqslant y_0(x) \leqslant e^{x^2}$ for $0 \leqslant x \leqslant \frac{1}{3}$.

For Volterra type equations, the interval on which convergence is obtained need not be as small as we have found here; see for example Tricomi (1957).

4. Approximation of the iterates

So far, in examining the iterative process, no account has been taken of the problems involved in the *practical* evaluation of the iterates $y_n(x)$. With this in mind, let us now reconsider the integral equation of Fredholm type, for which the iterative process is

$$y_{n+1}(x) = f(x) + \int_a^b g(x, t; y_n(t))dt. \qquad (21)$$

Wolfe (1969) transforms $[a, b]$ onto $[-1, 1]$ by means of a linear change of variable. Each iterate is approximated by a polynomial, expressed as a linear combination of Chebyshev polynomials. Thus the theoretical sequence $\{y_n(x)\}$ is approximated by the computed sequence, say $\{y_n^*(x)\}$. The functions y_n^* are polynomials of degree, say, N. The functions f and g are approximated by polynomials, say f^* and g^*; in the case of g^*, this is a polynomial in the two variables x and t.

Therefore the computed sequence $\{y_n^*\}$ satisfies the recurrence relation

$$y_{n+1}^*(x) = f^*(x) + \int_a^b g^*(x, t; y_n^*(t))dt, \qquad (22)$$

for $n = 0, 1, 2, \ldots$, with $y_0^*(x) = y_0(x)$. The integration is done exactly, neglecting rounding error. We may now write

$$g(x, t; y_n(t)) - g^*(x, t; y_n^*(t))$$
$$= [g(x, t; y_n(t)) - g(x, t; y_n^*(t))]$$
$$+ [g(x, t; y_n^*(t)) - g^*(x, t; y_n^*(t))]. \quad (23)$$

Let us suppose that f and f^* differ by less than some positive number δ, and that the same is true of g and g^*. By taking the degree of the approximating polynomials, N, sufficiently large, we can make δ as small as we please, if f and g are continuous. Then, from (21) and (22),

$$|y_{n+1}(x) - y_{n+1}^*(x)| \leqslant |f(x) - f^*(x)| + \int_a^b |g(x, t; y_n(t)) - g^*(x, t; y_n^*(t))| dt$$

Using (23) we obtain

$$||y_{n+1} - y_{n+1}^*|| \leqslant \delta(1 + b - a) + G||y_n - y_n^*||.$$

From this it follows that

$$||y_n - y_n^*|| \leqslant \delta(1 + b - a)/(1 - G) \quad (24)$$

for all values of n. This bound holds also for the Volterra integral equation and a similar result may be derived in the same way for the integrodifferential equations. This demonstrates that the accumulated error, obtained by working with polynomial approximations for the iterates, may be made as small as we please.

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