A comparison of some inverse Laplace transform techniques for use in circuit design

M. P. Atkinson and S. R. Lang

University of Cambridge Computer Laboratory, Corn Exchange Street, Cambridge CB2 3QG

At the University of Cambridge Computer Laboratory experiments are proceeding in the design of an integrated computer-aided circuit design system (Cheney *et al.*, 1971). We discuss here the search for a method of reliably obtaining the time response of a circuit when the frequency response is available.

(Received November 1971)

The environment of the Rainbow circuit design system facilitates the graphic or textual input of a circuit to the database and provides data manipulation programs which support a set of application programs including a program to obtain the frequency response $F(s)$ for any complex frequencies (Cheney *et al.*, and Etherton, 1971). This response is obtained without forming an analytic representation of the circuit's frequency response (or Laplace Transform) so we are precluded from using methods for finding the time response $f(t)$ which involve this analytic form. For any circuit which is of reasonable complexity the process of determining $F(s)$ for each s takes a time which is larger than the other steps in determining $f(t)$; therefore, we are comparing methods of obtaining the inverse Laplace transform of $F(s)$ with respect to the number of points in s at which $F(s)$ must be evaluated. In this context it should be noted that engineers usually require $f(t)$ for a sequence of points in t.

We now report our findings for four methods considered, we shall report on other methods in the near future.

Formalising the requirement

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \qquad (1a)$$

and

$$f(t) = \frac{1}{2\pi j} \int_{a-j\infty}^{a+j\infty} e^{st} F(s)\, ds \qquad (1b)$$

we require $f(t)$ (Bellman *et al.*, 1966).

The methods considered are various numerical approximations to the integration shown in equation (1b)

They are:

1. The method due to Dubner and Abate (1968) where the approximation for $f(t)$ is given by

$$\text{Error} + f(t) = \frac{2e^{at}}{T}\left[\tfrac{1}{2}\text{Re}\{F(a)\} + \sum_{k=1}^{N} \text{Re}\left\{F\left(a + \frac{k\pi j}{T}\right)\right\} \cos\left(\frac{k\pi t}{T}\right)\right] \qquad (2)$$

In this method a, T and N are free variables which were assigned values so that $T > 2t_{max}$ where t_{max} was the largest value of t at which $f(t)$ was required. N was chosen so that the last term in the summation above was smaller than the required error, and a was chosen so that $e^{-aT} <$ error required. (A discussion of the choice of a is given in Dubner and Abate (1968), Silverburg (1970), and Herkowitz (1968).) The modifications using FFT'S or setting $T = 2t$ can be seen by inspection to be prohibitively expensive in the application considered. Excluding these modifications we observe that only N evaluations of $F(s)$ are required for any number of points in t, $0 < t < t_{max}$.

2. The method due to Zakian (1969, 1970a, and 1970b) where the approximation for $f(t)$ is given by

$$\text{Error} + f(t) = \frac{1}{t} \sum_{i=1}^{N} K_i F\left(\frac{\alpha}{t}\right) \qquad (3)$$

Zakian (1970b) suggests a good set of $\alpha_i K_i$ is given by the solution of

$$\sum_{i=1}^{N} \frac{K_i k!}{(\alpha_i)^{k+1}} = 1 \qquad k = 0, 1, 2, \ldots 2N-1 \qquad (4)$$

We have therefore assumed only one variable of choice, namely N. This we let have the value 10 since a set of values for $\alpha_i K_i$ has been published for $N = 10$ (Zakian, 1969), and since this appears to be an optional choice.

Since α_i and K_i occur in conjugate pairs it was possible to reduce the summation of equation (3) into one of five terms, thus for each value of t, $F(s)$ was evaluated five times (Zakian, 1970a).

3. The method due to Stehfest (1970a and 1970b) where the approximation for $f(t)$ is given by

$$\text{Error} + f(t) = \frac{\log 2}{t} \sum_{i=1}^{N} V_i F\left(\frac{j \log 2}{t}\right) \qquad (5)$$

where

$$V_i = (-1)^{N/2+i} \sum_{k=(i+1)/2}^{\min(i,N/2)} \frac{k^{N/2}(2k)!}{(N/2-k)!\, k!(k-1)!\,(i-k)!\,(2k-i)!} \qquad (6)$$

Here also the choice available is only the choice of N, and the values of s at which $F(s)$ must be sampled are t-dependent so that $F(s)$ must be evaluated at Nq points if $f(t)$ is required at q points. An optimum choice of N was found to be 14.

4. The method due to Piessens (1969) where the approximation for $f(t)$ is given by

$$\text{Error} + f(t) = \frac{1}{t}\left[\sum_{k=1}^{N} B_k u_k F\left(\frac{u_k}{t}\right) + \sum_{k=1}^{N+1} C_k v_k F\left(\frac{v_k}{t}\right)\right] \qquad (7)$$

where u_k, v_k, B_k and C_k are calculated by a new method of Piessens based on the Gaussian quadrature formula.

Again the only variable of choice is N, and sets of corresponding values of u_k, v_k, B_k and C_k are suggested by Piessens (1969). The method requires $2N + 1$ evaluations of $F(s)$ for each value of t. However, an increase in the order of N to improve the accuracy so far achieved, requires only an extra $N + 1$ evaluations as the u, v, C and B occur in complex conjugate pairs. Satisfactory accuracy was obtained with $N = 5$.

To form a comparison we applied Zakian's method to various circuits (since only $N = 5$ was available for this method) and found that it gave an accuracy of four significant figures. We

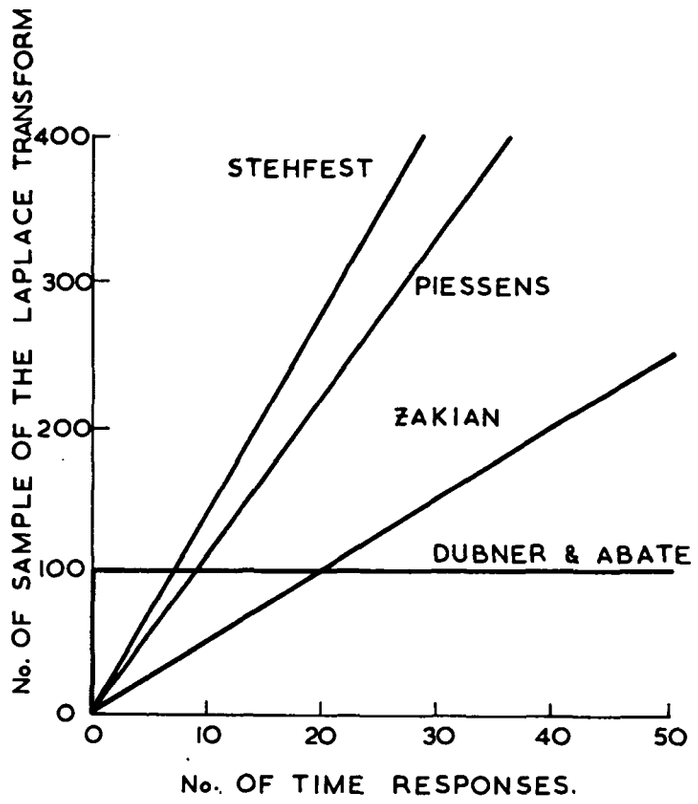

Fig. 1. Variation in the number of samples needed of the Laplace transform according to the number of time responses required and according to the method of inversion used

then ran the other three methods adjusting N in each case to obtain the same accuracy. These results are summarised in **Fig. 1.** where the number of evaluations of $F(s)$ is plotted against the number of values of t at which $f(t)$ is required averaged over the circuits considered.

Fig. 1 shows that at no point is there an advantage in using Stehfest's or Piessens's method, and that for many points in time Dubner and Abate is significantly the most efficient method. Further qualitative comments can be made.

Only the Dubner and Abate method is applicable to finding $f(t)$ at $t = 0$. Methods such as those of Zakian, Stehfest and Piessens where only one variable (N) is to be chosen have distinct advantages for automatic systems where the analytic form of the circuit is not known, since the choice of a in equation (2) requires knowledge of the positions of the poles (we may have these in the right half plane since active components are processed using nullator-norator models). The accuracy of the Dubner and Abate method can be increased by increasing the number of terms N in the summation, this requires only the evaluation of the extra terms and their addition to the partial summation, whereas in the methods of Zakian and Stehfest an increase in accuracy requires a fresh start, and in the Piessens method half the summation must be abandoned. It must be admitted though that the error for Dubner and Abate increases as $t \to t_{max}$ which has led to some difficulty in deciding whether to increase N or T in equation (2) to improve the accuracy. The error for Zakian increases as $t \to T_{crit}$ where T_{crit} is a property of the circuit.

In none of these methods is it possible to compute reasonable and reliable estimates of the error not knowing the analytic form of the circuit. So to implement a satisfactory automatic system we have had to rely on comparisons between methods. We shall report details of this implementation in due course.

References

BELLMAN, R., KALABA, R., and LOCHETT, J. (1966). *Numerical Inversion of the Laplace Transform*, American Elsevier, New York.

CHENEY, C., ETHERTON, M., HILES, J., LEMKE, H., and WISEMAN, N. (1971). Rainbow—An Integrated Computer Aided Design System, Proc. 8th Annual Design Automation Workshop, Atlantic City.

DUBNER, H., and ABATE, J. (1968). Numerical Inversion of Laplace Transforms, *JACM*, Vol. 15, No. 1, pp. 115-123.

ETHERTON, M. (1971). Data structures for a network design system, *Computer Journal* Vol. 14, No. 4, pp. 366-374.

HERKOWITZ, G. J. (1968). *Computer-Aided Integrated Circuit Design*, McGraw-Hill, pp. 138-143.

PIESSENS, R. (1969). New Quadrature Formulas for the Numerical Inversion of the Laplace Transform, *BIT*, Vol. 9, pp. 351-361.

Rainbow System Manual: Internal Document, University of Cambridge Computer Laboratory.

SILVERBURG, M. (1970). Improving the efficiency of Laplace-Transform Inversion for Network Analysis, *Electronic Letters*, Vol. 6, No. 4, pp. 105-106.

STEHFEST, H. (1970a). Algorithm 368, *CACM*, Vol. 13, No. 1.

STEHFEST, H. (1970b). Comment and Correction on Algorithm 368, *CACM*, Vol. 13, No. 10.

ZAKIAN, V. (1969). Numerical Inversion of Laplace Transform, *Electronic Letters*, Vol. 5, No. 6, pp. 120-121.

ZAKIAN, V. (1970a). Rational Approximation to Transform Function Matrix of Distributed System, *Electronic Letters*, Vol. 6, No. 15, pp. 474-476.

ZAKIAN, V. (1970b). Optimisation of Numerical Inversion of Laplace Transform, *Electronic Letters*, Vol. 6, No. 21, pp. 677-679.