The selection of ADI iteration parameters by numerical experiment for the solution of Poisson's equation over a circular area

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It is shown how a set of iteration parameters for the Peaceman-Rachford ADI process may be determined by numerical experiment so that the rate of convergence of a square mesh finite difference approximation to the solution of the Dirichlet problem over a circular area may be a maximum. It is found that the rate of convergence is very sensitive to the choice of parameters but may be made nearly as high as over a square.

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In numerical weather forecasting it is important to be able to solve two-dimensional elliptic partial differential equations as rapidly as possible. For example, in the operational numerical forecasting system in use at the Meteorological Office, Bracknell, one Poisson and a pair of simultaneous Helmholtz-type equations have to be solved at each time-step, and a 48-hour forecast may contain up to 192 time-steps. The operational forecast is at present made over a rectangle containing 1,927 (47 × 41) grid-points which is part of a polar stereographic projection of the northern hemisphere of the earth. For research purposes, this area has been extended to the whole of the hemisphere north of 15 deg. N approximated to as closely as possible (using the same grid-length) by a network of 2,801 points; a quadrant of this grid is shown in **Fig. 1.** In the operational forecast the Poisson equation is solved by the Peaceman-Rachford alternating direction implicit method using eight optimum parameters. When we came to extend the forecast to the circular area we wished to retain if possible the considerable advantages of the ADI method but had of course no theory to guide us in the selection of optimum parameters. It nevertheless seemed from statements in the literature that ADI methods would be likely to work over all areas of reasonable shape even though the matrices involved were not commutative, so it was decided to conduct some numerical experiments which led to the results described below. No theoretical discussion will be attempted.

Method

It was thought that a plausible method of selecting iteration parameters was to regard the quasi-circular grid of 2,801 points as equivalent to some square of N^2 points; for any such square p optimum parameters could then be selected by the method of Jordan as described by Wachspress (1963). It was thus necessary to determine, by numerical experiment, the values of p and N that maximised the rate of convergence of the ADI method of solving the particular finite-difference representation of the Poisson equation and its boundary conditions used in the numerical forecast. (It is perhaps worth while pointing out here that the 'boundaries' of regions used for numerical weather prediction are not boundaries in any true physical sense; rather they are lines on a map enclosing regions such that atmospheric conditions outside them are thought unlikely to influence events in the area of vital interest to the forecaster during the period of the forecast.)

It is easy to see that an iterative numerical process for solving a Poisson equation

$$\nabla^2\phi+\lambda = 0 \tag{1}$$

may be formulated as a process for solving Laplace's equation

$$\nabla^2 e_n = 0 \tag{2}$$

where e_n is the error of the nth approximation to ϕ. We can therefore assess the rate of convergence of the ADI solution of (1) by considering the convergence of the solution of (2) for various first guesses at e.

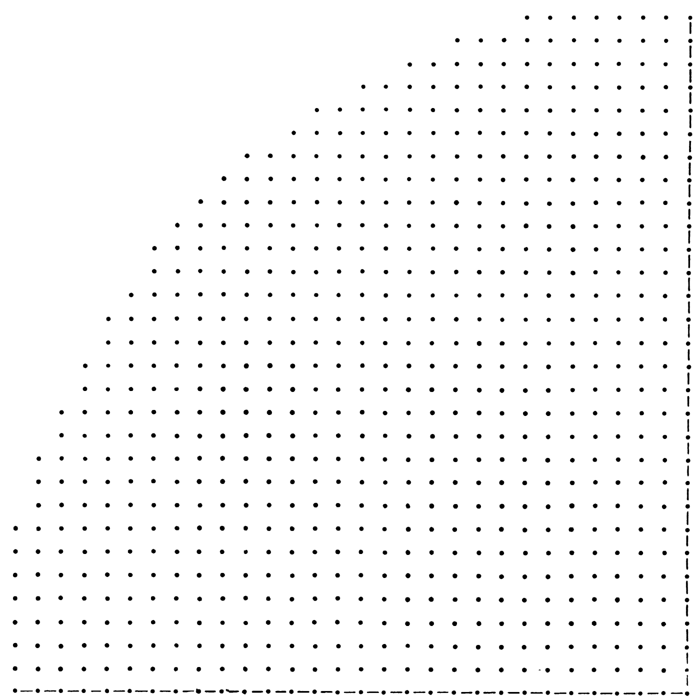

Fig. 1. Quadrant of quasi-circular network used in calculation. The whole network contains 2,801 points

If the operator ∇^2 is defined to be the usual 5-point finite-difference approximation to ∇^2, then the ADI process can be applied to solving the finite-difference analogue of (2), namely

$$\nabla^2 e = 0 \quad (3)$$

given that on the boundary $e = 0$, and the first guess at e (or e_0) is given at interior points by

$$e_0 = \sin\left(\frac{2\pi x}{mh}\right) \cdot \sin\left(\frac{2\pi y}{nh}\right) \quad (4)$$

where x and y are rectangular co-ordinates and h is the grid-length. m and n are integers which in the work described were each allowed to take any of the four values 2, 8, 32 and 128, giving 16 combinations; these combinations gave a spread of wave-length in the first guess field from the shortest that could be resolved by the grid to roughly twice the diameter of the area considered.

If the final 'error' field after application of the ADI process is called e_f, then a 'RMS error reduction factor' ε may be defined by

$$\varepsilon = \frac{(\Sigma e_f^2)^{\frac{1}{2}}}{(\Sigma e_0^2)^{\frac{1}{2}}}$$

where the summation is taken over all grid-points.

Results

It was found that the optimum value of N was about 52. This is illustrated in **Table 1** which gives values of $\varepsilon \times 10^5$ for a double application of five optimum Jordan parameters to initial guess fields for which the wave-length is the same in both directions ($n = m$).

Table 1

Values of $\epsilon \times 10^5$ as a function of N and wave-length of initial guess

N	Wave-length of initial guess field as multiple of grid-length			
	2	8	32	128
49	9·4	5·2	40·8	81·1
50	9·9	5·4	40·4	60·0
51	10·5	5·6	40·4	49·4
52	11·1	5·8	40·6	46·3
53	11·7	6·1	40·9	47·4
54	12·4	6·3	41·4	50·4
59	15·8	7·6	44·4	71·3

The variation of ε with N is slow for the short wave-lengths around $N = 52$, but for the long wave-lengths there is a definite minimum. Mr. W. A. Murray has shown (private communication) that the maximum and minimum eigenvalues of the ADI matrices for the quasi-circular region are the same as for the smallest square, with sides parallel to the axes, which encloses the region, i.e. $N = 59$; values of $\varepsilon \times 10^5$ for $N = 59$ are included in the table.

Table 2

Values of $\epsilon \times 10^5$ as a function of p and wave-length of initial guess

p	k	Wave-length of initial guess field as multiple of grid-length			
		2	8	32	128
3	3	145·8	115·9	138·2	299·4
4	2	119·7	46·9	165·0	206·9
5	2	11·1	5·8	40·6	46·3
6	1	318·4	333·5	905·7	1790·8
7	1	97·1	154·2	713·9	1401·2
8	1	29·8	81·3	568·2	1112·3
9	1	9·4	53·2	460·7	909·6
10	1	3·3	35·7	379·1	767·0

It is worth while noting that the radius of the circular area is about 29 grid-lengths, and that $29^2\pi \simeq (52 - 1)^2$; that is, the square equivalent to the circular grid was one of about the same area.

The variation of ε with p was much more marked than that with N as is shown in **Table 2**. In this table k is defined as the number of complete cycles of the ADI process used, and is introduced to keep the product pk within the range 6 to 10. (This was because a satisfactory error-reduction for the 47 × 41 rectangular network had been obtained with a cycle of eight optimum parameters.)

It is clear from this table that easily the best reduction of errors of all wave-lengths is obtained by a double cycle of five optimum Jordan parameters, this being much better for most wave-lengths than a single cycle of 10 parameters. Visual examination of grid-point prints of the final error fields confirmed this result.

A test was also carried out on a field of real meteorological data (stream function of the wind-field at 600mb); in this case ε was found to be $0{\cdot}64 \times 10^{-3}$.

It is of interest that the theoretical value of ε for application of a double cycle of five parameters to a 52 × 52 square is about 1×10^{-4}.

Conclusions

It is possible to solve a Poisson equation over a quasi-circular area by means of the Peaceman-Rachford ADI method nearly as accurately as over a square. However, the accuracy of the solution is very sensitive to the choice of iteration parameters which it is necessary to choose by numerical experiment.

Acknowledgements

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Reference

Wachspress, E. L. (1963). *J. Soc. Indust. Appl. Math.*, Vol. 11, No. 4, p. 994.