Values for Error_2 at major nodes are given for Day, Moore, and the present method in **Tables 4D, 4M, and 4** respectively. Again since Δu, Δp, $\Delta q, f_u, f_p, f_q$ all vary monotonically over D it is essential to use averages taken from a central region ($0{\cdot}3 \leqq x, y \leqq 1{\cdot}8$) for instance. The truncation error estimate in $u(2{\cdot}0, 2{\cdot}0)$ at node (39, 39) when expressed in the mode of Error_2 turns out to be: 2·35 which compares with the deviation from 100 of the terminal diagonal entry in Table 4.

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Correspondence

To the Editor
The Computer Journal

Sir,

The method described by M. G. Cox (A bracketing technique for computing a zero of a function, this *Journal*, Vol. 13, No. 1, pp. 101-102) should be compared with the method proposed by W. M. Stone (A form of Newton's method with cubic convergence, *Quart. Appl. Math.*, Vol. 11, No. 1, pp. 118-119) which uses the same information, f and f at the two boundary points. While Stone's method requires solving a quadratic and selecting the proper root it has cubic convergence while Cox's method only converges quadratically.

Yours faithfully,

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23 June 1970

Dr. Cox replies:

In reply to Professor Squire's letter I have carried out a theoretical and practical comparison of the methods of Stone (1953) and Cox (1970) and I give here the main results. I shall refer to these methods as S and C respectively.

Stone assumes that at the start of the nth iteration bounds a_n and b_n such that $f(a_n) f(b_n) < 0$ are available. He then shows that if x_{n+1} is the value obtained from his formula using a_n and b_n, then

$$x_{n+1} - \xi = \frac{(a_n - \xi)(b_n - \xi)}{b_n - a_n} P(x - \xi),$$

where ξ is the required zero and $P(x - \xi)$ is a power series containing quadratic terms and higher in $(a_n - \xi)$, $(b_n - \xi)$ and $(x_{n+1} - \xi)$. It is not immediately evident from Stone's paper exactly how his method is implemented. However, since a_n and b_n are to straddle the zero it appears that at the end of the $(n - 1)$th iteration the following strategy is employed. If $f(x_n)$ and $f(a_{n-1})$ have the same sign then new bounds are given by setting $a_n = x_n$ and $b_n = b_{n-1}$, otherwise $a_n = a_{n-1}$ and $b_n = x_n$.

It follows that at the start of the nth iteration either a_n or b_n is set equal to x_n. Assume for sake of definiteness that $a_n = x_n$. Hence b_n must be equal to an earlier value x_{n-r}, say, where $r \geqslant 1$. Let $\epsilon_n = x_n - \xi$. If we assume that $|\epsilon_n| < |\epsilon_{n-r}|$, then asymptotically

$$|\epsilon_{n+1}| = C|\epsilon_n|\epsilon^2_{n-r},$$

where $C \to$ a positive constant. After taking logarithms and solving the resulting difference equation we find

$$|\epsilon_{n+1}| = K|\epsilon_n|P,$$

where $K \to$ a positive constant and p is the real positive root of the equation $t^{r+1} - t^r - 2 = 0$. When $r = 1$, $p = 2$. When $r > 1$, $1 < p < 2$, and as $r \to \infty$, $p \to 1$. These results may be interpreted as follows. In the most favourable case the bounds are replaced alternately and consequently $p = 2$; i.e. the convergence is quadratic. If one bound remains fixed for r iterations ($r > 1$), then the convergence is subquadratic ($r = 2$ gives $p = 1{\cdot}696$, $r = 3$ gives $p = 1{\cdot}544$, $r = 4$ gives $p = 1{\cdot}451$, etc.). In particular, convergence is at best quadratic and certainly not cubic as claimed by Stone.

In my paper I showed that in the least favourable case when one bound remains fixed and the other is replaced at each iteration the convergence of C is quadratic. In the most favourable case, when bounds are replaced alternately, the analysis of Jarratt (1966) can be used to show that the order is $1 + 3^{\frac{1}{2}} = 2{\cdot}732$. Thus if p_s and p_c are the orders of S and C, respectively, then $1 < p_s \leqslant 2 \leqslant p_c \leqslant 2{\cdot}732$. It follows that if the order is taken as a means of comparison then C is theoretically superior to (or at least as good as) S.

It is possible however that S possesses certain desirable properties in practice. In order to provide a partial test of such a possibility I have applied S to the same class of test functions as considered in my paper, viz. the class of randomly generated polynomials $p_n(x)$ of degree n. This test was carried out 100 times for both $n = 10$ and $n = 30$ using initial bounds $a_0 = 0$ and $b_0 = 1$. Each test was terminated when two successive estimates of the zero differed by $\frac{1}{2} \times 10^{-8}$ or less. The table gives the average number of evaluations of the polynomial together with its derivative for the two methods.

Degree	C	S
10	4·95	9·41
30	5·15	19·84

The figures do not include the evaluations of $p_n(x)$ and $p_n'(x)$ at the initial bounds $x = 0$ and 1, since they are common to both methods. On the evidence of these tests we conclude that C appears to be superior to S in practice.

It is possible of course that Professor Square has a different interpretation of Stone's method, for which the performance is better than I have observed. If this is the case I should be most interested to have details.

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Yours faithfully,

M. G. COX

26 August 1970

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