Systematic generation of ordered sequences using recurrence relations

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The calculation of the number of members of certain sets can be achieved by analysing recurrence relations. In this note, attention is drawn to the use of such recurrence relations for deriving orderings of the members of the sets in a systematic way and for answering questions about places in the orderings. Examples for certain types of restricted permutations are given.

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1. Introduction

For some combinatorial problems which arise in practical computing contexts it is required to perform one or more of the following processes:

(i) compute how many members there are of a certain set;
(ii) list the members of the set in some convenient systematic order;
(iii) determine what place a specified member occupies in a given order;
(iv) determine what member occupies a given place in the order;
(v) select, at random, a member of the set.

Difference equations have often been found convenient routes for solving (i) but they have more rarely been interpreted to assist in (ii) to (v). Some such examples are given in the following sections.

2. Unrestricted permutations

Several algorithms for generating the arrangements of n distinct objects (here taken as the integers $1, 2, \ldots, n$) have been given in the literature (e.g. Johnson, 1963; Lehmer, 1964; Paige and Tompkins, 1960; Wells, 1961) and recently Ord-Smith (1970) has compared the efficiencies of the methods of generation. One proof of the elementary result that the number of such permutations, $U_n = n!$ notices that $U_1 = 1$ and $U_n = nU_{n-1}$, since each permutation of the first n-1 integers can yield n permutations of $1, 2, \ldots, n$ by inserting the integer n into any of the n-2 spaces between the integers or at the beginning or end of the (n-1) permutation. Thus, the method of derivation of the difference equation indicates recursive methods of listing the permutations; the order obtained by inserting the new item from the rightmost positions first are shown in **Fig. 1.**

$n = 2$	$n = 3$	$n = 4$
1 2	1 2 3	1 2 3 4
2 1	1 3 2	1 2 4 3
	3 1 2	1 4 2 3
	2 1 3	4 1 2 3
	2 3 1	. . . .
	3 2 1	. . . .
		2 3 1 4
		2 3 4 1
		2 4 3 1
		4 2 3 1
		. . . .
		. . . .

Fig. 1. Unrestricted permutations

Identification of the position of a given n-permutation in this ordering is itself obtained recursively; if the number of places from the right which the integer n occupies is r_n, the permutation is at position

$$r_n + n(V_{n-1} - 1) \tag{2.1}$$

where V_{n-1} is the position of the $(n-1)$-permutation obtained by deleting n in the ordering of all $(n-1)$-permutations. The converse problem, (iv), derives the permutation for a given position by finding successively the positions of n, $n-1$, $n-2, \ldots$. For example, for $n = 4$ we get the fourteenth permutation by noting $14/4 = 3 +$ remainder 2. Hence $\cdot\cdot\, 4 \,\cdot$; and $(V_3 - 1) = 3$ so that the 3-permutation is fourth in its order: then $4/3 = 1 +$ remainder 1 and so we have $\cdot\cdot\, 3$ for the position of the '3', giving $\cdot\cdot\, 43$ and finally $(V_2 - 1) = 1$ which yields 2 1 4 3 as the fourteenth in the order.

The ordering produced in this way is thus the same as that based upon the representation of integers in the form

$$N \equiv n! \left[\frac{a_n}{n!} + \frac{a_{n-1}}{(n-1)!} \cdots \frac{a_2}{2!}\right] \tag{2.2}$$

where $0 \leq a_2 \leq i$, where a $(1-1)$ correspondence between the integers $\{0, 1, \ldots, (n! - 1)\}$ and all n-permutations is obtained by regarding a_i as the number of digits less than i which appear to the right of it. Thus, the fourteenth permutation follows from $13 \equiv (a_2, a_3, a_4) = (1, 0, 1)$.

The same representation (2.2) is used in a different way by Johnson (1963) to generate all n-permutations successively by single interchanges of adjacent digits.

3. Restricted permutations with repetition

Consider the number U'_n of permutations of n objects drawn with whatever repetition is desired from k distinct objects

$U'_1 = 3$	$U'_2 = 9$	$U'_3 = 24$		
1	11	122	121	231
2	12	133	123	232
3	13	211	131	312
	21	233	132	313
	22	311	212	321
	23	322	213	323
	31	112	221	331
	32	113	223	332
	33			

Fig. 2. Listing of restricted permutations I

(1, 2, 3, . . .) such that no three adjacent objects in the permutations shall be the same.

These n-permutations can be divided into two sets according as their last two objects are the same or are different. Any n-permutation with the last two objects the same can be obtained from an $(n-2)$-permutation by attaching two like symbols to it, as long as they are distinct from the previous final object; there are thus $(k - 1)U'_{n-2}$ such n-permutations. Similarly all n-permutations of the other set are $(k - 1)U'_{n-1}$ in number and are obtained by attaching one different symbol to the end. Hence

$$U'_n = (k - 1)(U'_{n-2} + U'_{n-1}) \tag{3.1}$$

where clearly $U'_1 = k$, $U'_2 = k^2$ and an expression for U'_n can be derived.

Once again, a systematic ordering is derived naturally, but this time the ordering for n-permutations involves both $(n-1)$- and $(n-2)$-permutations. For example, for $k = 3$, the permutations are listed in some order
for $n = 1, 2$. The permutations for $n = 3$ can then be listed by attaching the possible identical pairs of objects in turn to the members of U'_1, followed by those of U'_2 with the possible single object added, and similarly for $n > 3$ (**Fig. 2**).

The identification of the place in the order occupied by a given one of these restricted n-permutations is attained by determining successively which permutation it is derived from, i.e. whether it comes from an $(n-1)$- or an $(n-2)$-permutation and so on until the permutations of one or two elements are reached. The diagram (**Fig. 3**) indicates the process for a five-permutation where D, D' denote different digits.

This algorithm is, of course, a generalisation of the evaluation of a number $d_n d_{n-1} \ldots d_1$ in the scale of t by nested multiplication

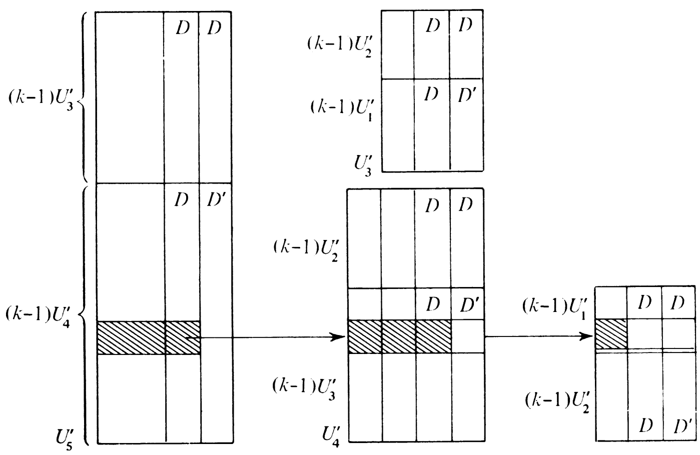

Fig. 3. Identification of a restricted permutation

$U'_1 = 3$	$U'_2 = 9$	$U'_3 = 24$		
1	11	112	123	232
2	12	113	131	233
3	13	221	132	311
	21	223	133	312
	22	331	211	313
	23	332	212	321
	31	121	213	322
	32	122	231	323
	33			

Fig. 4. Listing of restricted permutations II

$$(\ldots(t(td_n + d_{n-1}) + d_{n-2}) + \ldots) .$$

The reverse process of finding the permutation at a given place in the order consists of repeated division by $(k-1)$, subtraction of U_{n-2} if the quotient exceeds it and noting the remainders to permit the building of the permutation from whichever member of the U'_1 1-permutations or U'_2 2-permutations is appropriate.

Just as the unrestricted permutations of Section 2 could be composed from either left to right or in the opposite sense in an order, so can these restricted permutations. The argument for deriving the difference equation (3.1) differs a little. The first two digits of an n-permutation are either the same or different; if the same then we can attach to these digits any $(n-1)$-permutation which does not begin with the same digit as the initial pair—as U'_{n-2}/k of these will begin with each of the digits, there will be a total of $(k-1)U'_{n-2}/k$ $(n-1)$-permutations that can be used for each of the k pairs of like initial digits. A similar argument follows for the unlike digits at the start and we have

$$U'_n = k\left\{\frac{k-1}{k}(U'_{n-1} + U'_{n-2})\right\} . \tag{3.2}$$

If the order of the 1- and 2-permutations are as shown, the order of the 3-permutations follows for $k = 3$ (**Fig. 4**).

A further order, special to this example, comes from noticing the relationship of (3.1) to the Fibonacci numbers and their difference equation

$$f_n = f_{n-1} + f_{n-2}, f_1 = 1, f_2 = 2 . \tag{3.3}$$

Each restricted permutation can be regarded as a sequence of single or double digits and the number of such patterns is f_n by the previous arguments on the last two digits. For any given pattern, e.g. for seven digits, $S_1S_2\ D_1D_1S_3D_2D_2$ where the S's represent single digits and the D's double, the first digit may be selected in k ways and there are $(k-1)$ digits possible at every position except those occurring between two double digits (i.e. D's). Thus, for the pattern above there are $k(k-1)^4$ different permutations and these can be ordered in a natural way numerically; a number representation

$$\sum_{i=1}^{5} a_i(k-1)^{5-i},\ 0 \leq a_1 < k,\ 0 \leq a_i < k-1,\ i = 2, 3, 4, 5$$

is then readily available to identify or to construct the corresponding permutation. The first digit $S_1 = 1 + a_1$ and the i^{th} digit is the $(a_i + 1)^{th}$ available digit of $(1, 2, \ldots, k)$, equality with the $(i-1)^{th}$ digit not being permitted; thus $S_2 = 1 + a_2$ unless $a_2 \geq a_1$ when $S_2 = 2 + a_2$, and so on. Hence, if we are given an order for the f_n patterns, we derive from them an order for the n-permutations. For example, for $n = 3$, the order of the patterns could be $S_1D_1D_1$, $D_1D_1S_1$, $S_1S_2S_3$; then for $k = 3$ the derived order is the same as that shown in **Fig. 5**.

$S_1D_1D_1$			$D_1D_1S_1$			$S_1S_2S_3$					
1	2	2	1	1	2	1	2	1	2	3	1
1	3	3	1	1	3	1	2	3	2	3	2
2	1	1	2	2	1	1	3	1	3	1	2
2	3	3	2	2	3	1	3	2	3	1	3
3	1	1	3	3	1	2	1	2	3	2	1
3	2	2	3	3	2	2	1	3	3	2	3

Fig. 5. Listing of restricted permutations III

Other problems which lead to second order linear difference equations can be treated in the same way. For example, if V_n is the number of valid ALGOL expressions that can be constructed from the symbols +, −, ×, ↑, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 then, since the syntax shows that only

<expression> <sign> <digit>
or <expression> <digit>

are possible we obtain

$$V_n = 10\ V_{n-1} + 40\ V_{n-2} \qquad (3.4)$$

where $V_1 = 10$, $V_2 = 120$. The construction of the order for n-expressions is obtained by systematically adding all possible combinations <sign> <digit> to the $(n-2)$-expressions in their order and then all digits to the $(n-1)$-expressions. For this example, too, the number of valid arrangements of n symbols <sign> and <digit> is f_n, the Fibonacci number given by (3.3), so that the corresponding 'two dimensional' ordering may also be generated.

4. Key permutations

A more complex example is provided by a (slightly simplified) version of a problem posed by a lock and key manufacturer who required a list of the length of teeth of all keys satisfying the following conditions. Each key must have n teeth of different lengths 1, 2, . . . , n units but no adjacent teeth may differ by more than two units. In order to satisfy these conditions we need to construct the subset of those permutations of (1, 2, . . . , n) which satisfy the requirement on adjacent digits. Naturally, a listing of the key permutations could be made in a variety of ways (e.g. by enumeration of all permutations and rejection of unacceptable ones or by a backtrack procedure) but computation of the number of key permutations for general n can be obtained by establishing recurrence relations for different types of key permutations, and a listing then follows in a manner similar to the earlier examples.

Let:

I U_n = no. of key permutations of 1, 2, . . . , n which start $n, n-1, \ldots$

II V_n = no. of key permutations of 1, 2, . . . , n which start $n-1, n, \ldots$

III W_n = no. of key permutations of 1, 2, . . . , n which start $n, n-2, \ldots$ and end with $(n-1)$.

IV T_n = no. of key permutations of 1, 2, . . . , n which start $n, n-2$ and do not end with $(n-1)$.

V S_n = no. of key permutations of 1, 2, . . . , n which has the pair . . . $n-1, n$ . . . at *neither* end.

Then a key $(n+1)$-permutation of type I can be obtained only by putting the tooth of length $(n+1)$ before a key n-permutation of types I, III and IV. Hence

$$U_{n+1} = U_n + W_n + T_n\ .$$

An $(n+1)$-permutation of type II arises only from placing the tooth $(n+1)$ following the n of a type I of n teeth. Hence

$$V_{n+1} = U_n$$

Similarly

$$T_{n+1} = V_n\ .$$

n	3	4	5	6	7	8	9	10
N_n	6	12	20	34	56	88	136	234

Fig. 6. The number of key permutations

We need also to notice that for every key permutation of types I to V, there corresponds exactly one of the type obtained by reading from right to left. Hence we obtain

$$W_{n+1} = W_n$$

by inserting the $(n+1)$ tooth at the right-hand end of a type III (and it is obvious that $W_n = 1$), and reading in reverse and further

$$S_{n+1} = S_n + V_n$$

The total number of key n-permutations is thus

$$N_n = 2(U_n + V_n + W_n + T_n + S_n)$$

Solution of the difference equations shows that

$$N_n = O(\alpha^n)$$

where $\alpha(\doteqdot 1{\cdot}47)$ is the root of largest modulus of

$$x^3 - x^2 - 1 = 0$$

Fig. 6 shows a few values of N_n

Since $W_n = 1$, it is perhaps convenient to place the permutation of type III first in the listing and to follow it with types I, II, IV and V. Thus, half the listing (i.e. L. to R. only) would appear as in **Fig. 7**.

A simpler problem which can be treated in the same way is the ordering of the set of permutations of the first n integers such that no element is more than 2 units greater than both its neighbours; it follows from a similar argument that the number in this set is the integer part of

$$(1+\sqrt{2})\ (2+\sqrt{2})^{n-2}/2\ .$$

5. Random selection

Once the number, N, of members in a set is known and an ordering defined which allows, for given r, the rth member to be constructed, the task of making random selections from the set is theoretically trivial—being reduced just to the selection of integer r $(1 \leq r \leq N)$ with probability $1/N$. However, if N is at all large (e.g. greater than the largest integer contents of a single word in storage) the source of 'random' numbers available may not be sufficiently well defined; for example, the execution of the process

$$[N\ \xi_i] + 1$$

where $\{\xi_i\}$ are the 'random' numbers from the generator

	$n = 3$		$n = 4$		$n = 5$
III	3 1 2	III	4 2 1 3	III	5 3 1 2 4
I	3 2 1	I	4 3 1 2	I	5 4 2 1 3
II	2 3 1		4 3 2 1		5 4 3 1 2
		II	3 4 2 1		5 4 3 2 1
		IV	4 2 3 1		5 4 2 3 1
		V	2 4 3 1	II	4 5 3 1 2
					4 5 3 2 1
				IV	5 3 4 2 1
				V	3 5 4 2 1
					1 3 5 4 2

Fig. 7. Listing of key permutations

assumed uniformly distributed in (0, 1) may not even be able to produce all the integers 1, 2, . . . , N, let alone to give them equal probabilities. Again, the random selection of r would need to be followed by the construction of the corresponding set member and it can be little more work to make several calls upon the random number generator and use the smaller integers produced from these calls to construct the set member. The difference equation approach is suitable for this step by step construction (e.g. for unrestricted permutations see Page, 1967). For the U'_n restricted n-permutations of Section 3 we note that a fraction $p_n = U'_{n-2}/(U'_{n-1} + U'_{n-2})$ begin with two equal digits; hence if $p_n < \xi$, where ξ is a standard uniform variate we start the n-permutation with the pair DD with probability $1/k$, and otherwise with the single digit D, again with probability $1/k$. We continue the construction, choosing whether a single digit or pair of equal ones follows with the correct probability, $U'_{r-2}/(U'_{r-1} + U'_{r-2})$ and then the actual digit with probability $1/k(k-1)$.

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Book review

Elementary Linear Algebra, by Bernard Kolman, 1970; 255 pages. (*Collier-MacMillan Ltd.*, £4·50)

On commencing this book, it seems at first to be wholly admirable, giving a simple, logical and fairly rigorous introduction to finite-dimensional linear spaces, bases and linear transformations, having first introduced matrices and used the methods of elementary row reduction to echelon form of a matrix of a system of simultaneous linear equations to obtain the usual results on consistency and solutions. There is a first chapter on set theory and functions, though this language is not much used subsequently. Determinants are introduced and evaluation is given both by use of cofactors and alternatively by use of row-reduction methods. In the remainder of the book, use is made of eigenvectors to obtain similarity transformations of a matrix to diagonal form and these methods are then applied to symmetric matrices and quadratic forms and to obtaining orthogonal transformations in Euclidean space and, finally, to dealing with linear systems of ordinary differential equations with constant coefficients. The whole text is very adequately illustrated with a good selection of numerical examples.

However, this book seems to be a dangerous one to recommend for first reading by a student. It confines itself to finite dimensional vector spaces with real scalars but these restrictions are often not brought out explicitly enough. Thus we find the statement that a one-one linear transformation between two spaces of equal dimension is necessarily onto and, later, the surprising statement that the eigenvalues of a matrix are the *real* (my italics!) roots of its characteristic equation. A student could easily believe these to be true generally. This also leads to some illogical statements when a brief attempt is made to cover the use of complex scalars. Another fault is that, although there is some attempt to give numerical methods, those given are sometimes not very practical and there is no mention of iterative methods. For example, to obtain eigenvectors it is suggested that the characteristic equation should be solved by repeated bisection methods and these roots then used to obtain eigenvectors; the book limits this method to matrices of order less than 5! A final surprising omission in the last chapter is any consideration of normal modes.

There are few misprints in the book—I noted three affecting the mathematics, the omission of the word 'not' on line 8, p. 194 being perhaps the most important.

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