Lagrangian interpolation at the Chebyshev points $x_{n,\nu} \equiv \cos(\nu\pi/n)$, $\nu = 0(1)n$; some unnoted advantages

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Besides many applications of the Chebyshev points $x_{n\nu} \equiv \cos(\nu\pi/n), \nu = 0(1)n$, in approximation, interpolation by Chebyshev series, numerical integration and numerical differentiation, there are advantages in their use in the barycentric form of the Lagrange interpolation formula and in checking by divided differences. When $n = 2^m$, we obtain $x_{2^m,\nu}$ with less than half the number of square roots that are required to find the other Chebyshev points $x'_{2^m,\nu} \equiv \cos[(2\nu - 1)\pi/2^{m+1}]$, $\nu = 1(1)2^m$. Also, the barycentric interpolation formula may be applied to the solution of a near-minimax problem so as to avoid extensive calculation of auxiliary polynomials, and in a numerical differentiation procedure that conveniently by-passes direct differentiation of the interpolation polynomial.

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1. Previously noted properties and advantages

The Chebyshev points $x_{n,\nu} \equiv \cos(\nu\pi/n)$, $\nu = 0(1)n$, are remarkable for a wide variety of useful properties in interpolation, near-minimax approximation, numerical integration and numerical differentiation, appearing in a fairly large volume of current literature in numerical analysis.

Early writers, then Lanczos (1956, pp. 229-239, 245-248) and later Elliott (1965) discuss the use of $x_{n,\nu}$ for trigonometric interpolation and the closely related interpolation by series of Chebyshev polynomials. Lanczos (1956, p. 477) also cites $x_{n,\nu}$ as suitable in his τ-method for the numerical solution of differential equations. The Clenshaw-Curtis quadrature method, including variations and applications, is discussed in Clenshaw and Curtis (1960), Filippi (1964), Elliott (1965), Fraser and Wilson (1966), Wright (1966), and O'Hara and Smith (1968). The closed Chebyshev quadrature formula

$$\int_{-1}^{1} (1 - x^2)^{-1/2} f(x)\, dx = \frac{\pi}{n} \sum_{\nu=0}^{n}{}'' f(x_{n,\nu}) - \frac{\pi f^{(2n)}(\xi)}{2^{2n-1}(2n)!}, \ -1 < \xi < 1 \qquad (1)$$

where in $\sum''$ the first and last terms are halved, is treated extensively in Chawla (1968 and 1970). For the use of Chebyshev series interpolation, based upon $x_{n,\nu}$ for solving non-linear differential equations, see Clenshaw and Norton (1963) and Norton (1964), and for solving integral or integro-differential equations, see Elliott (1963) and Wolfe (1969). For near-minimax polynomial approximation, employing $x_{n,\nu}$, see Fraser (1965, pp. 310-313), Gavrilyuk and Mazanovskaya (1966) and Meinardus (1967, pp. 72-74). For numerical differentiation the points $x_{n,\nu}$ are optimal in the sense of computational stability, by virtue of the property that out of all the sets of $n + 1$ fixed base points for Lagrangian interpolation, they give the least upper bound for the sum of the absolute values of the kth derivative, $1 \leqslant k \leqslant n$, of the Lagrange coefficients, over the range $-1 \leqslant x \leqslant 1$. A proof which relies almost entirely upon an essential theorem of Duffin and Schaeffer (1941), is given in Rivlin (1969, pp. 117-118); see also Berman (1964).

For Lagrangian interpolation the following has been noted in the literature:

The points $x_{n,\nu}$ are the $n + 1$ zeros of $(x^2 - 1)\, U_{n-1}(x)$, where $U_{n-1}(x) = \sin n\theta/\sin\theta$, $x = \cos\theta$, is the Chebyshev polynomial of the second kind, with leading coefficient 2^{n-1}. Also, we note here that

$$(x^2 - 1)\, U_{n-1}(x)/2^{n-1} = [\cos(n + 1)\theta - \cos(n - 1)\theta]/2^n \qquad (2)$$

so that $|(x^2 - 1)\, U_{n-1}(x)/2^{n-1}| \leqslant 1/2^{n-1}$, which is attained only for odd n. As a consequence of (2), in the remainder term of the $(n + 1)$ point Lagrange interpolation formula for $f(x)$, $-1 \leqslant x \leqslant 1$, based upon $x_{n,\nu}$, namely

$$\prod_{\nu=0}^{n} (x - x_{n,\nu}) f^{(n+1)}(\xi)/(n + 1)!, \ -1 < \xi < 1,$$

we have

$$\left| \prod_{\nu=0}^{n} (x - x_{n,\nu}) \right| \leqslant 2^{-n+1}.$$

This is near minimal, being no more than twice the best upper bound of 2^{-n} when $x'_{n+1,\nu} \equiv \cos[(2\nu - 1)\pi/(2n + 2)]$, $\nu = 1(1)\, n + 1$, is in place of $x_{n,\nu}$.

The computational stability of Lagrangian interpolation for any fixed set of $n + 1$ base-points, as n increases, is measured by an upper bound for the sum of the absolute values of the Lagrange coefficients, for $-1 \leqslant x \leqslant 1$. For $x_{n,\nu}$, an upper bound according to Berman (1963), is $8 + (2/\pi)\ln(n + 1)$, which is comparable to that for $x'_{n+1,\nu}$, namely $1.9 + (2/\pi)\ln(n + 1)$, obtained by a slight refinement of the argument in Rivlin (1969, pp. 93-96).

Berman's upper bound implies that when $f(x)$ is continuous in $[-1, 1]$ and its modulus of continuity $\omega(\delta)$ satisfies the Dini-Lipschitz condition $\lim_{\delta \to 0} \omega(\delta) \ln \delta = 0$, the Lagrange interpolation polynomial, based upon $x_{n,\nu}$, converges uniformly to $f(x)$ as $n \to \infty$. The proof is identical with that for the Chebyshev points $x'_{n+1,\nu}$ given in Natanson (1955, pp. 389-392). Also for convergence when $f(x)$ is either absolutely continuous, or has bounded variation, or has an absolutely convergent Chebyshev expansion, but only for $x'_{n+1,\nu}$, see Krylov (1956), and Johnson and Riess (1970). However, the latter indicates (p. 355) that their methods are also applicable to the problem of convergence of the polynomials that interpolate at $x_{n,\nu}$.

Thus it is apparent, from the existing literature, that the Lagrange interpolation formulas based upon the Chebyshev points $x_{n,\nu}$ and $x'_{n+1,\nu}$ have practically the same advantages of minimal remainder, computational stability and uniform convergence.

2. Additional advantages in Lagrangian interpolation

The purpose of this present note is to point out five additional advantages in Lagrangian interpolation based upon $x_{n,v}$, $v = 0(1)n$, in particular, when the interpolating polynomial is expressed in barycentric form.

The Lagrangian interpolation polynomial of the nth degree which equals $f(x_v)$ at any $n + 1$ points x_v, $v = 0(1)n$, say $L_n(x)$, is given by

$$L_n(x) = \sum_{v=0}^{n} \frac{\phi_{n+1}(x) f(x_v)}{(x - x_v)\, \phi'_{n+1}(x_v)} ,$$

where

$$\phi_{n+1}(x) \equiv \prod_{v=0}^{n} (x - x_v) \tag{3}$$

From the uniqueness of $L_n(x)$ we obtain the barycentric form of (3), which is

$$L_n(x) = \sum_{v=0}^{n} \frac{A_v f(x_v)}{x - x_v} \Big/ \sum_{v=0}^{n} \frac{A_v}{x - x_v} ,$$

where A_v are any $n + 1$ quantities that are proportional to the divided difference coefficients

$$1/\phi'_{n+1}(x_v) \tag{4}$$

A detailed discussion of the computational advantages in calculating $L_n(x)$ by (4), apart from the advantages in some particular choice for x_v, is given in Winrich (1969).

When in (4) the points x_v are chosen to be $x_{n,v}$, in which case $\phi_{n+1}(x)$ is given by the left member of (2), we obtain the divided difference coefficients $1/\phi'_{n+1}(x_{n,v})$ by differentiating the right member of (2) with respect to $x = \cos\theta$ and then setting $x = x_{n,v}$ or $\theta = v\pi/n$, $v = 0(1)n$. We find

$$\frac{1}{\phi'_{n+1}(x_{n,v})} = \frac{2^n}{(n+1)\sin(n+1)\theta/\sin\theta - (n-1)\sin(n-1)\theta/\sin\theta}\Bigg|_{\theta = v\pi/n} \tag{5}$$

from which, by direct substitution for $v \neq 0, n$, and by taking limits for $v = 0, n$, we get

$$\frac{1}{\phi'_{n+1}(x_{n,0})} = \frac{2^{n-2}}{n} ;$$

$$\frac{1}{\phi'_{n+1}(x_{n,v})} = (-1)^v \frac{2^{n-1}}{n} , \ v = 1(1)\, n - 1 ;$$

$$\frac{1}{\phi'_{n+1}(x_{n,n})} = (-1)^n \frac{2^{n-2}}{n} . \tag{6}$$

Therefore in (4) for $x_v = x_{n,v}$, by choosing

$$A_v = \frac{n}{2^{n-1}\, \phi'_{n+1}(x_{n,v})} \tag{7}$$

we obtain

$$A_0 = \tfrac{1}{2};\ A_v = (-1)^v,\ v = 1(1)\, n - 1;\ A_n = \tfrac{1}{2}(-1)^n \tag{8}$$

so that finally

$$L_n(x) = \sum_{v=0}^{n}{}'' \frac{(-1)^v f(x_{n,v})}{x - x_{n,v}} \Big/ \sum_{v=0}^{n}{}'' \frac{(-1)^v}{x - x_{n,v}} \tag{9}$$

The first advantage in Lagrangian interpolation at $x_{n,v}$ is the simplicity of (9). For, in addition to the advantages in the barycentric form *per se*, especially for larger values of n, that have been already cited in Winrich (1969), comparison of (4) with (9) shows for the latter no storage and retrieval of A_v. Furthermore, replacement of the $n + 1$ division operations for $A_v/(x - x_v)$ in (4) by $n + 1$ reciprocal operations for $1/(x - x_{n,v})$ in (9) might save time (the amount depending upon the structure of the machine language and program employed in division) when we avoid many-digit values of A_v. In this discussion we discount the factor $\frac{1}{2}$ in $\sum''$ because of the binary structure permeating most machine operations.

There is also an alternative method of computing the right member of (9), which takes advantage of the absence of A_v, and where we reduce further the number of operations at the expense of more storage.

From $\phi_{n+1}(x) = (x^2 - 1)\, U_{n-1}(x)/2^{n-1}$ and (7), we have

$$1 \Big/ \sum_{v=0}^{n}{}'' \frac{(-1)^v}{x - x_{n,v}} = (x^2 - 1)\, U_{n-1}(x)/n \tag{10}$$

Now if the program for $L_n(x)$ might include the storage of a table of $(x^2 - 1)\, U_{n-1}(x)/n$, at a fine interval in x, obtaining $L_n(x)$ requires just $n + 1$ divisions and 1 multiplication, instead of $n + 1$ reciprocals, $n + 1$ multiplications and 1 division. For (9) the user may decide whether or not it is better to employ this alternative method. However, for (4) in general, due to the presence of A_v, there is no saving in the number of operations by storing a table of

$$1 \Big/ \sum_{v=0}^{n} [A_v/(x - x_v)] ,$$

because the numerator would still require $n + 1$ multiplications and $n + 1$ divisions.

It is interesting to point out here that also in the case where the x_v are equal to the Chebyshev points

$$x'_{n+1,v} \equiv \cos\left[(2v - 1)\, \pi/(2n + 2)\right],\ v = 1(1)\, n + 1,$$

and whenever n is *even*, we do not have to store any A_v after we have stored the $x'_{n+1,v}$. The reason is that the A_v are proportional to the $x'_{n+1,v}$, save for sign and a different ordering in the v (for the special case, when $n + 1 = 2^m$, see Salzer (1969)). But there we still have the extra work of performing divisions instead of reciprocals, not counting the work in retrieving the $x'_{n+1,v}$, in the proper order, for the role of A_v.

Applying (9) to the Chebyshev polynomial $T_n(x) \equiv \cos(n \arccos x)$, for which $T_n(x_{n,v}) = (-1)^v$, we obtain this pretty identity which the writer has not encountered elsewhere:

$$T_n(x) = \sum_{v=0}^{n}{}'' \frac{1}{x - x_{n,v}} \Big/ \sum_{v=0}^{n}{}'' \frac{(-1)^v}{x - x_{n,v}} \tag{11}$$

Meinardus (1967) comes close to formulating (9) in two places. On p. 44 occurs just the bare statement, without proof, that

$$\sum_{v=0}^{n}{}'' (-1)^v H(x_{n,v}) = 0$$

whenever $H(x) = x^{n+1} + a_{n-1} x^{n-1} + \ldots + a_0$. This implies

$$\sum_{v=0}^{n}{}'' (-1)^v x^r_{n,v} = K\delta^r_n ,\ r = 0(1)n,$$

where $K \neq 0$ since $\det \|x^r_{n,v}\| \neq 0$. But since $K/\phi'_{n+1}(x_{n,v})$ (knowledge of K unnecessary here) satisfies the preceding system of equations whose solution is unique, $\frac{1}{2}, \ldots, (-1)^v, \ldots, \frac{1}{2}(-1)^n$, are proportional to the divided difference coefficients. On pp. 74-76 this proportionality is established explicitly, K determined to be $n/2^{n-1}$, by employing a complex variable argument. However, Meinardus does not indicate the crucial final step to (9), in which is the practical application of that proportionality.

A second advantage of interpolation at $x_{n,v}$ is the extreme simplicity and stability of a checking formula for $f(x_{n,v})$, $v = 0(1)n$. Whenever $f(x)$ is a polynomial of the $(n - 1)$th degree or less, or can be approximated to the desired accuracy by some $(n - 1)$th degree polynomial which we do not have to know, any quantity proportional to the nth divided difference of $f(x_{n,v})$ vanishes, save for the cumulation of errors due to roundoff. In particular, if we employ the barycentric coefficients in the 'alternating trapezoidal sum' an overall check on the correctness of $f(x_{n,v})$ is furnished by

$$\sum_{v=0}^{n}{}'' (-1)^v f(x_{n,v}) = 0 \tag{12}$$

Of course, when (12) does not hold we cannot localise any error or combination of errors in $f(x_{n,v})$. But very much in favour of (12), is the fact that it involves merely a summation, without a single multiplication, in contrast to difference or divided difference checks based upon functional values at other points. Thus to illustrate the convenience and stability of (12), we compare it with ordinary differencing by supposing that for $n = 100$, instead of $f(x_{100,v})$, we have 101 equally spaced values of $f(x_v)$, $v = 0(1)100$, and wish to check them by taking the 100th difference. The multiplier of $f(x_{50})$ alone would be around 10^{29}, whether in direct multiplication by ${}_{100}C_{50}$ in the formula for the 100th difference, or as a result of the $100 + 99 + \ldots + 1 = 5050$ subtractions (instead of the 100 addition-subtractions in using (12)) in finding successive differences up to the 100th.

For the third advantage, suppose that we require very high accuracy in the barycentric formula (9), and n is replaced by 2^n. It is shown in Salzer (1969, p. 271) that, for the 2^n-point barycentric formula based upon $x'_{2^n v}$, $v = 1(1)2^n$, the zeros of $T_{2^n}(x)$, we may calculate $x'_{2^n,v}$ recursively by

$$x'_{2^m,v} = \pm[(1 + x'_{2^{m-1},v'})/2]^{1/2},\ v = 1(1)2^m,$$
$$v' = 1(1)2^{m-1},\ m = 2(1)n,$$

beginning with

$$x'_{2^1,v'} = \pm(1/2)^{\frac{1}{2}},\ v' = 1, 2 \tag{13}$$

On p. 385 it is pointed out that only $2^n - 1$ square roots are needed for $(2^n - 1)$th degree accuracy. Now here for (9), to calculate $x_{2^n,v}$, $v = 0(1)2^n$, begin with the middle and end points $1, 0, -1$, corresponding to $v = 0,\ 2^{n-1},\ 2^n$ resp., and then apply (13) only as far as $n - 1$ instead of n, to fill in successively $x'_{2^m,v}$, $v = 1(2)2^m$, $m = 1(1)n - 1$, the zeros of $T_2(x)$, $T_4(x), \ldots, T_{2^{n-1}}(x)$. Then it is apparent that to obtain $x_{2^n,v}$, $v = 0(1)2^n$, a total of only $2^{n-1} - 1$ square roots are needed for 2^nth degree accuracy in (9). (Of course all this discussion is apart from that concerning the operations needed for getting $f(x_{2^n,v})$.) To summarise, less than half the number of square roots yields one degree higher accuracy when the interpolation is at the Chebyshev points $x_{2^n,v}$, $v = 0(1)2^n$, instead of $x'_{2^n v}$, $v = 1(1)2^n$.

The fourth advantage is in the solution of a near-minimax problem which is really one of approximation, rather than one of interpolation. But a convenient form of the solution is expressible as the barycentric interpolation formula for a suitably modified $f(x_{n,v})$. The near-minimax problem, for the $n + 1$ points $x_{n,v}$ is to find the polynomial of $(n - 1)$th degree, say $P_{n-1}(x)$, such that

$$f(x_{n,v}) - P_{n-1}(x_{n,v}) = (-1)^v \lambda,\ v = 0(1)n \tag{14}$$

In Meinardus (1967, pp. 72-74), there is a solution for $P_{n-1}(x)$ which gives formulas for the polynomial coefficients of $f(x_{n,v}) + f(x_{n,v+1})$ in terms of $U_{n-1}(x)$ and $T_n(x)$. In his notation, $x_{n,v} = -\cos(v\pi/n)$. Furthermore, Gavrilyuk and Mazanovskaya (1966), in a rather extensive tabulation, give the coefficients of these polynomial multipliers of $f(x_{n,v}) + f(x_{n,v+1})$. But oddly enough, computing $P_{n-1}(x)$ by those formulas involving $U_{n-1}(x)$ and $T_n(x)$ can be avoided completely by noting the general solution for any x_v, $v = 0(1)n$, given in Meinardus (1967, p. 73), which for $x_v = x_{n,v}$ (our notation for $x_{n,v}$) reads

$$P_{n-1}(x) = \sum_{v=0}^{n} \frac{\phi_{n+1}(x)\,(f(x_{n,v}) - (-1)^v\lambda)}{(x - x_{n,v})\,\phi'_{n+1}(x_{n,v})} \tag{15}$$

where

$$\lambda = \sum_{v=0}^{n} \lambda_v f(x_{n,v}) \tag{16}$$

and

$$\lambda_v = \frac{1}{\phi'_{n+1}(x_{n,v}) \displaystyle\sum_{\mu=0}^{n} \frac{(-1)^\mu}{\phi'_{n+1}(x_{n,\mu})}},\ v = 0(1)n \tag{17}$$

Since (6), (7) and (17) imply that $\lambda_v = A_v/n$, we have

$$\lambda = \frac{1}{n}\sum_{v=0}^{n}{}'' (-1)^v f(x_{n,v}) \tag{18*}$$

Now $P_{n-1}(x)$, given by (15), is calculated much more easily by (18) followed by the barycentric formula (9) in which $f(x_{n,v})$ is replaced by $f(x_{n,v}) - (-1)^v \lambda$.

The fifth advantage is in the use of the barycentric formula to facilitate numerical differentiation. The optimal character of the points $x_{n,v}$ for numerical differentiation by taking the kth derivative of (3), is expressible by the result that

$$\max \sum_{v=0}^{n} \left|d^k[\phi_{n+1}(x)/(x - x_v)\,\phi'_{n+1}(x_v)]/dx^k\right|,$$

for $-1 \leqslant x \leqslant 1$, is minimal for $x_v = x_{n,v}$, $v = 0(1)n$, when it is equal to $n^2(n^2 - 1^2)\ldots(n^2 - (k-1)^2)/1\cdot3\cdots(2k-1)$ (see Rivlin (1969, pp. 117-118)). The remarkable feature of this result is that the same set of points $x_{n,v}$ is optimal for every k, $1 \leqslant k \leqslant n$. The practical importance of this optimality is apparent when we differentiate $L_n(x)$ based upon $x_{n,v}$ and compare it with the result of differentiating an $L_n(x)$ based upon points x_v that are equally spaced. In the latter case, especially for very large n, we get impractical and unstable formulas due to the enormous magnitude of the coefficients of $f(x_v)$.

To calculate the kth derivative of (3) when $x_v = x_{n,v}$, $v = 0(1)n$, for the right member in the form of that of (3) or (9), for $-1 \leqslant x \leqslant 1$, $k \geqslant 1$, by direct differentiation with respect to x, might involve too much computation, which we may be able to by-pass. Now $d^k L_n(x)/dx^k$, $k \geqslant 1$, is also obtainable by $(n - k)$th degree Lagrangian interpolation from its values at any $n + 1 - k$ points. Thus we could combine the above mentioned optimal feature of numerical differentiation with the convenience of the barycentric form of the Lagrange interpolation formula. For the important values of n, and $k = 1, 2, \ldots$, we precompute from (3) or (4) for $x_v = x_{n,v}$, $v = 0(1)n$, the derivatives $d^k L_n(x)/dx^k$ for the $n + 1 - k$ values of $x = x_{n-k,v'} \equiv \cos[v'\pi/(n-k)]$, $v' = 0(1)n - k$. Then the barycentric formula (9), with $n - k$ in place of n, is applied to find $d^k L_n(x)/dx^k$ from $d^k L_n(x_{n-k,v'})/dx^k$, $v' = 0(1)n - k$. Thus we have

$$\frac{d^k L_n(x)}{dx^k} = \sum_{v'=0}^{n-k}{}'' \frac{(-1)^{v'} d^k L_n(x_{n-k,v'})/dx^k}{x - x_{n-k,v'}} \Bigg/ \sum_{v'=0}^{n-k}{}'' \frac{(-1)^{v'}}{x - x_{n-k,v'}} \tag{19}$$

The quantities $d^k L_n(x_{n-k,v'})/dx^k$ are found conveniently by differentiating (3) for $x_v = x_{n,v}$, so that the coefficient of $f(x_{n,v})$ becomes $d^k[\phi_{n+1}(x)/(x - x_{n,v})\phi'_{n+1}(x_{n,v})]$, where $\phi_{n+1}(x) = (x^2 - 1)\,U_{n-1}(x)/2^{n-1}$ and $1/\phi'_{n+1}(x_{n,v})$ is given by (6). Then in each of the $n + 1$ coefficients we set $x = x_{n-k,v'}$, $v' = 0(1)n - k$. Abbreviating $L_{n,v}(x) \equiv \phi_{n+1}(x)/(x - x_{n,v})\,\phi'_{n+1}(x_{n,v})$ as L, $d^k L_{n,v}(x)/dx^k$, $k = 1, 2, \ldots$ as $L', L'', \ldots$, and $\sum_{\mu=0,\mu\neq v}^{n} (x - x_{n,\mu})^{-r}$, $r \geqslant 1$, as $\sum_r$, where in $L, L', L'', \ldots$ and

*The referee has pointed out that (18) follows also from (12) and (14), by operating with $\sum_{v=0}^{n}{}''(-1)^v \ldots$ upon both members of (14).

$\sum_r$ the n, v and x are understood, by repeated use of $L' = L \sum_1$, we find for $k = 1(1)4, \ldots$, and any n, v and $x \neq x_{n,\mu}$,

$$L' = L\sum_1,\ L'' = L(\sum_1^2 - \sum_2),\ L''' = L(\sum_1^3 - 3\sum_1\sum_2 + 2\sum_3),$$
$$L'''' = L(\sum_1^4 - 6\sum_1^2\sum_2 + 8\sum_1\sum_3 + 3\sum_2^2 - 6\sum_4), \ldots \quad (20)$$

As long as $x_{n-k,v'} \neq x_{n,\mu}$ (which is nearly always the case) we substitute $x_{n-k,v'}$ directly into (20). Then after we find

$d^kL_n(x_{n-k,v'})/dx^k =$

$$\sum_{v=0}^{n} \left[\frac{d^k}{dx^k} \left\{ \frac{\phi_{n+1}(x)}{(x - x_{n,v})\,\phi'_{n+1}(x_{n,v})} \right\}_{x = x_{n-k,v'}} \cdot f(x_{n,v}) \right],$$
$$v' = 0(1)n - k, \quad (21)$$

we are ready for (19). However, for certain combinations of n, k, v' and $\mu \neq v$ we have $x_{n-k,v'} = x_{n,\mu}$, e.g., any n, any k, $v' = n - k$, $\mu = n$, or $n = 8$, $k = 4$, $v' = 3$, $\mu = 6$. Then in (20), since $\lim_{x \to x_{n,\mu}} L^{\overbrace{''\cdots'}^{k}}$ exists for every k, and L has a simple zero at $x_{n,\mu}$, the $(\ldots)$ factor, when expanded as a finite series of the form $\alpha_0(x) + \alpha_1(x)/(x - x_{n,\mu}) + \alpha_2(x)/(x - x_{n,\mu})^2 + \ldots$ (where $\alpha_r(x)$ is really $\alpha_{n,v,k,\mu,r}(x)$), must have $\alpha_r(x) \equiv 0$, $r > 1$. Since $L\alpha_0(x)$ vanishes at $x_{n,\mu}$, we find

$$L^{\overbrace{''\cdots'}^{k}}\Big|_{x = x_{n,\mu}} = \frac{L}{(x - x_{n,\mu})}\Big|_{x = x_{n,\mu}} \cdot \alpha_1,$$
$$\text{where } \alpha_1 \equiv \alpha_1(x_{n,\mu}) \quad (22)$$

which is really a concise expression for

$$\frac{d^k}{dx^k} \left[\frac{\phi_{n+1}(x)}{(x - x_{n,v})\,\phi'_{n+1}(x_{n,v})} \right]_{x = x_{n-k,v'}} = \left[\frac{\phi_{n+1}(x)}{(x - x_{n,v})\,(x - x_{n-k,v'})\,\phi'_{n+1}(x_{n,v})} \right]_{x = x_{n-k,v'}}$$
$$\times\ \alpha_{n,v,k,\mu,1}(x_{n-k,v'}),\ x_{n-k,v'} = x_{n,\mu}, \mu \neq v \quad (22')$$

Following are the expressions for $\alpha_1 \equiv \alpha_1(x_{n-k,v'}) \equiv \alpha_{n,v,k,\mu,1}(x_{n-k,v'})$ obtained from (20), for $k = 1(1)4$:

$$k = 1,\ \alpha_1 = 1;\ k = 2,\ \alpha_1 = 2\bar{\sum}_1;\ k = 3,\ \alpha_1 = 3\bar{\sum}_1^2 - 3\bar{\sum}_2;$$
$$k = 4,\ \alpha_1 = 4\bar{\sum}_1^3 - 12\bar{\sum}_1\bar{\sum}_2 + 8\bar{\sum}_3,\ \text{where } \bar{\sum}_r \equiv \sum_{\sigma = 0, \sigma \neq v, \mu}^{n} (x_{n-k,v'} - x_{n,\sigma})^{-r},\ r \geqslant 1, \quad (23)$$

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