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Correspondence

To the Editor
The Computer Journal

Sir

The recent paper by L. B. Smith ('Drawing Ellipses, . . .', Vol. 14, No. 1) suggests that a good criterion for approximating a convex curve by an inscribed polygon with a fixed number of vertices is that the N-gon have maximal area. In the case of conic sections this criterion leads to highly efficient algorithms, as Mr. Smith so clearly illustrates.

The paper gives a Lemma which states that an N-gon inscribed into a convex curve has maximal area if and only if the tangent to the curve at each vertex of the N-gon is parallel to the chord determined by the two adjacent vertices. The justification of the Lemma considered only the possibility of moving just one vertex at a time, so it is not surprising to find counter-examples which necessitate moving two or more vertices of the polygon. The triangle joining the midpoints of the sides of any given triangle satisfies the conditions of the Lemma, yet it is not maximal. The area of the inscribed triangle cannot be increased by altering any one of its vertices. In the following discussion an inscribed polygon such that the slope at each of its vertices is parallel to the chord of its two neighbouring vertices will be called a *locally maximum* polygon.

A locally maximum N-gon for a given arbitrary convex curve can be determined iteratively starting from an initial set of N points on the curve by moving one point after another to a point further away (if possible) from the chord of its two neighbouring vertices. Convergence is guaranteed and fairly rapid.

Some curves have but one locally maximum N-gon, in which case the conditions of the Lemma prove sufficient. The curve defined by $x = -1 + 3\cos\theta$, $y = 1 + \sin\theta$, for $0 \leqslant \theta \leqslant \pi/2$, and its rotations about the origin into the other quadrants, is one such example. It has but one locally maximum quadrilateral, with a vertex in each quadrant corresponding to the value of θ approximately 64°7′, as shown in **Fig. 1**.

The curve $x^4 + y^4 = 1$ has just two locally maximum octagons. One of them has vertices $(\pm 1, 0)$, $(0, \pm 1)$, and the points $(\pm c, \pm c)$, where $c^4 = 1$. This octagon, of area $4c$, is 'unstable', in the sense that if any one of its vertices is shifted slightly, then either one of its two neighbours can be shifted slightly to increase the area to *more* than $4c$. This means that if the vertices are shifted successively so as to increase the area at each step, they must ultimately approach the only other locally maximum octagon, namely the points $(\pm x, \pm y)$ and $(\pm y, \pm x)$, where $x^4 = (3 + \sqrt{5})/6$ and $y^4 = (3 - \sqrt{5})/6$. This octagon has area $2\sqrt{3}$. Once again, the lack of sufficiency of the conditions of the Lemma do not interfere with finding a maximal quadrilateral. The curve is shown in **Fig. 2**.

It is possible for a convex curve to possess many distinct maximal N-gons, with none of them unstable. Consider the convex curve determined by the two parabolas, $y^2 = 16 + 64x$ and $y^2 = 16 - x$, which intersect at $(0, \pm 4)$. A locally maximum 65-gon may have from 5 to 65 vertices on $y^2 = 16 - x$, the remaining vertices lying on $y^2 = 16 + 64x$. The points on either parabola must have ordinates distributed equally from -4 to $+4$. The more vertices on

Continued on page 116

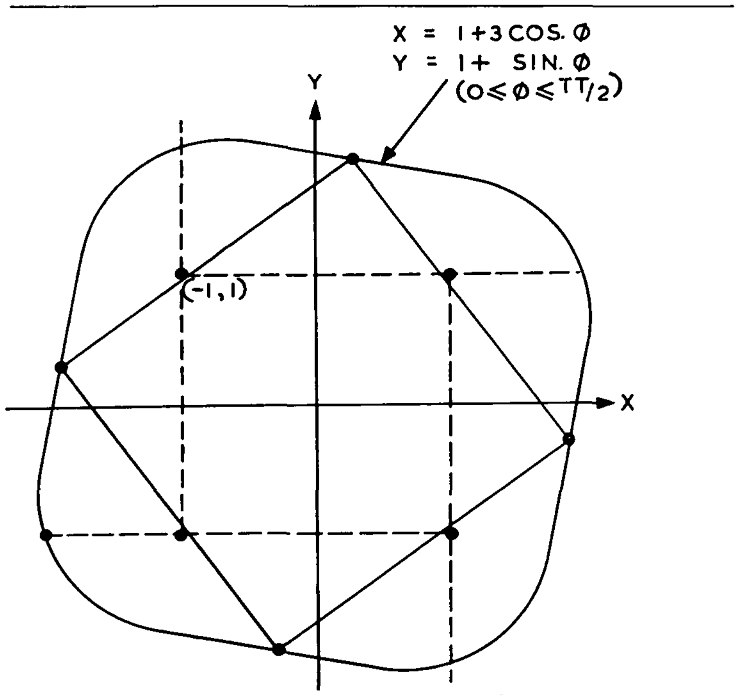

Fig. 1

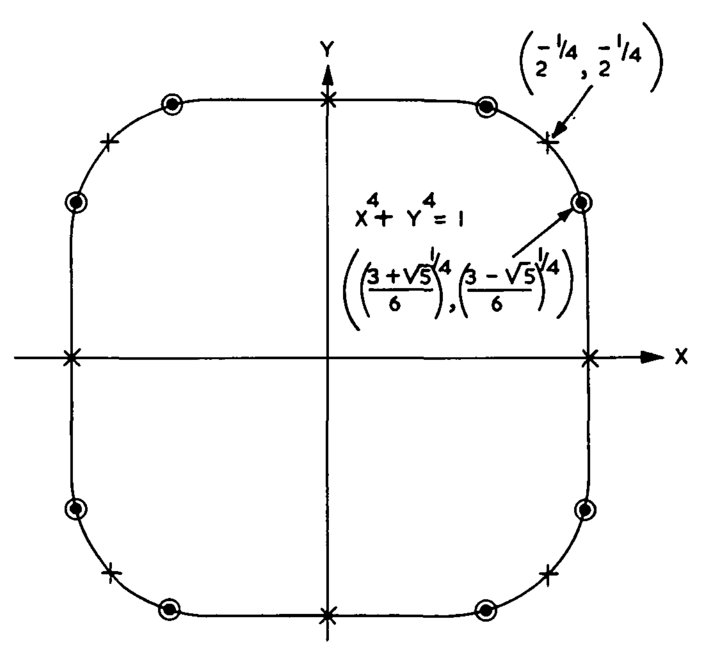

Fig. 2

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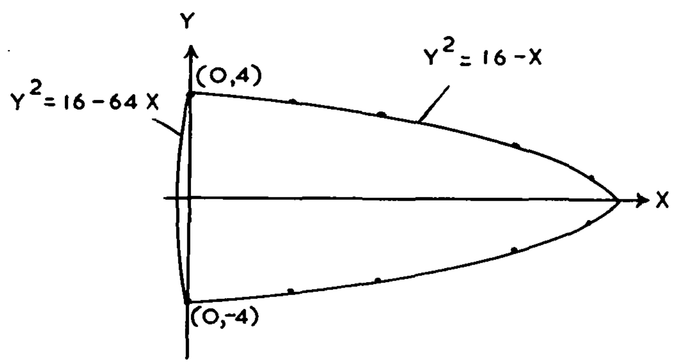

61 DISTINCT, STABLE, LOCALLY MAXIMUM 65-GONS EXIST, WITH FROM 5 TO 65 VERTICES ON Y = 16 - X.

Fig. 3

$y^2 = 16 - x$, the larger the area of the 65-gon. Each of the 61 locally maximum 65-gons are stable, in the sense that if small enough perturbations are made of each of their points (even simultaneously), then the iterative process of adjusting the middle points of various triplets of vertices must converge back to the initial configuration. The sharp angles at the intersections $(0, \pm 4)$ prove to be impassable barriers to the migrations of vertices of N-gons for $N \leqslant 65$ (see **Fig. 3**).

Any point of a circle may be the vertex of a regular inscribed polygon. The circle may be projected onto any ellipse, so that the regular inscribed polygon is projected onto a locally maximum polygon of the ellipse (note that the projection preserves tangency and parallelism). It is probably characteristic of the ellipse (and circle) that any of its points may be used as a vertex of a locally maximum polygon of any order.

Yours faithfully,
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6 December 1971

To the Editor
The Computer Journal

Sir

Calculation of a double-length square root from double-length number using single precision techniques

I write to comment on the letter by D. W. Honey (this *Journal*, Vol. 14, Nov. 1971, p. 443) where he describes a method which he attributes to his colleague, Mr. J. Grabau. The method given is, however, quite well known, being Newton's method with rearrangement of terms to exhibit the correction to be made at any stage. The usual form of Newton's method for finding $\sqrt{a}$ is

$$x_{i+1} = \tfrac{1}{2}\left(x_i + \frac{a}{x_i}\right)$$

which can be rewritten as

$$x_{i+1} = x_i + \frac{a - x_i^2}{2x_i}$$

to show the correction. Mr. Honey's (or Mr. Grabau's) technique is therefore seen to be equivalent to one more step of the Newton process after the single-length result has been obtained.

However, *it is necessary to take care when this method is being used* in fixed-point arithmetic, as overflow could result if the 'wrong' single-length square root is taken. It is not enough to take the unrounded (rounded down) value, because this leads to a value x satisfying

$$0 \leqslant a - x^2 \leqslant 2x\,,$$

and this can obviously give overflow. No such difficulty can arise if we take the rounded value, because this satisfies

$$-x \leqslant a - x^2 \leqslant x\,,$$

with a correction of at most $\frac{1}{2}$ unit, although it may be of either sign. In Mr. Honey's example, therefore, he *should* have used 14 as his initial guess at $\sqrt{192}$, which would have led to 13·86 as the better approximation, instead of 13·88. Since $(13{\cdot}86)^2 = 192{\cdot}0996$ this gives an error which is about $\frac{1}{7}$ of that quoted. Indeed, it is not difficult to show that the maximum relative error using the rounded single-length approximation will be about $\frac{1}{4}$ of the error that could arise from using the unrounded version. Choosing the rounded approximation is thus noticeably more accurate for the same amount of work.

Yours faithfully,
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14 December 1971

Mr. Honey replies:

I am obliged to Professor P. A. Samet for his letter commenting on 'Grabau's Method' for obtaining a double precision result using single precision techniques.

I think that I may have misled the readers by the lack of emphasis on the single precision. Professor Samet is quite correct in his observation that Newton's method is involved, although I had not appreciated this fact at the time. My main concern was that single precision techniques are used throughout the process and is something often overlooked by software designers with non-mathematical background.

I am also obliged for Professor Samet's further comment re 'overflow' (again sometimes overlooked), and his development of my worked example in decimal in which a rounded value is taken in preference to an unrounded value—a technique I shall remember in future.

I am sure that several readers will have gained benefit from our minor correspondence—which is its basic purpose. Thank you, Professor Samet, once again.