mental data), and finally the Deming results be used as starting guesses for the present algorithm. Further, in some experiments with synthetic, highly accurate data, we found that the values of Δ_x and Δ_a in the calculation of derivatives had to be chosen extremely small (e.g. $\sim 10^{-10}$) to obtain convergence. Using real data, however, we have not encountered such problems.

Our method is somewhat heuristic because of the great difficulty in performing numerically what 'exact' theory would demand. Comparison with previously reported general least squares procedures leads us to believe that the present method is more than competitive. We concur with O'Neill *et al.* in their conclusion that Deming's method will often provide adequate results without further calculation. Our iterative scheme, however, almost always produces a reduction of 10 to 40% or more in parameter standard deviation estimates when compared with ordinary linearized estimates, along with a lower sum of squares, guaranteed to be at least a relative minimum when our convergence criteria are satisfied. Even though the bias of nonlinear equation estimates may not always be negligible, we still feel that such reduction is useful. Whether or not an experimenter wishes more precision than Deming's method provides will depend upon his unique situation. Perhaps most important, he cannot be assured of a *least* squares solution with Deming's method, and there may be situations where significant differences arise in parameter estimates obtained by the two methods.

Acknowledgement

We wish to thank Dr. Eric L. Jones for his most valuable suggestions and discussions of this work. We much appreciate permission to use the preliminary *Kr* data of C. A. Swenson and M. Anderson.

References

BOX, M. J. (1971). Bias in nonlinear estimation, *J. Roy. Stat. Soc., Series B*, Vol. 33, pp. 171-190.

DAVIS, P. J., and POLONSKY, I. (1964). Numerical interpolation, differentiation, and integration, in *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series No. 55, US Government Printing Office, pp. 875-924.

DEMING, W. E. (1943). *Statistical Adjustment of Data*, New York: Wiley.

DRAPER, N. R., and SMITH, H. (1966). *Applied Regression Analysis*, New York: Wiley.

HALL, K. R., and CANFIELD, F. B. (1967). Optimal recovery of virial coefficients from experimental compressibility data, *Physica*, Vol. 33, pp. 481-502.

JONES, E. L. An efficient, second-order algorithm for the generalised nonlinear least squares problem, *The Computer Journal*, to be submitted.

KOPAL, Z. (1955). *Numerical Analysis*, London: Chapman and Hall; see pp. 104-110.

MACDONALD, J. R. (1964). Accelerated convergence, divergence, iteration, extrapolation, and curve fitting, *J. Appl. Phys.*, Vol. 35, pp. 3034-3041.

MACDONALD, J. R. (1969). Review of some experimental and analytical equations of state, *Rev. Mod. Phys.*, Vol. 41, pp. 306-349.

MACDONALD, J. R., and POWELL, D. R. (1971). Discrimination between equations of state, *J. Res. Natl. Bureau of Standards, Series A*, Vol. 75A, pp. 441-453.

O'NEILL, M., SINCLAIR, I. G., and SMITH, F. J. (1969). Polynomial curve fitting when abscissas and ordinates are both subject to error, *The Computer Journal*, Vol. 12, pp. 52-56.

PEARSON, K. (1901). On lines and planes of closest fit to systems of points in space, *Phil. Mag.*, Vol. 2, pp. 559-572.

SOUTHWELL, W. H. (1969). Fitting experimental data, *J. Comp. Phys.*, Vol. 4, pp. 465-474.

WENTWORTH, W. E. (1965). Rigorous least squares adjustment, *J. Chem. Ed.*, Vol. 42, pp. 96-103; 162-167.

WOLBERG, J. R. (1967). *Prediction Analysis*, New Jersey: Van Nostrand.

YORK, D. (1966). Least-square fitting of a straight line, *Can. J. of Phys.*, Vol. 44, pp. 1079-1086.

Book review

Numerical Initial Value Problems in Ordinary Differential Equations, by C. William Gear, 1971; 253 pages. (*Prentice-Hall*, £6·50)

This text has 12 chapters on: 1, Introduction (and Euler Method); 2, Higher order one-step methods; 3, Systems of equations and equations of order greater than one; 4, Convergence, error bounds, and error estimates for one-step methods; 5, The choice of step size and order; 6, Extrapolation methods (of Bulirsch and Stoer, etc.); 7, Multivalue or Multistep methods—introduction; 8, General multistep methods, order and stability; 9, Multivalue methods; 10, Existence, convergence, and error estimates for multivalue methods; 11, Special methods for special problems (mainly stiff equations); 12, Choosing a method.

The treatment is in general good especially in the more practical Chapters 5, 6, 11 and 12. Complete FORTRAN routines are described and given for fourth order Runge Kutta with automatic step-length control, and for a variable order, variable step-length multivalue method with optional provision for stiff equations. A FORTRAN version of Bulirsch and Stoer's ALGOL procedure is also given. These large programs are photographically reproduced from computer printout. Smaller programs are type-set and much easier to read.

The discussion of one-step methods is shorter than that of Henrici (1962), but all the essential theoretical points are covered.

Multistep methods are treated from the less familiar multi*value* point of view (Gear, 1967), although this is scarcely used in Chapters 6 and 7. These multi*value* methods are exemplified by the finite difference and Nordsieck forms which are equivalent to the Adams-Bashforth-Moulton method. The stored quantities at each step are linear combinations of the usual y_r and $f(y_r, t_r)$ but the same approximating polynomial is used for all equivalent methods. Possible advantages are economy of arithmetic, especially when changing step length, and that the method may depend on a smaller number of previous steps, since the k stored values may approximate y_r and $f(y_r, t_r)$ for $0 \leqslant r \leqslant ((k-1)/2)$. Methods investigated include those using a fixed number of corrector iterations, as well as iteration to convergence. The only particular multistep or multivalue methods discussed are those of Milne and Adams-Bashforth-Moulton.

Chapter 10 includes a host of theorems relating the root condition, stability, consistency, convergence and asymptotic error form. The Dahlquist theory on the maximal order of stable multistep methods is also given. I did not find this theory easy to understand, partly because the author treats systems of pth order equations involving $(p-q)$ other derivatives, necessitating norms involving these two suffices, and partly because he does not always make clear exactly how the steps in the proof follow from the given hypotheses.

Rather than the usual one used by Gear, I prefer the (to me) more obvious definition of consistency of order r, for a pth order equation:

$$\Sigma \alpha_i z(t_{n-i})/h^p - \Sigma \beta_i f(z(t_{n-i}) \; t_{n-i}) = z^{(p)}(t_n) - f(z(t_n), t_n) + O(h^r).$$

The requirement that $\Sigma \beta_i = 1$ and the index of the order h term are then automatic.

Very many stability concepts—asymptotic stability, absolute and relative stability regions, stiff A stability, and several others—are introduced and clearly explained. Figures 11.2 and 11.3 illustrate the difficulty of solving stiff equations, and the advantage of the backward Euler method very well.

I noticed rather more than the usual number of misprints, and in

Continued on page 169

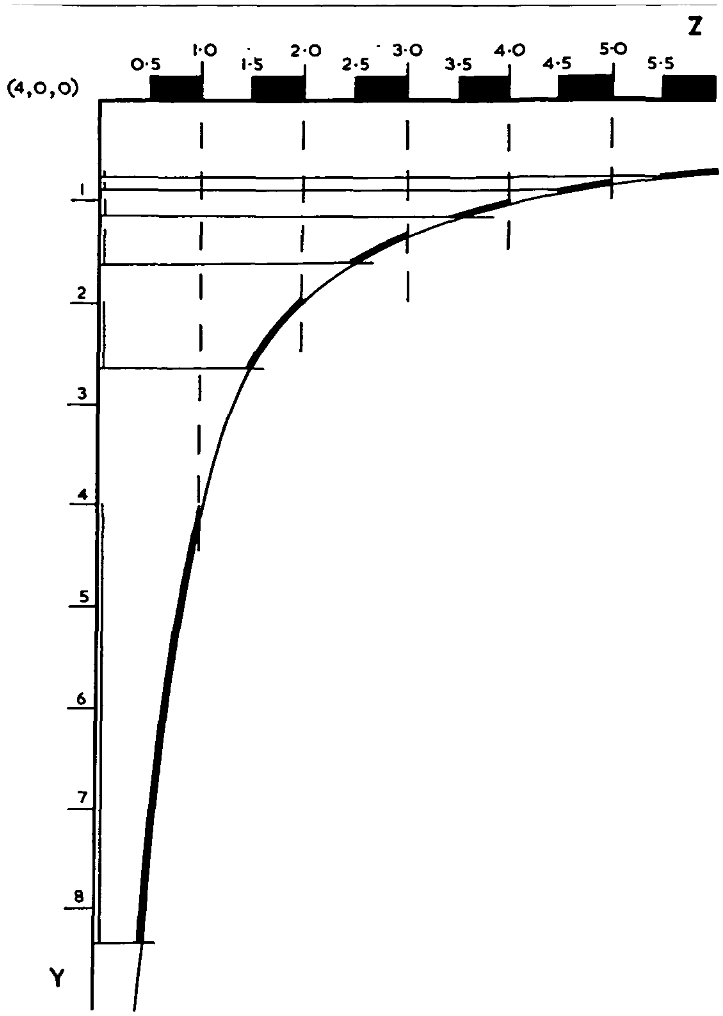

Fig. 10

The same effect is produced by the normal form which rotates the surface round the Y axis, thus altering its position relative to the Z plane.

The reason that particular patterns occur for particular decimal places is as follows. In taking the first decimal place a measure is made along the Z axis in tenths of a unit. If a character is printed only when the digit in the first decimal place is greater than four, then the bars along the Z axis in **Fig. 10** are mapped on to the Y axis, as shown, via the curve. When the second decimal place is taken, then the bars on the Z axis will appear every 0·05 of a unit and be 0·05 wide. The rapid decrease in the width of these bars explains the rapid increase in the complexity noticed as further decimal places are taken. In fact, the bars shown on the Z axis extend along the Z plane parallel to the X axis. The effect of rotating the surface is to alter its position relative to these bars.

Conclusion

The process of producing Character Maps is really very simple yet there is a wide range of patterns that can be generated. The use of the normal form to calculate the point values for the unit Map needs a minimum of data and is more efficient where multiplication is faster than division.

Similar patterns can be produced by the use of other infix operators. The use of complex combinations of prefix and infix functions results in some striking symmetric patterns. Interesting possibilities lie in the use of non-asymptotic surfaces—such as cubes—and rotating them about an axis in three dimensions.

Acknowledgements

I would like to thank Colin Emmett of the Computer Arts Society for introducing me to the technique; and Simon Armstrong for directing me to J. L. Dawson's article that presents the original idea.

Reference

Dawson, J. L. (1966). Art by computer, *Eureka* (Magazine of the Archimedeans), Arts School, Bene't Street, Cambridge.

Book reviews (*continued from page 155*)

Section 4.6 a term $O(h^{r+1})$ is differentiated r times to give zero at $h = 0$!

The book does not attempt an exhaustive review of the subject. It is arranged so that the more theoretical sections can be easily omitted. I would have preferred the material on multivalue methods to have been so arranged.

The text 'is addressed to two audiences: the men with problems to solve and the student numerical analyst'. I believe it is the best book available for these people.

References

Gear, C. W. (1967). The Numerical Integration of Ordinary Differential Equations, *Math. Comp.*, Vol. 21, pp. 146-156.

Henrici, P. (1962). Discrete Variable Methods for Ordinary Differential Equations, Wiley, New York and London.

J. M. Watt (Liverpool and Texas)

Artificial Intelligence and Heuristic Programming, by N. V. Findler and B. Meltzer (editors), 1971; 327 pages. (*Edinburgh University Press*, £6·00)

In August 1970 a most successful Advanced Study Institute on Artificial Intelligence and Heuristic Programming was held in Menaggio, Italy with the support of NATO. Every participant with whom I have spoken has praised the lectures, the discussions and the general intellectual liveliness of the occasion. This volume consists of a series of articles based on these lectures, and forms a very worthwhile reference book to the thinking of prominent workers in the area. It contains survey articles describing current problems and motivations and there is also some detailed descriptions of recent work.

The areas covered with the contributors of articles to this book are: Theorem proving (J. A. Robinson and B. Meltzer), Problem Oriented Languages (E. W. Elcock and N. V. Findler), Problem Solving (E. Sandewall, D. Michie and J. Pitrat), Integrated Systems (B. Raphael), Natural Language and Picture Processing (R. K. Lindsay, R. F. Simmons, J. Palme and M. Clowes) and Cognitive Studies (M. Kochen and L. W. Gregg (abstract only)).

The articles vary widely in length and style but all are very readable, an unusual phenomenon in a collection such as this. Where authors introduce formalisms the articles tend to be rather too condensed to enable the reader fully to understand the motivations; more reading would be needed. The basic ideas and thinking, however are all there. Many articles give the impression that their author, a well-known practitioner in the area, has really tried to stand back, take thought, and present a reasoned approach to his topic—most articles have a useful bibliography.

This book is recommended reading for anyone interested in a brief, but reasonably comprehensive, account of the major areas of Artificial Intelligence and of current thinking in the subject. It should also be very useful background material for undergraduate and postgraduate Computer Science students—although computers are hardly mentioned clearly the representation of information is one of the key topics of concern.

D. C. Cooper (Swansea)