Simultaneous trigonometric approximation of the function and its first derivative

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We study simultaneous cosine trigonometric approximations involving the function and its first derivative over sets of equidistant sampling points. A numerical algorithm is indicated for use in an automatic computer.

Using an example we compare this method with the classical cosine trigonometric one (where no derivative information is considered) and outline its flexibility in graphical applications.

(Received March 1972)

1. Introduction

Trigonometric approximations of observational functions defined over sets of discrete points have been known for a long time. From Lipka (1918) we learned that trigonometric approximations over sets of equidistant sampling points using the discrete orthonormality properties (2.I) were already known in 1900. Recently this was also extended to sets of unequally-spaced sampling points (see Oliveira-Pinto, 1967; Newbery, 1970).

Elsewhere we show that for sets of equidistant sampling points, the trigonometric polynomials are just one of the infinitely many sets of *generalised polynomials* with equidistant zeros (see Oliveira-Pinto, 1972) that approximate the discrete data values in an optimal way.

In spite of such optimal approximating properties over sets of equidistant sampling points these trigonometric polynomials are known to suffer from the following disadvantage: If we try to approximate $Z(x)$, defined by $Z_s = Z(x_s), s \to 0, 1, \ldots, q$ they tend to oscillate strongly *in between* data values everytime that a part of $Z(x)$ which is presumed to be flat is followed by a section with a sharp bend in it. Let us look at an example:

In **Fig. 2** we show the result of a trigonometric interpolation over the 21 sampled values of the Runge test function

$$Z(x) = \frac{1}{1 + x^2}, \quad x = \pm 10, \pm 9, \pm 8, \ldots 0 .$$

Around the points corresponding to $x = \pm 2{\cdot}5$ where the sharp bend begins these oscillations are clearly visible. To control such undesirable oscillations we may decide looking at this example to prescribe at each sampling point $x_{s\to 0,1,\ldots,20}$ a suitable slope $Z^{(1)}_{s\to 0,1,\ldots,20}$. By doing this, we stress the shape of the approximating curve in between data values if not completely, at least in their neighbourhood. A possible result of such a strategy is given in **Fig. 4.** There, to the data of Fig. 2, another 21 slopes were conveniently added.

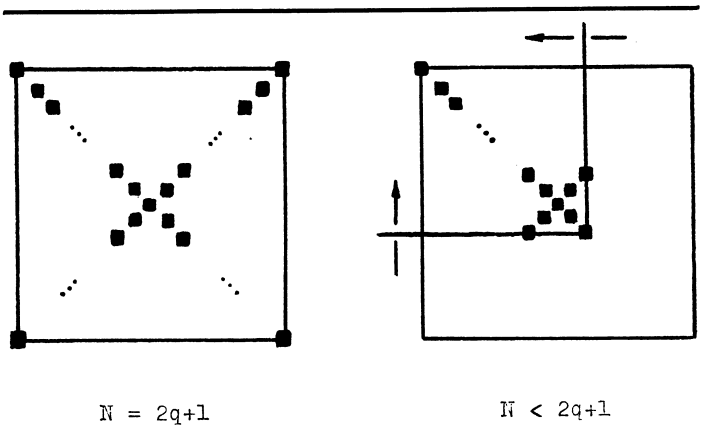

Fig. 1. The system of 'normal equations'

It is, then, our intention to explore in this paper the possibility of introducing known or fictitious derivative information to let us say, 'correct' such trigonometric approximations.

First we introduce a simple method to compute cosine trigonometric approximations over sets of equidistant sampling points for the function $Z_{s\to 0,1,\ldots,q}$ and first derivative values $Z^{(1)}_{s1\to 0,1,\ldots,q_1}$. Afterwards, using the method, we present and discuss several numerical examples.

The case of unequally-spaced sampling points is left to a future paper.

In the following, to simplify the notation, we presume that the range of the independent variable x has been scaled to the traditional sampling interval $[0, \pi]$.

2. Simultaneous trigonometric approximation

Let us suppose given the data values $Z_{s\to 0,1,\ldots,q}$, on the sampling points

$$x_s = \pi \frac{s + \frac{1}{2}}{q + 1}, \quad s \to 0, 1, \ldots, q \tag{2.A}$$

the slopes $Z^{(1)}_{s1\to 0,1,\ldots,q_1}$ over the set of possibly different points

$$x_{s1} = \pi \frac{s + \frac{1}{2}}{q_1 + 1}, \; s1 \to 0, 1, \ldots, q_1 \tag{2.B}$$

and the positive integer $N < q + q_1 + 2$. We intend to approximate both $Z_{s\to 0,1,\ldots,q}$ and $Z^{(1)}_{s1\to 0,1,\ldots,q_1}$ simultaneously using the linear approximating form

$$\Phi_N(A_k, x) = \tfrac{1}{2} A_0 + A_1 \cos x + \ldots + A_N \cos Nx, \; x \in [0, \pi] \tag{2.C}$$

in such a way that for the systems of positive weights $H_{s\to 0,1,\ldots,q}$, $H_{s1\to 0,1,\ldots,q_1}$ and for a positive weighting parameter $\Lambda > 0$,

$$J_N(A_k) = \sum_s H_s[Z_s - \Phi_N(A_k, x_s)]^2 + \Lambda \sum_{s1} H_{s1}[Z^{(1)}_{s1} - \Phi^{(1)}_N(A_k x_{s1})]^2 \tag{2.D}$$

is minimum in the $A_{k\to 0,1,\ldots,N}$.

Usually we expect to have $x_{s1\to 0,1,\ldots,q_1}$ as a sub-set of $x_{s\to 0,1,\ldots,q}$ but this may not be so. The minimisation of (2.D) leads to the solution of the linear system in the A_k:

$$\sum_k{}' A_k[\langle \cos_k, \cos_r \rangle]_H - [\langle Z, \cos_r \rangle]_H = 0 \quad k \to 0, 1, \ldots, N \quad r \to 0, 1, \ldots, N \tag{2.E}$$

with

*This work was partly financed by the grant No. 663 of the Science Department of the Gulbenkian Foundation.

$$[\langle \cos_k, \cos_r \rangle][\ldots]_H = \sum_s H_s \cos(kx_s)\cos(rx_s) + \Lambda \sum_{s1} H_{s1} \cos^{(1)}(kx_{s1}) \cos^{(1)}(rx_{s1}) \quad (2.F)$$

$$[\langle Z, \cos_r \rangle][\ldots]_H = \sum_s H_s Z_s \cos(rx_s) + \Lambda \sum_{s1} H_{s1} Z_{s1}^{(1)} \cos^{(1)}(rx_{s1}) \quad (2.G)$$

where the quantities (2.F) and (2.G) can be computed directly from the available data. The prime in the summation of (2.E) means that the term for $k = 0$ is halved.

The solution of this system is highly simplified due to the orthogonal properties of the sequence of trigonometric polynomials $\cos(rx)$ and $\cos^{(1)}(rx)$ $r \to 0, 1, \ldots, N$ over the sets (2.A) and (2.B).

Let us study in detail such properties when $q_1 \leqslant q$ (the case $q_1 \geqslant q$ is similar).

In the following it will occasionally be simpler to use Q and Q_1 to represent

$$Q = q + 1 \text{ and } Q_1 = q_1 + 1 \quad (2.H)$$

For $N \leqslant q_1$ the discrete orthogonality conditions (see e.g. Lipka, 1918, p. 176 or Lanczos, 1938) $k \to 0, 1, \ldots, N$ $r \to 0, 1, \ldots, N$ give directly.

$$\sum_s H_s \cos(kx_s)\cos(rx_s) = \delta_{k,r} \text{ with } H_s = \frac{2}{q+1} \quad (2.I)$$

$$kr \sum_{s1} H_{s1} \sin(kx_{s1}) \sin(rx_{s1}) = kr\delta_{k,r} \text{ with } H_{s1} = \frac{2}{q_1 + 1}$$

where $\delta_{k,r}$ is the usual Kronecker symbol δ (except $\delta_{0,0} = 2$) and due to (2.F) the system (2.E) reduces to

$$A_r = \frac{[\langle Z, \cos_r \rangle]_H}{1 + r^2\Lambda}, \quad r \to 0, 1, \ldots, N \quad (2.J)$$

For $N > q_1$ the classical properties (2.I) do not hold anymore. Nevertheless we still have

$$\sum_s H_s \cos(kx_s) \cdot \cos(rx_s) = \delta_{k,r} - \delta_{k,2q-r+2} - {}_{k,2q+r+2} + \delta_{k,4q-r+4} + \ldots \quad (2.K)$$

$$\sum_{s1} H_{s1} \sin(kx_{s1}) \cdot \sin(rx_{s1}) = \delta_{k,r} + \delta_{k,2q_1-r+2} - \delta_{k,2q_1+r+2} - \delta_{k,4q_1-r+4} + \ldots$$

and due to (2.F) and (2.H) the system (2.E) can still be reduced to

$$\begin{cases} A_0 - 2A_{2Q} + \ldots = [\langle Z, \cos_0 \rangle]_H & (2.L) \\ [A_r - A_{2Q-r} - \ldots] + r\Lambda\,[rA_r + (2Q_1 - r)\,A_{2Q_1-r} - \ldots] \\ \qquad = [\langle Z, \cos_r \rangle]_H \quad r \to 1, 2, \ldots, N \end{cases}$$

Let us now consider the important case where $q_1 = q$, i.e. when at every sampling point $x_{s \to 0,1,\ldots,q}$ we have one function value Z_s and one (known or 'estimated') slope $Z_s^{(1)}$. In this case the system (2.L) where the quantities (2.G) are known, is just

$$A_0 = [\langle Z, \cos_0 \rangle]_H \quad (2.M)$$

$$[1 + r^2\Lambda]\,A_r - [1 - r(2Q - r)\,\Lambda]\,A_{2Q-r} = [\langle Z, \cos_r \rangle]_H \quad (2.N)$$

because the number of unknowns $A_{r \to 0,1,\ldots,N}$ cannot be greater than the number of conditions, i.e. $N < 2Q$.

For $N = 2q + 1$, (2.N) is a very interesting system of equations since it is a cross-diagonal one. It can be easily solved taking two equations at a time, e.g. the first and the last to obtain A_1 and A_{2Q-1} the second and the $(2Q - 2)$th to obtain A_2 and A_{2Q-2}; etc.

To obtain in general A_r and A_{2Q-r} with $r \neq Q$, we have

$$[1 + r^2\Lambda]A_r - [1 - r(2Q - r)\Lambda]A_{2Q-r} = [\langle Z, \cos_r \rangle]_H \quad (2.O)$$

$$[1 + (2Q - r^2)\Lambda]A_{2Q-r} - [1 - r(2Q - r)\Lambda]A_r = [\langle Z, \cos_{2Q-r} \rangle]_H$$

The determination of A_Q is instead given directly by

$$A_Q = \frac{[\langle Z, \cos_Q \rangle]_H}{2Q^2\Lambda} \quad (2.P)$$

For $N < 2q + 1$ the system (2.N) becomes a mixture of a diagonal one plus a cross-diagonal one, and it may look like the one represented schematically in **Fig. 1.**

The solution of (2.N) for $N < 2q + 1$ is therefore very similar to the previous case—$N = 2q + 1$—only the number of systems (2.O) is reduced, because we have directly $A_{k \to N+1, N+2, \ldots} = 0$.

It is also interesting to note that the value of the determinant Δ of (2.O) is independent of the subscript r. In effect it can be easily proved that it is always given by:

$$\Delta = 4(q + 1)^2\Lambda \quad (2.Q)$$

Thus, taking

$$\Lambda = \frac{1}{4(q+1)^2} \quad (2.R)$$

then we simply have $\Delta = 1$.

To conclude we may say that the time needed to compute the coefficients A_k of (2.C) for the approximation of data involving function and first derivative values over the same set of equivalent sampling points, is almost the same if we had used instead the double number of function values and no derivatives.

In fact what takes time to compute are the quantities (2.G) and for the double number of function values we will have the first sum in (2.G) with the double number of terms instead of the second sum. Therefore the number of arithmetic operations is exactly the same.

3. Numerical examples

It is always a problem of conscience to choose a suitable example to illustrate a scheme of numerical approximation. In principle the example should be as simple as feasible for easy understanding, as realistic as possible for general acceptance, and above all, it should not be an example which shows the method to best advantage!

Having this in mind, we have chosen the example originally introduced by Runge and afterwards presented with slight variations by other mathematicians like Mineur (1952), p. 426, and Lanczos (1961), p. 12.

It is defined here by

$$Z(X) = \frac{1}{1 + X^2} \quad (3.A)$$

and its first derivative with respect to $X - Z^{(1)}(X)$, for the equidistant sampling points $X_s \to -10, -9, \ldots +9, +10$.

After the normalisation of the X interval $[-10, +10]$ to $[0, \pi]$ the function may be written:

$$Z(x) = \frac{1}{1 + 100\left[\left(2x - \frac{\pi}{21}\right) \Big/ \left(\pi - \frac{\pi}{21}\right) - 1\right]^2} \quad (3.B)$$

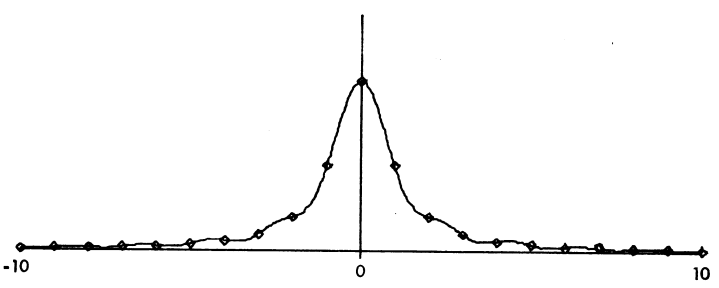

Fig. 2. Cosine trigonometric approximation of order $N = 20$

for

$$x_s = \pi \frac{s + \frac{1}{2}}{21}, s \to 0, 1, \ldots, 20 \qquad (3.C)$$

The expression (3.B) has no practical interest because the corresponding data values are not affected by the scaling of the independent variable and therefore can be obtained directly from (3.A). The same is unfortunately not true with the first derivative data values because

$$\frac{dZ}{dx} = \frac{dX}{dx} \cdot \frac{dZ}{dX} \text{ with } \frac{dX}{dx} = \frac{21}{\pi} \cdot \qquad (3.D)$$

The 21 function values $Z_{s\to 0,1,\ldots,20}$ and the already scaled 21 derivative values $Z^{(1)}_{s\to 0,1,\ldots,20}$ define therefore the basic table that we are going to use for our numerical experiments.

Experiment 1

For the first trial we have taken only the 21 function values $Z_{s\to 0,1,\ldots,20}$, in order to obtain a classical cosine trigonometric approximation of order $N = 20$, as shown in Fig. 2. It will be used as a reference for the following trials.

There, when a flat part of $Z(X)$ is followed by a relatively sharp bend, the characteristic oscillation between data values is observed and it often forces the rejection of this type of approximation. Its maximum error $\varepsilon = 0{\cdot}04$ is obtained around the points $X = \pm 1{\cdot}5$ of (3.A).

Experiment 2

Here we have only about half, i.e. 11 function values over the same range together with 11 derivative values for the same sampling points.

With these data values we obtained a simultaneous trigonometric approximation of the form (2.C) of order $N = 21$. **Fig. 3** shows the plot of the approximation obtained.

The main reason why we have taken 11 + 11 data values is to allow comparison between this approximation and the approximation of Experiment 1 based on practically the same number of data values—21.

In Fig. 3 we notice that the replacement of half of the function

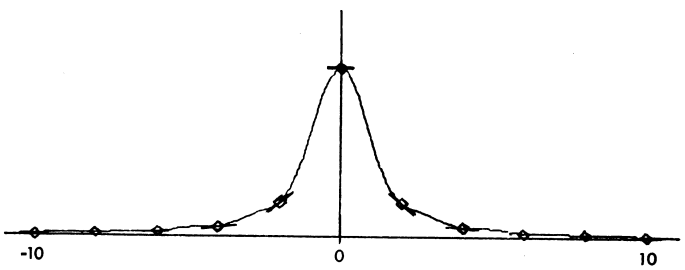

Fig. 3. Simultaneous cosine trigonometric approximation of order $N = 21$

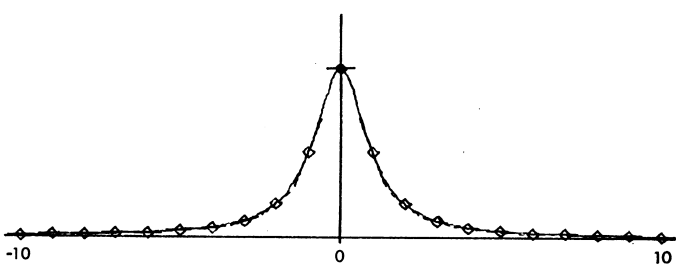

Fig. 4. Simultaneous cosine trigonometric approximation of order $N = 41$

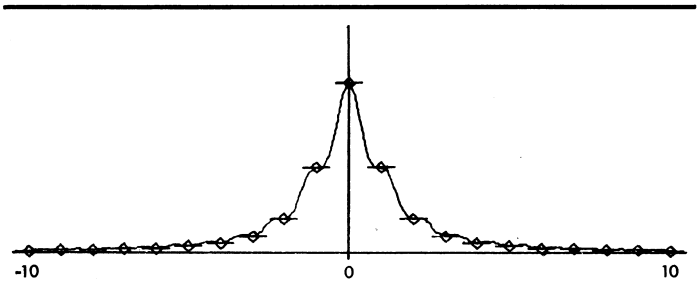

Fig. 5. Simultaneous cosine trigonometric approximation of order $N = 41$

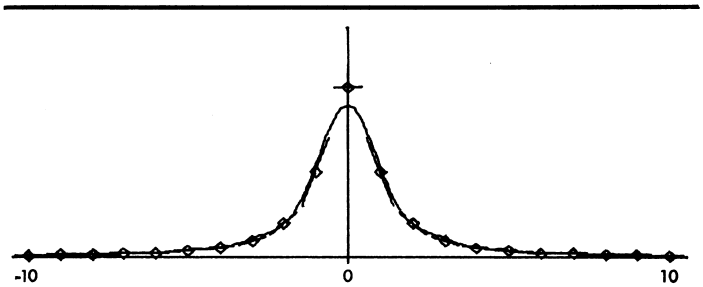

Fig. 6. Simultaneous cosine trigonometric fitting of order $N = 20$

Table 1. Extended cosine trigonometric approximations of order $N = 41$

$A(K)$	$A(K)$
+0·28224625	+0·28224625
0·00000000	0·00000000
−0·22474225	−0·21436343
0·00000000	0·00000000
+0·16436098	+0·14935408
0·00000000	0·00000000
−0·12274146	−0·10618714
0·00000000	0·00000000
+0·09053968	+0·07466387
0·00000000	0·00000000
−0·06740378	−0·05316136
0·00000000	0·00000000
+0·04978659	+0·03791377
0·00000000	0·00000000
−0·03703824	−0·02767452
0·00000000	0·00000000
+0·02736028	+0·02071720
0·00000000	0·00000000
−0·02035023	−0·01633101
0·00000000	0·00000000
+0·01502281	+0·01371964
0·00000000	0·00000000
−0·01116922	−0·01247240
0·00000000	0·00000000
+0·00822904	+0·01224825
0·00000000	0·00000000
−0·00610597	−0·01274905
0·00000000	0·00000000
+0·00447353	+0·01383726
0·00000000	0·00000000
−0·00329269	−0·01516551
0·00000000	0·00000000
+0·00237051	+0·01661292
0·00000000	0·00000000
−0·00169215	−0·01756797
0·00000000	0·00000000
+0·00114354	+0·01769786
0·00000000	0·00000000
−0·00071458	−0·01572148
0·00000000	0·00000000
+0·00033935	+0·01071817
0·00000000	0·00000000

values by an equivalent number of derivative values has decreased the oscillation of Fig. 2, improving therefore the general appearance of the interpolation obtained.

Experiment 3

In this attempt we have used complete information about the function, i.e. 21 + 21 function and first derivative values.

We have obtained another trigonometric interpolating expression now of order $N = 41$ presented in **Fig. 4.** The approximation in between data values is extremely good with a maximum error $\varepsilon \leqslant 0{\cdot}006$ around $X = \pm 0{\cdot}5$ of (3.A). At least to the eye the characteristic oscillations are non-existent, which is remarkable.

The first half of **Table 1** gives the coefficients A_k of the corresponding approximating function (2.C).

Experiment 4

To show the flexibility of this type of approximation as a designing tool, when the slopes $Z^{(1)}_{s\to 0\ 1,\ldots,q}$ are not exactly known, we prepared **Fig. 5.**

There we have taken the 21 function values of Experiment 1, together with another 21 arbitrary derivative values.

At each sampling point we took in fact slopes with zero value and the step-like function turned up as shown in the picture. We could, with several trials, bring it to look like the plot of Fig. 4.

Table 1 presents in its second half, the coefficients A_k of the corresponding approximating function.

Experiment 5

To conclude these experiments, we present not a trigonometric interpolation as we did in the previous tentatives, but a trigonometric least square fitting.

We have taken exactly the same data as used in Experiment 3 but we have asked for an approximation of the same order we did in Experiment 1, i.e. $N = 20$.

Now it is important to define the weighting parameter Λ used in the process. It was defined by relation (2.R) and the result is plotted in **Fig. 6.** From this plot we can appreciate how important can be the introduction of derivative conditions in trigonometric least square fitting.

In effect, the derivative information of the approximating function (2.C) is almost entirely preserved, therefore no unnecessary oscillations are introduced, but the curve does not get through all function values, that are interpreted as containing an error.

The relative importance of this derivative information can be stressed or relaxed by increasing or decreasing the corresponding weighting parameter Λ.

Acknowledgement

I would like to thank Dr. J. C. P. Miller for his most valuable criticism on the work leading to this paper.

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