A further note on top-down deterministic languages*

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Two new families of languages, the $\mathscr{F}(k)$ and $\mathscr{U}(k)$ languages, are introduced each of which is, in some sense, a generalisation of top-down deterministic languages. This leads us to new characterisations of s-languages and $LL(1)$ languages. We include a characterisation of the unambiguous context-free languages, generalisations of the equivalence relation on s-grammars to s-separable sets, a summary of the non-closure results for $LL(k)$, $F(k)$ and $U(k)$ languages, and it is shown that non-degenerate hierarchies exist for the families of $\mathscr{F}(k)$ and $\mathscr{U}(k)$ languages.

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Introduction

A survey of the approaches to top-down deterministic languages has been given in Wood (1969a). However, since that time, some earlier unpublished work by Schorre (1965) and Tixier (1967) has come to light. The aim of this paper is to investigate their approach, relating it to the $LL(k)$ languages of Lewis and Stearns (1968). At the same time we take this opportunity to generalise some results of Korenjak and Hopcroft (1966) and to include a survey of the non-closure results for $LL(k)$ languages, some of which are new.

In Section 2 we introduce separability, in Section 3 f-separability and f-quasi-separability, in Section 4 we deal with generalisations of the equivalence relation for s-grammars and in Section 5 are found the various non-closure counterexamples.

1. Notation

We use ϕ to denote the empty set. A grammar G is a 4-tuple

$$G = (N, T, S, P)$$

where N is a finite set of *nonterminal* symbols, T is a finite set of *terminal* symbols, S in N is the *sentence symbol*, and P is a finite set of *rules* (or productions) of the form $X \to x$, X in N and x in $(N \cup T)^*$. Let $V = N \cup T$. If $X \to x$ in P then x is an *alternative* of X.

In the usual manner the free monoid generated by a set of symbols A, is denoted by A^*, similarly $A^+ = AA^*$. ε denotes the *empty word*. We have the binary relations $\underset{G}{=>}$, $\underset{G}{=>}^+$, $\underset{G}{=>}^*$ or more usually $=>$, $=>^+$, $=>^*$, on words over V^*, which define derivations over G. The *language* generated by a word w is the set $\{x\colon w =>^* x,\ x \text{ in } T^*\}$, written both as $L(w)$ or ,$\tilde{w}$ in this paper. The language generated by the grammar G, denoted $L(G)$, is $L(S)$. The *length* of a word x in V^* is denoted by $|x|$, and is the number of symbols in x, $|\varepsilon| = 0$. For $k > 0$, for all x in V^* let $k\colon x$ be x if $|x| \le k$, otherwise x_1 where $|x_1| = k$ and $x = x_1x_2$.

We say a grammar is *admissible* if for all X in V there exist derivations $S =>^* uXv$ and $X =>^* x$, where u, v in V^*, x in T^*. Henceforth grammar means admissible grammar. The reader is assumed to be familiar with the concept of ambiguity and (left) derivations (see Ginsburg, 1966).

Definition

For $k > 0$, X in N is said to be $LL(k)$ if for all u, v, v', u_1, u_1', x, x', such that

$$S =>^* uXv => uxv =>^* uu_1,$$
$$S =>^* uXv' => ux'v' =>^* uu_1', \text{ and}$$
$$k\colon u_1 = k\colon u_1' \text{ then } x = x', \text{ where}$$

u, u_1, u_1' in T^*, v, v' in V^* and $X \to x$, $X \to x'$ in P. Similarly a grammar G is $LL(k)$ if for all X in N, X is $LL(k)$. L is an $LL(k)$ *language* if it is generated by some $LL(k)$ grammar. Let $\mathscr{L}(k)$ be the family of $LL(k)$ languages and $\mathscr{L} = \bigcup_{\text{all } k} \mathscr{L}(k)$.

Definition

A grammar G is an ε-*free grammar* if for all $X \to x$ in P, x in V^+. An ε-free $LL(1)$ grammar is an *s-grammar*†, similarly as *s-language* is a language that can be generated by an s-grammar.

A set X is ε-*free* if ε not in X.

Definition

A nonterminal X *has a cycle* (or is *cyclic*) if there exists a derivation $X =>^+ uXv$, u, v in V^*, $uv \ne \varepsilon$. If $uv = \varepsilon$ then X *has a loop*. A grammar G has a cycle (loop) if at least one nonterminal in G has a cycle (loop). If X, Y in N have cycles and there exists no derivation $X =>^+ uYv$, u, v in V^*, then X, Y are said to have *disjoint cycles*.

2. Separability and context-free grammars

Schorre (1965) introduces the concept of separability which is further explored by Tixier (1967). We extend the notion of separability in order to apply it to arbitrary context-free grammars. By 'set' we will mean 'a set of words over some terminal alphabet T'.

Definition

The *left quotient* (*right quotient*) of a set X by a set Y is $\{u\colon vu \text{ in } X, v \text{ in } Y\}$ ($\{u\colon uv \text{ in } X, v \text{ in } Y\}$) denoted by $Y\backslash X$ (X/Y).

Definition

The *proper left quotient* of a set X by a set Y is $\{u\colon vu \text{ in } X,\ v \text{ in } Y,\ u \ne \varepsilon\}$ denoted by $Y.\backslash X$. Similarly we define *proper right quotient* denoted by $X/.Y$.

Definition

Let (A, B) denote an ordered pair of sets. We say (X, Y) is *separable* if $X.\backslash X \cap Y/.Y = \phi$. Let

$I(X, Y)$ denote the set $X.\backslash X \cap Y/.Y$.

Remarks

1. It follows that if ε is in a set X then

$$X - \{\varepsilon\} \subseteq X.\backslash X \text{ and } X - \{\varepsilon\} \subseteq X/.X.$$

2. If x in T^* then $\tilde{x}.\backslash\tilde{x} = \tilde{x}/.\tilde{x} = \phi$.

Definition

A nonterminal X is *separable* if for all $X \to x$ in P, where x

†The original definition for s-grammars, which is equivalent to this one, is: an s-grammar is an ϵ-free $LL(1)$ grammar in Greibach normal form.

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is in VV^+ for all Y, Z in V such that $x = uYZv$, u, v in V^*, $(\tilde{Y}, \tilde{Z})$ is separable. A grammar is *separable* if all X in N are separable.

Lemma 1

Given a grammar G and Y, Z in N such that $(\tilde{Y}, \tilde{Z})$ is separable then it follows that:

(i) for all $Y =>^+ yW$, y in V^*, W in V, $(\tilde{W}, \tilde{Z})$ is separable,
(ii) for all $Z =>^+ Uz$, z in V^*, U in V, $(\tilde{Y}, \tilde{U})$ is separable,
(iii) for all W and for all U as above, $(\tilde{W}, \tilde{U})$ is separable.

Proof:

We will prove (i) in detail, (ii) and (iii) follow similarly.

(i) Assume the contrary, then $I(\tilde{W}, \tilde{Z}) \neq \phi$ therefore there exists at least one p in $I(\tilde{W}, \tilde{Z})$. Thus there exists p_1 in $\tilde{W}$ such that $p_1 p$ is in $\tilde{W}$, therefore qp_1 and qp_1p are in $\tilde{Y}$ for some q in $\tilde{y}$. Therefore p is in $I(\tilde{Y}, \tilde{Z})$ giving a contradiction. The result follows.

Corollary 2

If G is a separable grammar then for all X, Y in V such that there exists a derivation $Z =>^+ uXYv$, $(\tilde{X}, \tilde{Y})$ is separable.

The converse of Lemma 1 obviously does not hold.

Definition

For $n \geqslant 2$, an n-tuple of sets $(X_1, \ldots, X_n)$ is *R separable* if for all i, $1 \leq i < n$, $(X_i, X_{i+1} \ldots X_n)$ is separable.

Definition

A nonterminal X is *R separable* if for all $X \to X_1 \ldots X_n$ in P, $n \geq 2$, $(\tilde{X}_1, \ldots, \tilde{X}_n)$ is R separable. A grammar is *R separable* if all X in N are R separable.

We now give a lemma that demonstrates the behaviour of R separability under substitution.

Lemma 3

Given a grammar G and some $X \to X_1 \ldots X_n$ in P, $n \geq 2$, such that $(\tilde{X}_1, \ldots, \tilde{X}_n)$ is R separable then it follows that:

(i) for all i, $1 \leq i < n$, for all $X_i =>^+ yY$, y in V^*, Y *in* V, $(\tilde{Y}, \tilde{X}_{i+1}, \ldots, \tilde{X}_n)$ is R separable.
(ii) if for some i, $1 \leq i \leq n$, $X_i \to Y_1 \ldots Y_m$ is in P, $m \geq 2$ and $(\tilde{Y}_1, \ldots, \tilde{Y}_m)$ is R separable, then $(\tilde{Y}_1, \ldots, \tilde{Y}_m, \tilde{X}_{i+1}, \ldots, \tilde{X}_n)$ is R separable.

Proof:

(i) follows directly from Lemma 1.

(ii) Assume the contrary, then there exists a j, $1 \leq j \leq m$, such that $(\tilde{Y}_j, \tilde{Y}_{j+1} \ldots \tilde{Y}_m \tilde{X}_{i+1} \ldots \tilde{X}_n)$ is not separable. Let $\tilde{y}$ denote $\tilde{Y}_{j+1} \ldots \tilde{Y}_m$ and $\tilde{x}$ denote $\tilde{X}_{i+1} \ldots \tilde{X}_n$. This implies there exists a q in $I(\tilde{Y}_j, \tilde{y}\tilde{x})$, a p in $\tilde{Y}_j$ and an r in $\tilde{y}\tilde{x}$ such that pq is in $\tilde{Y}_j$ and qr in $\tilde{y}\tilde{x}$. Now $r = r'_1 r'_2$ where r'_1 in $\tilde{y}$, r'_2 in $\tilde{x}$.

(*a*) $qr = qr_1r_2$ where qr_1 in $\tilde{y}$, r_2 in $\tilde{x}$.

If $r_2 = r'_2$ then q in $I(\tilde{Y}_j, \tilde{y})$ which is a contradiction as $(\tilde{Y}_j, \tilde{y})$ is separable.

If $|r_2| > |r'_2|$ let $r_2 = r_{21}r'_2$ then r_{21} in $I(\tilde{X}_i, \tilde{x})$ as both pqr'_1 in $\tilde{Y}_j\tilde{y}$ and pqr_1 in $\tilde{Y}_j\tilde{y}$. If $|r_2| < |r'_2|$ a similar argument holds.

(*b*) $qr = q_1q_2r$ where q_1 in $\tilde{y}$, q_2r in $\tilde{x}$. As r'_2 in $\tilde{x}$ we have $q_2r'_1$ in $\tilde{x}/.\tilde{x}$ and as p,pq in $\tilde{Y}_j$, q_1, r'_1 in $\tilde{y}$ we have $q_2r'_1$ in $\tilde{Y}_j\tilde{y}.\backslash \tilde{Y}_j\tilde{y}$.

Both cases lead to $I(\tilde{X}_i, \tilde{x}) \neq \phi$, a contradiction.

Corollary 4

If G is an R separable grammar then G is separable. This is really a corollary to the definition of R separability.

Corollary 5

If G is an R separable grammar then for all X_i in V such that there exists a derivation $Z =>^+ X_1 \ldots X_n$, $n \geq 2$, $(X_1, \ldots, X_n)$ is R separable.

Definition

We say an n-tuple $(X_1, \ldots, X_n)$, $n \geq 2$, is *L separable* if for all i, $1 < i \leq n$, $(X_1 \ldots X_{i-1}, X_i)$ is separable. We can extend the definition to nonterminals and grammars. We now give two examples which illustrate separability.

Example 1

Let $X \to X_1X_2X_3$ be a rule of some grammar, where $\tilde{X}_1 = \{a\}^* = \tilde{X}_3$ and $\tilde{X}_2 = \{a\}$.
Then $(\tilde{X}_1, \tilde{X}_2)$ and $(\tilde{X}_2, \tilde{X}_3)$ are separable but $(\tilde{X}_1, \tilde{X}_2\tilde{X}_3)$ and $(\tilde{X}_1\tilde{X}_2, \tilde{X}_3)$ are not separable.
Thus $(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3)$ is not L or R separable (note that the grammar is ambiguous).

Example 2

Let X be as above, where $\tilde{X}_1 = \{a, \varepsilon\}$, $\tilde{X}_2 = \{b, \varepsilon\}$ and $\tilde{X}_3 = \{a\}$. Then $(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3)$ is both L and R separable and X is therefore separable.

This leads to the following lemma.

Lemma 6

A grammar is L separable if and only if it is R separable.

Proof:

This is not included as it is very similar to that used in Lemma 3.

Definition

A grammar is *s(uper)-separable* if it is R and hence L separable.

Corollary 7

If G is an s-separable grammar then for all X_i in V such that there exists a derivation $Z =>^+ X_1 \ldots X_n$, $n \geq 2$, $(\tilde{X}_i, \ldots, \tilde{X}_j)$ is s-separable for all i, j, $i \geqslant 1$, $j \leqslant n$, $i < j$.

Remark

s-separability formalises an intuitive notion of unambiguity, i.e. if $(\tilde{X}, \tilde{Y})$ is separable then we know that there is no subword v, for which there exists x in $\tilde{X}$, y in $\tilde{Y}$ such that xv in $\tilde{X}$ and vy in $\tilde{Y}$. Therefore it follows that each word x in $\tilde{X}\tilde{Y}$ can be uniquely partitioned into x_1x_2 with x_1 in $\tilde{X}$, x_2 in $\tilde{Y}$. Thus s-separability is a necessary condition for unambiguity, that it is not sufficient is shown by the next example.

Example 3

Let $G = (\{S, X_1, X_2, Y_1, Y_2\}, \{a, b\}, S, \{S \to X_1Y_1 | X_2Y_2, X_1 \to a, X_2 \to a, Y_1 \to b, Y_2 \to b\})$.

G is trivially s-separable and ambiguous.

We are now in a position to prove the following necessary and sufficient condition for unambiguity.

Theorem 8

A grammar G is unambiguous if and only if the following property is satisfied:

(i) for all X in N, for all $X \to x_1$, $X \to x_2$ in P, $\tilde{x}_1 \cap \tilde{x}_2 = \phi$, $x_1 \neq x_2$.
(ii) G is s-separable. This is *Property A*.

Proof:

As necessity follows trivially consider sufficiency. Assume the property holds and G is ambiguous. There exists at least one word x in $L(G)$ such that x has at least two distinct left derivation sequences, let these be $\{v_i\}$ and $\{w_i\}$, where $v_0 = w_0 = S$ and $v_p = w_q = x$. There exists j, $0 \leq j < \min(p, q)$ such that

$$v_i = w_i,\ 0 \leq i \leq j \text{ and } v_{j+1} \neq w_{j+1}.$$

Let $v_j = aXv$, $v_{j+1} = ax_1v$, $w_{j+1} = ax_2v$, a in T^*, X in N, v in V^*, where $X \to x_1$, $X \to x_2$ in P, $x_1 \neq x_2$.

Now either $x_1 =>^* a_1$, $x_2 =>^* a_1$, $v =>^* a_2$, a_1, a_2 in T^*, where

$$x = a\,a_1a_2. \text{ Thus } \tilde{x}_1 \cap \tilde{x}_2 \neq \phi$$

or

$$x_1 =>^* a_1,\ x_2 =>^* a'_1,\ a_1, a'_1 \text{ in } T^*,\ |a_1| < |a'_1|$$

say, and $v =>^* a_2$, $v =>^* a'_2$, a_2, a'_2 in T^* with $a_1a_2 = a'_1a'_2$. Thus $I(\tilde{X}, \tilde{v}) \neq \phi$, which by Corollary 7 contradicts the second condition. The theorem follows.

Definition

G is a *binary grammar* if for all $X \to x$ in P, $x = \varepsilon$ or $x = X_1X_2$, X in N, X_1, X_2 in V.

Schorre (1965) proved the following result.

Corollary 9

A binary grammar is unambiguous if and only if the following property is satisfied:

(i) for all X in N, for all $X \to x_1$, $X \to x_2$ in P, $x_1 \neq x_2$, $\tilde{x}_1 \cap \tilde{x}_2 = \phi$.

(ii) G is separable. This is *Property B*.

Proof:

(X, Y) is s-separable if and only if (X, Y) is separable.

Remark

The proof of Theorem 8 can be followed through by replacing s-separability with separability, and thus Lemma 6 is not necessary for this proof. Thus a direct proof of Corollary 9 would be much shorter than that originally given by Schorre (1965). Property B can be applied whenever at most two nonterminals appear together in the right sides of the rules in a grammar, so we obtain the following result.

Definition

A grammar is in ε-$(k, 2)$-normal form, $k \geq 1$ if for all $X \to x$ in P, either x in T^*, $|x| \leq k$ or $x = ay$, a in T^*, $|a| = k$, y in $N \cup NN$.

It is known (Wood, 1970), that every grammar can be put in ε-$(k, 2)$ normal form.

Corollary 10

An ε-$(k, 2)$ normal form grammar is unambiguous if and only if Property B holds.

Definition

A grammar is an *operator grammar* if for all $X \to x$ in P, x in $T^* (NT^*)^*$ (Greibach, 1965).

Corollary 11

An operator grammar is unambiguous if for all X in N, for all $X \to x_1$, $X \to x_2$ in P, $x_1 \neq x_2$, $\tilde{x}_1 \cap \tilde{x}_2 = \phi$.

3. f-separability and k: separability

Definition

(X, Y) is *f-separable* if

$$f(X.\backslash X) \cap f(Y/.Y) = \phi$$

where f is a map from T^+ into some set D.

Letting $f(u) = u$ for all u in T^+ then we have separability. We extend the definition to give *f-s-separability* in the obvious way. We are particularly interested in the following special map.

Definition

(X, Y) is k:*separable* if $k\colon (X.\backslash X) \cap k\colon (Y/.Y) = \phi$. We define k:*s-separable* in a similar way; let $I_k(X, Y)$ denote the set $k\colon (X.\backslash X) \cap k\colon (Y/.Y)$.

Definition

Given an integer k, $k > 0$ a grammar G is a $U(k)$ *grammar* if the following property is satisfied:

(i) for all X in N, for all $X \to x_1$, $X \to x_2$ in P, $x_1 \neq x_2$, $k{:}\tilde{x}_1 \cap k{:}\tilde{x}_2 = \phi$.

(ii) G is k:s-separable.

Definition

(X, Y) is *f-q(uasi)-separable* if

$$f(X.\backslash X) \cap f(Y) = \phi$$

where f is a map from T^+ into some set D. Similarly define k:*q-separable* and k:*q-s-separable*.

Definition

Given an integer k, $k > 0$, a grammar G is an $F(k)$ *grammar* if the following property holds:

(i) for all X in N, for all $X \to x_1$, $X \to x_2$ in P, $x_1 \neq x_2$ $k{:}\tilde{x}_1 \cap k{:}\tilde{x}_2 = \phi$.

(ii) G is k:*q-s-separable*.

$F(k)$ grammars are a generalisation of the *RCF* (regular context free) grammars of Tixier (1967) and the *FCR* (first character recognition) grammars of Schorre (1965).

Definition

A language is a $U(k)$ or $F(k)$ *language* if it is generated by a $U(k)$ or $F(k)$ grammar. Let $\mathscr{U}(k)$ and $\mathscr{F}(k)$ denote the families of $U(k)$ and $F(k)$ languages. Further let $\mathscr{U} = \{L$: there exists a $k > 0$, such that L is in $\mathscr{U}(k)\}$ and similarly define $\mathscr{F}$.

Let us look at some basic properties of these families. From the definitions and Theorem 8 we have:

Corollary 12

For all L, L in $\mathscr{U}$, L is unambiguous.

However, because of the definition of q-separability we have:

Letting $X = \{c, \varepsilon\}$ and $Y = \{ca^ib^ic^j,\ a^ib^jc^j\colon i, j \geq 1\}$ we have $\{ca^ib^ic^j,\ ca^ib^jc^j\colon i, j \geq 1\} \subset XY$ and (X, Y) is 2: q-separable. Further as both X and Y can be generated by 1: q-s-separable rules we obtain the following result.

Theorem 13

$\mathscr{F}$ contains (inherently) ambiguous languages. We now exhibit a language which is not in $\mathscr{F}$.

Theorem 14

$L_1 = \{a^i, a^ib^i\colon i \geq 1\}$ is not in $\mathscr{F}$, and therefore $\mathscr{F}$ is a proper subset of the family of context-free languages.

Proof:

If G is such that $L(G) = L_1$ then G must have two disjoint cycles and therefore at least three nonterminals. Otherwise $L(G)$ would contain words of the form a^ib^j, $i > j$, thus we need at least one nonterminal for each cycle and one nonterminal that branches to either cycle, let this be X. Then X has at least two alternatives, one of which leads to words of the form a^i and the other to words of the form a^ib^i; let these be $X \to x_1$ and $X \to x_2$. Then for any $k > 0$, $k\colon \tilde{x}_1 \cap k\colon \tilde{x}_2 \neq \phi$.

Corollary 15

L_1 is not in $\mathscr{U}$.

It turns out that $\mathscr{U}$ and $\mathscr{F}$ are incomparable, an unexpected result.

Theorem 16

$L_2 = \{a^ib\,c^j\colon i \geq j \geq 1\}$ is in $\mathscr{U}$ but not in $\mathscr{F}$.

Proof:

The rules $S \to AB$, $A \to aA|\varepsilon$, $B \to aBc|b$ are l:s-separable; therefore L_2 is in $\mathscr{U}(1)$.

There are three distinct ways in which L_2 can be generated, these are:

(a) form $a^ia^jb\,c^j$
(b) form $a^ja^ib\,c^j$
(c) form $a^jb(c|\varepsilon)^j$

Each of these can be shown to be non-generable by an $F(k)$ grammar for any $k > 0$; in a similar manner to the proof of Theorem 14. We will prove (a) only. There must be two disjoint cycles, one to generate a^i, one to generate $a^jb\,c^j$ and there must be a nonterminal that branches to both cycles, so that $a^ia^jb\,c^j$ can be formed. Let this be the rule $X \to AB$, where $\tilde{A} = \{a^i\}$, $\tilde{B} = \{a^ib\,c^i\}$ then $\tilde{A}.\backslash\tilde{A} = \{a^i\}$ and therefore

$$k\colon (\tilde{A}.\backslash\tilde{A}) \cap k\colon \tilde{B} \neq \phi \text{ for any } k > 0.$$

We now show every $LL(k)$ grammar is $U(k)$.

Theorem 17

For all $k > 0$, each $LL(k)$ grammar is $U(k)$.

Proof:

Each $LL(k)$ grammar trivially satisfies the first condition for a grammar to be $U(k)$, therefore it remains to show that each $LL(k)$ grammar is k:s-separable. Proceed by contradiction. Given G, an $LL(k)$ grammar assume it is not a $U(k)$ grammar. Then there exists at least one rule $Y \to xXy$ such that $(\tilde{X}, \tilde{y})$ is not k:separable. This implies: b in $I_k(\tilde{X}, \tilde{y})$. If bc in $I(\tilde{X}, \tilde{y})$, c in T^* then G is ambiguous, therefore we must have bc in $\tilde{X}.\backslash\tilde{X}$ and bd in $\tilde{y}/.\tilde{y}$, c, d in T^*, $|b| = k$ and $cd \neq \varepsilon$.

Thus

abc in $\tilde{X}$ for some a in $\tilde{X}$ and

bde in $\tilde{y}$ for some e in $\tilde{y}$.

We have

$$Y => xXy =>^* u_1Xy =>^* u_1ay, u_1 \text{ in } T^*, \quad (1)$$

and

$$Y => xXy =>^* u_1Xy =>^* u_1abcy, \quad (2)$$

which in turn implies there exists Z in N such that

$$X =>^* a_1Z => a_1z_1 =>^* a_1a_2 \quad (3)$$

$$X =>^* a_1Z => a_1z_2 =>^* a_1a_2b\,c, \quad (4)$$

where

$a_1a_2 = a$, a_1, a_2 in T^*, $z_1 \neq z_2$, $Z \to z_1$, $Z \to z_2$ in P.

Thus letting $S =>^* uYv$, combining (1) with (3) and (2) with (4) and noticing that in (1) $y =>^* bde$, we have that Z is not $LL(k)$.

By an almost identical proof, we also have:

Corollary 18

For all $k > 0$ $\mathscr{L}(k) \subset \mathscr{F}(k)$.

We now compare $\mathscr{L}(1)$, $\mathscr{U}(1)$, $\mathscr{F}(1)$.

Theorem 19

(i) $\mathscr{L}(1) = \mathscr{F}(1)$

(ii) $\mathscr{L}(1) \neq \mathscr{U}(1)$

Proof:

(i) Because of Corollary 18 we need only show that $1{:}q$-s-separability implies $LL(1)$. Assume it does not. Then given G, an $F(1)$ grammar there exists at least one X in N, which is not $LL(1)$. Thus

$$S =>^* uXv => ux_1v =>^* uu_1,$$
$$S =>^* uXv' => ux_2v' =>^* u\,u_1', u, u_1' \text{ in } T^*,$$
$$v, v' \text{ in } V^*, X \to x_1, X \to x_2 \text{ in } P,$$
$$1{:}\,u_1 = 1{:}\,u_1' \text{ but } x_1 \neq x_2.$$

(a) ε in X.

Then as either $u_1 = \varepsilon$ and $u_2 = \varepsilon$ implies ε in $\tilde{x}_1 \cap \tilde{x}_2$ or $\tilde{X} - \{\varepsilon\} \subset \tilde{X}.\backslash\tilde{X}$ therefore $1{:}\,u_1$ in $1{:}(\tilde{X}.\backslash\tilde{X})$ and $1{:}\,u_1'$ in $1{:}\,\tilde{v}$.

(b) ε not in $\tilde{X}$.

$$1{:}\,\tilde{x}_1 \cap 1{:}\,\tilde{x}_2 \neq \phi.$$

In both cases we have a contradiction.

(ii) $L_3 = \{a^i(b|bbd)^i: i \geq 1\}$ is known to be $LL(2)$ but not $LL(1)$ (Rosenkrantz and Stearns, 1970). It can be generated by a $U(1)$ grammar, however. Let

$$G = (\{S, A, B, C\}, \{a, b, d\}, S, P)$$

where

$P = \{S \to aA,$
$\quad A \to aAB|B,$
$\quad B \to bC,$
$\quad C \to bd|\varepsilon\}.$

As $\tilde{B}/.\tilde{B} = \phi$ the grammar is trivially $U(1)$, but not $F(1)$. We have the weaker characterisation:

Corollary 20

A grammar is an s-grammar iff it is an ε-free $U(1)$ grammar.

Thus the $\mathscr{U}(k)$ languages can be thought of as a generalisation of s-languages while the $\mathscr{F}(k)$ languages are a generalisation of the $LL(1)$ languages.

Because L_1 in Theorem 14 is deterministic but not in $\mathscr{U}$ we are lead to the following result.

Theorem 21

$\mathscr{U}$ and the family of deterministic languages are incomparable.

Proof:

Let $L_3 = \{ww^R: w \text{ in } \{a, b\}^*\}$, then L_3 is not deterministic but it is generated by the rules

$$S \to aSa|bSb|\varepsilon$$

which are trivially $U(1)$ and $F(2)$.

Corollary 22

$$\mathscr{F} \cap \mathscr{U} \supset \mathscr{L}.$$

We now show that a nontrivial hierarchy exists for both the $\mathscr{F}(k)$ and $\mathscr{U}(k)$ families.

Theorem 23

For any $k \geq 1$, $L_4(k) = \{a^i(b|b^{k+1}d)^i: i \geq 1\}$ not in $\mathscr{F}(k)$.

Proof:

Any grammar generating $L_4(k)$ must have a cycle for some nonterminal X, say, i.e.

$$X =>^+ uXv$$

where $u =>^* a^m$, $v =>^* (b|b^{k+1}d)^m$, $m \geq 1$. This implies: b^kd in $\tilde{X}.\backslash\tilde{X}$ and b^k in $k{:}\,\tilde{v}$. Thus

$$k{:}(\tilde{X}.\backslash\tilde{X}) \cap k{:}\,\tilde{v} \neq 0.$$

However it is $F(k + 1)$.

Corollary 24

$\mathscr{F}(k) \subset \mathscr{F}(k + 1)$ for all $k \geq 1$.

Theorem 25

For any $k \geq 1$, $L_5(k) = \{a^i(d|b^kd|db^{k+1})^i: i \geq 1\}$ not in $\mathscr{U}(k)$, but in $\mathscr{U}(k + 1)$. Thus $\mathscr{U}(k) \subset \mathscr{U}(k + 1)$, for all $k \geqslant 1$.

Proof:

As in Theorem 23 we find an X in N such that $X =>^+ uXv$, b^{k+1} in $\tilde{X}.\backslash\tilde{X}$ and b^k in $\tilde{v}/.\tilde{v}$; thus $k{:}(\tilde{X}.\backslash\tilde{X}) \cap k{:}(\tilde{v}/.\tilde{v}) \neq \phi$, therefore $L_5(k)$ is not $U(k)$, but it is $U(k + 1)$.

Operations on $U(k)$ and $F(k)$ languages

By noting that $L_1 = \{a^i, a^ib^i: i \geq 1\}$ is neither $U(k)$ nor $F(k)$ for any $k > 0$, but that it is the union of two languages which are both $U(1)$ and $F(1)$, we can construct the various non-closure results given in **Table 1**, in a similar manner to those given for $LL(k)$ languages in Section 5.

4. Relations and context-free grammars

Korenjak and Hopcroft (1966) (henceforth let KH denote this reference) solved the equivalence problem for s-grammars by considering a relation on V^+ (in fact, an equivalence relation). Attempts since then have been made, without success, to extend this method to solve the equivalence problem for $LL(k)$ grammars, although this problem has been solved by a different route in Rosenkrantz and Stearns (1970). Tixier extended the relation to $LL(1)$ grammars; we now extend it to s-separable sets. Wood (1971) gives some further results.

Definition

A nonterminal X in N has the *prefix property* if $X =>^* x = yz$ ($z \neq \varepsilon$) then $X \not=>^+ y$. A set X is a *prefix set* if for all x in X there exists no y, $y \neq \varepsilon$, such that xy in X. We note that X is a prefix set iff $X.\backslash X = \phi$.

The following result is taken from KH.

Lemma 26

Given G, an s-grammar, every X in N has the prefix property.

Proof:

Assume the contrary then there exist words x, y in $\tilde{X}$, z in T^*, $z \neq \varepsilon$, such that $x = yz$. Now there is a left derivation sequence

$$S => A_1x_1 => A_1A_2\,x_2 => \ldots => A_1 \ldots A_{n-1}x_{n-1},$$

where $x_{n-1} \to A_n$ in P and $A_1 \ldots A_n = y$. Because of the determinism of s-grammars x must have the same derivation sequence, but as $x_{n-1} \to A_n$ in P there cannot be any $x_{n-1} \to A_nx_n$ in P. Therefore the result follows.

If S has the prefix property then $L(G)$ is said to have the prefix property. Therefore every s-language is a prefix set.

Definition

If X is a set of words let $sh(X)$ denote the length of a shortest word in X, defined as follows: if

$$X = \phi, sh(X) = -1$$

otherwise, $sh(X) = |x|$, x in X such that there exists no y in X, $|y| < |x|$.

Definition

A grammar G is in *(1,2)-normal form* if for all $X \to x$ in P either

x in T, x in TN or x in TNN.

Definition

If X, Y are sets of words from T^* and for all x, x in X if and only if x in Y then write $X \equiv Y$ (i.e. $X \equiv Y$ if and only if $X = Y$). We extend this to the catenation product of sets of words and to words over V (i.e. $x \equiv y$ if and only if $\tilde{x} = \tilde{y}$). We say $X \equiv Y$ is an *equivalence pair*.

As in KH we have:

Lemma 27

The relation '$\equiv$' is a congruence relation under catenation product.

Proof:

(i) reflexive: $X \equiv X$,
(ii) symmetric: $X \equiv Y$ implies $Y \equiv X$,
(iii) transitive: $X \equiv Y$ and $Y \equiv Z$ implies $X \equiv Z$,
(iv) catenation: $X \equiv Y$ and $W \equiv Z$ implies $XW \equiv YZ$.

Considering the corresponding sets of words (i)-(iv) follow trivially.

Remark

Note that the above lemma holds for any sets of words, however they are generated.

We now assume that for any set X, X is nonempty.

Definition

We say $\pi(X_1, \ldots, X_n)$ iff $(X_1, \ldots, X_n)$ is s-separable.

Lemma 28

If $\pi(X, Z)$, $\pi(W, Y)$, $W \equiv X$ and $WY \equiv XZ$ then $Y \equiv Z$.

Proof:

Assume otherwise. There exists a shortest word a that contradicts $Y \equiv Z$; without loss of generality assume a in Y.

Let b be a shortest word in W (and hence in X); then

$$ba \text{ in } WY \text{ and } ba \text{ in } XZ.$$

Now no proper prefix of b can be in X, by construction, therefore ba_1 in X, a_2 in Z, $a_1a_2 = a$, $a_1 \neq \varepsilon$. Further a_2 in Y as a is a shortest word that contradicts $Y \equiv Z$. Therefore b, ba_1 in W, a_1a_2, a_2 in Y contradicts $\pi(W, Y)$.

Corollary 29. *Left cancellation*

If $\pi(W, Z)$, $\pi(W, Y)$ and $WY \equiv WZ$ then $Y \equiv Z$.

Corollary 30

If $\pi(X, Z)$, $\pi(W, Y)$, $Y \equiv Z$ and $WY \equiv XZ$ then $W \equiv X$.

Corollary 31. *Right cancellation*

If $\pi(W, Y)$, $\pi(X, Y)$ and $WY \equiv XY$ then $W \equiv X$.

A useful operation is substitution of equivalences, which preserves equivalences.

Lemma 32

If $\pi(X, Y)$, $\pi(W, Y)$ and $XY \equiv Z$ then $X \equiv W$ iff $WY \equiv Z$.

Proof:

if: $XY \equiv Z$ and $WY \equiv Z$ then $XY \equiv WY$ and $X \equiv W$.
only if: $X \equiv W$ and $Y \equiv Y$ then $XY \equiv WY$ giving $WY \equiv Z$.

KH give two transformations on equivalence pairs; we extend these in a natural way.

Definition. The A-transformation

Given an ε-free set X we let $X(a)$ denote the subset of X defined as $\{u_1 : u \text{ in } X, u = au_1\}$.

Given the equivalence pair $X_1 \ldots X_n \equiv Y_1 \ldots Y_m$ form $X_1(a)$ and $Y_1(a)$ for all a in T. We replace the equivalence pair by a set of new equivalence pairs

$$X_1(a)X_2 \ldots X_n \equiv Y_1(a)Y_2 \ldots Y_m \text{ for all } a \text{ in } T.$$

We have trivially:

Lemma 33

In the above definition

$$X_1 \ldots X_n \equiv Y_1 \ldots Y_m \text{ iff for all } a \text{ in } T$$
$$X_1(a)\, X_2 \ldots X_n \equiv Y_1(a)\, Y_2 \ldots Y_m.$$

Remarks

(i) Note that $X(a) = \{a\}\backslash X$.
(ii) If the equivalence pair is over $V^+ \times V^+$ then the A-transformation is carried out as a left substitution, followed by a collecting of terms, as more than one alternative of X_1 may begin with a specific terminal symbol.
(iii) Note that $X_1(a)$ or $Y_1(a)$ for some a in T may contain the empty word. For s-languages (and grammars) we have ε in $X_1(a)$ iff ε in $Y_1(a)$ for any a in T. However, this is obviously not true for $LL(k)$ languages (and grammars) in general.

From Remark (i) above we can infer the more general result.

Corollary 34

If $X_1 \ldots X_n \equiv Y_1 \ldots Y_m$ and $X \subseteq T^*$ then

$$X\backslash(X_1 \ldots X_n) \equiv X\backslash(Y_1 \ldots Y_m).$$

We now generalise the B-transformation of KH.

Definition. The B-transformation

Let $X_1 \ldots X_n \equiv Y_1 \ldots Y_m$, a in X_1 and $aZ \subseteq Y_1 \ldots Y_l$, $l \geq 1$, for some set Z such that there exists no set Z_1, $Z \subset Z_1$ with $aZ_1 \subseteq Y_1 \ldots Y_l$, then replacing

$$X_1 \ldots X_n \equiv Y_1 \ldots Y_m \text{ by}$$
$$X_2 \ldots X_n \equiv ZY_{l+1} \ldots Y_m \text{ and } X_1Z \equiv Y_1 \ldots Y_l$$

we have the B-transformation.

Theorem 35

$$X_1 \ldots X_n \equiv Y_1 \ldots Y_m \text{ iff}$$
$$X_2 \ldots X_n \equiv ZY_{l+1} \ldots Y_m \text{ and } X_1Z \equiv Y_1 \ldots Y_l.$$

Proof:

As in KH.

Further, s-separability is preserved.

Considering equivalences on s-grammars we have the following corollaries.

Noting that Z will have the form $Z_1 \ldots Z_p$, Z_i in V, $p \geq 0$, we have:

Corollary 36

$$0 \leq p \leq |a| + 1, \text{ if } Y_1 \ldots Y_q \not\Rightarrow^+ ay, \text{ for } q < l.$$

Definition

Given a grammar G_i, let

$$t_i = \max(\{sh(\tilde{X}) : X \text{ in } N_i\}) \text{ and let } t = \max(\{t_i\}),$$

all i.

This leads to the following corollary.

Corollary 37

If $X_1 \ldots X_n \equiv Y_1 \ldots Y_m$ is an equivalence on s-grammars G_1 and G_2, then

(i) $sh(X_1 \ldots X_n) = sh(Y_1 \ldots Y_m)$
(ii) $1 \leq m \leq nt$, i.e. the length of the right side is bounded by the length of the left side.

Then we have the special case of KH.

Corollary 38

If $n \leq t + 3$ then the left sides generated by the B-transformation have length at most $t + 2$, and therefore the right sides have length at most $t(t + 2)$.

Remark

So far the underlying properties of equivalence pairs have not been examined in much detail. Consider the following simple question for prefix sets:

if

$$X_1X_2 \equiv Y_1Y_2 \text{ and } sh(X_1) = sh(Y_1)$$

then is it true that $X_1 \equiv Y_1$?

The results above do not answer this and other related questions, therefore the remainder of the section will investigate these problems.

Definition
Given two sets X, Y then we say:
(i) $X \cdot < Y$, *X is left string contained in* Y, if for all x in X, xy in Y for some y in T^*,
(ii) $X \cdot > Y$, *X left string contains* Y, if for all x in X, there exists y in Y and z in T^* such that $x = yz$,
(iii) $X < \cdot\, Y$, *X is right string contained in* Y, if for all x in X, yx in Y for some y in T^*,
(iv) $X > \cdot\, Y$, *X right string contains* Y, if for all x in X, there exists y in Y and z in T^*, such that $x = zy$.

Similarly we can define $x \cdot <, \cdot >, < \cdot, > \cdot\, y$ for words x, y in T^*.
We also have:

Definition
Given two sets X, Y then
(i) $X \cdot \subset Y$ if $X \cdot < Y$ and $Y \cdot > X$,
(ii) $X \supset \cdot\, Y$ if $X > \cdot\, Y$ and $Y < \cdot\, X$.
We now have the following theorem, which relates equivalence pairs, shortest words and prefix sets.

Theorem 39
Given $X_1 X_2 \equiv Y_1 Y_2$, where each set is a prefix set we have:
(i) if $sh(X_1) = sh(Y_1)$ then $X_1 \equiv Y_1$
(ii) if $sh(X_1) < sh(Y_1)$ then $X_1 \cdot \subset Y_1$,
(iii) if $sh(X_1) > sh(Y_1)$, then $X_1 \cdot \supset Y_1$.

Proof:
Let $\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2$ be shortest words in X_1, X_2, Y_1, Y_2, and $\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{Y}_2$ be the corresponding sets of shortest words.
(i) Choose the shortest word which contradicts $X_1 \equiv Y_1$, let this be x_1. It is true that
$$\bar{X}_1 = \bar{Y}_1 \text{ and } \bar{X}_2 = \bar{Y}_2.$$
Now $x_1\bar{x}_2$ in $Y_1 Y_2$. This implies X_1 is not prefix, therefore $X_1 \equiv Y_1$.
(ii) We have $\bar{X}_1 \cdot \subset \bar{Y}_1$ and $\bar{X}_2 \supset \cdot\, \bar{Y}_2$. First show $X_1 \cdot < Y_1$. Let x_1 be the shortest word in X_1 such that there exists no y in T^* such that $x_1 y$ in Y_1. Immediately we have
$$x_1\bar{x}_2 \text{ in } Y_1 Y_2 \text{ implies } x_{11} \text{ in } Y_1,\ x_{12}\bar{x}_2 \text{ in } Y_2$$
where $x_{11}x_{12} = x_1$, giving
$$x_{11}\bar{y}_2 \text{ in } X_1 X_2 \text{ and as } \bar{X}_2 \supset \cdot\, \bar{Y}_2 \text{ we have}$$
$$v_1 \text{ in } X_1,\ v_2\bar{y}_2 \text{ in } X_2, \text{ where } v_1 v_2 = x_{11},$$
giving a contradiction of the prefix condition.
Secondly, show $Y_1 \cdot > X_1$ which then implies the result. Assume there is a y in Y_1 for which there exists no x in X_1, such that $y \cdot > x$. Then:
$$y\bar{y}_2 \text{ in } X_1 X_2, \text{ giving}$$
$$y_1 \text{ in } X_1,\ y_2\bar{y}_2 \text{ in } X_2 \text{ as } sh(Y_2) < sh(X_2).$$
Now $y \cdot > y_1$, therefore contradiction hence the result.
(iii) is proved in a similar way to (ii).
We have in fact a stronger result.

Theorem 40
Given sets X_1, X_2, Y_1, Y_2, where X_1, Y_1 have the prefix property, $sh(X_1) = sh(Y_1)$ and $sh(X_2) = sh(Y_2)$ then
$$X_1 X_2 \equiv Y_1 Y_2 \text{ iff } X_1 \equiv Y_1 \text{ and } X_2 \equiv Y_2.$$
The following question arises.
Does Theorem 40 hold without the prefix property?

Example 4
Let
$$X_1 = \{ab^i: i \geq 0\},\ X_2 = \{d\},$$
$$Y_1 = \{a\},\ Y_2 = \{b^i d: i \geq 0\}.$$
X_1 does not have the prefix property.
$$sh(X_1) = sh(Y_1),\ sh(X_2) = sh(Y_2) \text{ and } X_1 X_2 \equiv Y_1 Y_2$$
but $X_1 \neq X_2$ and $Y_1 \neq Y_2$.
The most we can say is:

Lemma 41
Given sets X_1, X_2, Y_1, Y_2 with $sh(X_1) = sh(Y_1)$ and $sh(X_2) = sh(Y_2)$ then $X_1 X_2 \equiv Y_1 Y_2$ if $X_1 \equiv Y_1$ and $X_2 \equiv Y_2$.
This is just a trivial restatement of the catenation property.

Remark
In Example 4 (above), $\pi(X_1, X_2)$ and $\pi(Y_1, Y_2)$ imply that in order to prove a theorem similar to Theorem 39 for s-separable sets either extra conditions are needed or a weakening of the statement of the theorem. We choose the latter course giving the following very weak version of Theorem 39 for s-separable sets.

Theorem 42
Given $X_1 X_2 \equiv Y_1 Y_2$, $\pi(X_1, X_2)$, $\pi(Y_1, Y_2)$ we have:
(i) if $sh(X_1) = sh(Y_1)$ then $X_1 \cdot > Y_1$ & $Y_1 \cdot > X_1$,
(ii) if $sh(X_1) < sh(Y_1)$ then $Y_1 \cdot < X_1$,
(iii) if $sh(X_1) > sh(Y_1)$ then $X_1 \cdot > Y_1$.

The proof of this theorem follows immediately from the assumptions. We now state a conjecture, which, if true, would be the expected weakened version of Theorem 39. However its proof or disproof is non-trivial.

Conjecture
Given $X_1 X_2 \equiv Y_1 Y_2$, $\pi(X_1, X_2)$, $\pi(Y_1, Y_2)$ we have:
(i) if $sh(X_1) = sh(Y_1)$ then $X_1 \cdot \subset Y_1$ or $Y_1 \cdot \subset X_1$,
(ii) if $sh(X_1) < sh(Y_1)$ then $X_1 \cdot \subset Y_1$,
(iii) if $sh(X_1) > sh(Y_1)$ then $Y_1 \cdot \subset X_1$.

We now compare $\cdot \subset$ with the set inclusion relation.

Lemma 43
$X \cdot \subset Y$ & $Y \cdot \subset X$ does not imply $X = Y$, although the converse result holds.

Proof:
Let $X = \{b^i: i \geqslant 0\}$, $Y = \{b^{2i}: i \geqslant 0\}$, then
$$X \cdot \subset Y \,\&\, Y \cdot \subset X \text{ but } X \neq Y.$$
We have a weaker version of Lemma 28.

Lemma 44
If $X_1 X_2 \equiv Y_1 Y_2$, $\pi(X_1, X_2)$, $\pi(Y_1, Y_2)$, $X_1 \cdot \subset X_1$ then $X_2 > \cdot\, Y_2$.
This result follows trivially having once noted that:

Lemma 45.
If $X \cdot \subset Y$ then $sh(X) \leq sh(Y$ and
if $X \cdot \subset Y$ & $Y \cdot \subset X$ then $sh(X) = sh(Y)$.

Proof:
$X \cdot \subset Y$ implies $Y \cdot > X$ which implies that $sh(Y) \geqslant sh(X)$.
The second result follows immediately.

Table 1 Comparison of $LL(k)$, $F(k)$, $U(k)$ and deterministic languages

CLOSED UNDER	$LL(k)$	$DPDL$	$U(k)$ OR $F(k)$
Union	no	no	no
Concatenation	no	no	no
Concatenation with R	no	yes	no
Closure	no	no	no
Reversal	no	no	no
Intersection	no	no	no
Complement	no	yes	?
Intersection with regular set	no	yes	no
Substitution	no	no	no
ϵ-free substitution	no	no	no
Gsm mappings	no	yes	no
ϵ-free gsm mappings	no	yes	no
Inverse deterministic gsm mappings	?	yes	?
Quotient with regular set	no	yes	no
Homomorphism	no	no	no
ϵ-free homomorphism	no	no	no

5. Operations and LL(k) languages

In Table 1 we compare $LL(k)$ languages with the deterministic context-free languages of Ginsburg and Greibach (1966). The results for $LL(k)$ languages are non-closure results; this with the known result (Rosenkrantz and Stearns, 1970) that the $LL(k)$ languages form the largest known class for which the equivalence problem is decidable, their misbehaviour is surprising. Most of the results detailed below appeared previously in Korenjak and Hopcroft (1966), Rosenkrantz and Stearns (1970), Tixier (1967) and Wood (1969b).

Boolean operations

Lemma 46

$\mathscr{L}$ is not closed under (i) union, (ii) intersection, (iii) complement.

Proof:

(i) Let $L_1 = \{a^i : i \geq 1\}$, $L_2 = \{a^i b^i : i \geq 1\}$ then $L_1 \cup L_2$ is not $LL(k)$ for any $k > 0$.

(ii) Let $L_3 = \{a^i(b|c)\, a^i(b|c) : i \geq 1\}$,
$L_4 = \{a^i b\, a^j b, a^i c\, a^j c : i, j \geq 1\}$,
then $L_5 = L_3 \cap L_4 = \{a^i b\, a^i b, a^i c\, a^i c : i, j \geq 1\}$ which is not $LL(k)$ for any k.

(iii) Let $L_6 = \{a^i b^j : j \geq i \geq 1\}$ then $\{a, b\}^* - L_6$ is not $LL(k)$ for any k,

this is proved in Rosenkrantz and Stearns (1970). Because L_1 and L_4 are regular sets we have the following.

Corollary 47

$\mathscr{L}$ is not closed under union or intersection with a regular set.

Letting $L_7 = \{a^i b\, a^j c, a^i c\, a^j b : i, j \geq 1\}$ then $L_3 - L_6 = L_5$ and as $\{a, b\}^* - L_6$ is not $LL(k)$ we have:

Corollary 48

$\mathscr{L}$ is not closed under subtraction or subtraction with a regular set.

Mappings

Let $L_8 = cL_1 \cup dL_2$, then L_8 is $LL(1)$.

Define a homomorphism σ, that maps d onto c and the other symbols onto themselves, then

$$\sigma(L_8) = L_9 = \{ca^i, ca^i b^i : i \geq 1\}$$

which is not $LL(k)$ for any k. We have shown

Lemma 49

$\mathscr{L}$ is not closed under ε-free homomorphism. Further as homomorphism is a special case of substitution we have

Corollary 50

$\mathscr{L}$ is not closed under ε-free(finite) substitution, homomorphism or (finite) substitution.

Similarly we can define a *gsm* mapping that performs the homorphism σ, therefore we also have

Corollary 51

$\mathscr{L}$ is not closed under ε-free *gsm* mappings. Finally we note that as $\mathscr{L}$ is not closed under ε-free homomorphism, it is not closed under k-limited erasing.

Products and quotients

Let $L_{10} = \{c\} \cup cL_2$ which is $LL(1)$; then

$$L_{10}L_1 = \{ca^i, ca^i b^i a^j : i, j \geq 1\} \text{ is not } LL(k).$$

Trivially letting $L_{11} = L_1 \cup cL_2$ we have

$$\{c, cc\}\, L_{11} = \{ca^i, cca^i, cca^i b^i, ccca^i b^i : i \geq 1\}$$

is not $LL(k)$. We have just shown the following.

Lemma 52

$\mathscr{L}$ is not closed under product, pre-product with a regular set or post-product with a regular set. We have in fact shown a stronger result namely:

Corollary 53

$\mathscr{L}$ is not closed under pre-product with a finite set. However we do have (the proof left to the reader).

Lemma 54

$\mathscr{L}$ is closed under post-product with a finite set. Let $L_{12} = L_1 \cup cL_2 \cup \{c\}$ we have that

L_{12}^* contains words of the form

$ca^n x$ and $ca^n b^n y$ which means that L_{12}^* is not $LL(k)$;

this gives:

Lemma 55

$\mathscr{L}$ is not closed under catenation closure.

$\{c, \varepsilon\} \backslash L_{11} = L_{11} \cup L_2$ and is therefore not $LL(k)$ giving an expected result:

Lemma 56

$\mathscr{L}$ is not closed under left quotient, left quotient with a regular set or left quotient with a finite set.

Tixier (1967) has shown the strongest possible result:

Lemma 57

$\mathscr{L}$ is closed under left quotient with a single word. Letting $L_{13} = L_2/(\{ab^i : i \geq 1\} \cup \{\varepsilon\})$

$\{a^i b^i, a^{i+1} : i \geq 1\}$

which is not $LL(k)$ for any k. Therefore we have

Lemma 58

$\mathscr{L}$ is not closed under right quotient or right quotient with a regular set.

However, as expected by the post-product result:

Lemma 59

$\mathscr{L}$ is closed under right quotient with a finite set. We now examine those operations introduced by Ginsburg and Greibach (1966) which preserve the deterministic languages.

Definition

$\text{Init}(L) = \{u : uv \text{ in } L \text{ for some } v \text{ in } T^*\}$, thus

$\text{Init}(L) = L/T^*$, the set of all initial subwords of words in L.

Taking $L = \{a^i b^i : i \geq 1\}$, $\text{Init}(L) = \{a^i b^j : i \geq j \geq 0\}$ which is not $LL(k)$ for any k, giving

Lemma 60

Init does not preserve the $LL(k)$ condition.

Definition

$RDIV(L_1, L_2) = \{u : uL_2 \subseteq L_1\}$.

Lemma 61

$RDIV$ does not preserve $\mathscr{L}$.

Proof:

Let $L_1 = \{a^{i+1} b^i (ab^j c|c) : i, j \geq 1\}$ and

$$L_2 = \{a\, b^i c : i \geq 1\}$$

then $RDIV(L_1, L_2) = \{a^i, a^{i+1} b^i : i \geq 1\}$ which is not in $\mathscr{L}$.

Miscellaneous operations

Lemma 62

$\mathscr{L}$ is not closed under reversal.

Proof:

Let $L = (L_1 \cup L_2)^R$ which is $LL(1)$ but L^R is not $LL(k)$.

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Book review

Progress of Cybernetics, Vols. I, II and III, by J. Rose (editor), 1970; 521 pages. (*Gordon & Breach*, Vol. 1 £10·25 or $24·50), Vol. 2 and Vol. 3 £8 or $19·50 (each), complete set £20·75 or $50·00).

The number and diversity of the contributions to a large compilation precludes the reviewer from attempting to assess them individually. His proper task is rather to select systematic trends; a problem in fact of pattern recognition. Where an editor has exerted cybernetic influence on the compilation, it is also proper to enquire whether the editorial statement of aims has been duly reflected in the actual material.

The Editor of the present three volumes states in the preface that an objective of the 1969 Congress of Cybernetics was 'To establish cybernetics as an interdisciplinary science on solid foundations without the spurious accretions of the last two decades'; also that 'it is intended to demonstrate the scope and maturity of cybernetics, though a few papers bear the stamp of a rather exotic approach. These somewhat fatuous contributions were included in order to bring to the surface certain undesirable accretions. A mature science has to be able to live and cope with those who are trying to jump on the band-wagon and use it as a vehicle for their exuberant claims'. We ask therefore whether (exluding 'fatuous contributions' which the reader can skip without outside aid) the contents do reflect a mature science. Specifically, do they represent sound contributions to the scientific method of systematic observation, formation of hypotheses, quantitative development of the consequences of these hypotheses, and confrontation with new observations?

The present reviewer judges these matters in part by scoring positively for pages containing relevant mathematical development or experimental results and comparisons. He scores negatively for such features as material that is either not new or (non-exclusively) is trivial, repetitions of well-worn diagrams, the mutual taking in of quotational washing, photographs of opulent apparatus accompanied by minimal experimental results, and above all for acres of qualitative discourse. It is a question of whether the author is actually doing the subject, or just talking about doing the subject.

By these criteria 'Progress of Cybernetics' scores somewhere around half marks. There is indeed a good sprinkling of sound scientific building-blocks, enough perhaps for one volume or even a little more; experiments, theoretical developments, developments of technical capabilities, and the combination of these things into coherent scientific strategies.

The opening section is called 'Main Papers'. These are rather longer than the general run of contributions, appear to have an invited status and are largely of a review nature. The density of plus scores is not maximal in this Section; one would not for example mistake the general format for the undoubtedly mature Reports of Progress in Physics. Among others, however, Ashby has an interesting discussion of information flow in tasks like tight-rope walking or driving in a large city (the connection is only too obvious), Beer has some deservedly unkind things to say about our economic and social institutions, and Glushkow gives a solid account of data processing in specific natural sciences.

Section I (which follows the Main Papers) is called 'The Meaning of Cybernetics' so one fears the worst but does not always find it. Muses, going as far afield as operator algebras and epistemology, still keeps the appearance of rigor. Section II, 'Neuro- and bio-cybernetics', includes a contribution by Levy on computer simulation of neurological systems, by Andrew on the results of simulation of self-organising systems with significance feedback, by Arigoni on the algebra of intelligence, by Moore *et al.* on a model of a visual system, by Taylor on visual size-illusions, by Gambardella on auditory time-frequency analysis (but some concepts appear to have been anticipated by P. M. Woodward), and by Auslander and Sharma on computer simulation of hormone levels. These unselectively chosen examples illustrate the broad international flavour of the Congress.

Volume 2 opens with Section III, 'Cybernetics and Industry (automation)'. With a few exceptions, the contributions in the Section are good, solid, quantitative and practical, perhaps even sometimes stolid; the plus scores are here too numerous to mention individually. Section IV, 'Social and economic consequences of Cybernetics', has the additional rubric 'including management' *pace* Beer's assertion (q.v.) that there ain't no such animal, and overall scores about $\beta-$. However, Billeter-Frey criticises current economic models for leaving out some of the most important feed-back connections, Winkelbauer analyses co-operative games (or, how to maximise your divi) and Vaida has a paper on ALGOL 60 implementation and translation which would not be out of place in *The Computer Journal*.

'Cybernetics and artifacts' (Section V) is wide-ranging, including even computer sculptures, and contains several interesting articles, notably a pouring of cold water by Bagley on any assumption of an easy road to artificial intelligence.

Volume 3 contains Section VI, 'Cybernetics and natural sciences', and Section VII, 'Cybernetics and social science'. Neither received many plus marks from your reviewer, Section VII in particular posing the implicit question whether there are indeed as yet any social *sciences*. Two specific and quantitative contributions are by Chiaraviglio on computer modelling of DNA sequences, and by Malitza and Zidaroiu on random decision processes. Goffman treats the spread of the ideas of symbolic logic by means of a theory developed for epidemics, Irtem gives an intriguing hint about how to 'change' natural laws. There is also the aimable intelligence given by Kerschner that the term Cybernetics had been used before Wiener not only by Ampére in 1843 but also by Platon. Kerschner also tells us that all but 2^{-4} of the professional 'political scientists' in the world are American; no comment.

The reviewer draws two general conclusions. The first is that these volumes suffice neither to prove nor to deny the assertion that Cybernetics is now a mature science. The second is that while one may have reservations about the usefulness of publishing conference reports in general, this is not a bad example of its kind. (The reviewer would not personally pay £20·75 for it, however.)

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