An evolutionary approach to the concept of randomness

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The Von Mises and Kolmogorov definitions of randomness are discussed in terms of the complexity of binary sequences. An evolutionary approach is then described and some results presented.

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Problems of feature extraction in pattern recognition can sometimes be related to the problem of defining a measure of structure or lack of structure in a body of data (Jermann, 1970). Although it is intuitively clear what is meant by 'lack of structure' or randomness, it is difficult to set down a precise definition. In this paper two already existing approaches to this problem (1, 2) are described and finally the complexity of a binary sequence is discussed using the concept of an evolutionary procedure.

1. *Von Mises's Definition* (Von Mises, 1957)
'An infinite binary sequence possesses the property of randomness if the relative frequency of 1's (to 0's) tends to a certain limiting value which remains unchanged by the omission of a certain number of the elements and the construction of a new sequence from those which are left. The formula for omission must leave an infinite number of retained elements and it must not use the attributes of the selected elements.'

This definition of randomness is very close to what is intuitively meant by the word; if it is at all possible to detect structure in a binary sequence then it should also be possible to construct a selection procedure which changes the relative frequencies of the zeros and ones. In other words, if a sequence can be *seen* to be non-random, then it is certainly non-random according to the definition.

Although the general intent of Von Mises's definition conforms with what is meant by randomness, the lack of a precise formulation has led to severe criticism (Church, 1940; Wald, 1937; Martin-Löf, 1966; Loveland, 1966).

2. *Kolmogorov definition*
Kolmogorov (1965) and Chaitin (1966, 1970) have independently suggested that computing machines be applied to the problem of defining what is meant by a random or patternless finite sequence

The length n of a binary string $a = a_1 a_2 \ldots a_n$ will be denoted by $l(a)$.

Let A be an algorithm transforming a pair of binary strings p, x into a binary string $a = A(p, x)$.

The *conditional complexity* of a for given x with respect to A is defined as

$$K_A(a|x) = \begin{cases} \{\min l(p) | A(p, x) = a\}, \\ +\infty \text{ if there does not exist } p \text{ such that } A(p, x) = a \end{cases}$$

p can be thought of as a program which when fed into a machine A causes it to compute a by means of the given data x (**Fig. 1.**).

Inserting the empty string for x in $K_A(a|x)$ gives $K_A(a)$, the complexity of a with respect to A.

The random or patternless sequences are those having the greatest complexity, or alternatively, those which necessarily require the longest programs when produced by a computing machine. Those sequences that can be obtained by putting into a computer a small program are those that possess a pattern and follow a law. It can be shown that 'most' finite binary sequences of length n require minimal programs of about length n to generate them. These sequences are considered to be the random sequences (Chaitin, 1970; Solomonoff, 1964).

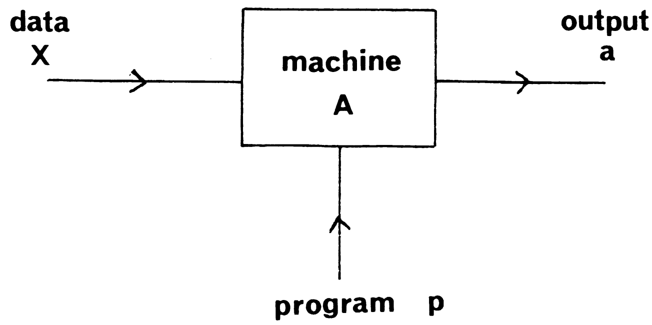

Fig. 1. A computing machine

The Kolmogorov definition provides a conceptually satisfactory solution to the problem. Patterned finite sequences are just those sequences which follow a simple law; unstructured finite sequences follow a complex law, which could possibly incorporate the sequence itself in a 'table-look up' scheme.

Kolmogorov has pointed out a disadvantage in his concept of randomness; it does not allow for the 'difficulty' of preparing a program which generates the sequence a. Indeed, the theory gives no indication of how the minimal program p is obtained.

Effective definitions

The definitions discussed above provide concepts which conform with what is intuitively meant by the word 'randomness'. However, both definitions are not effectively computable‡ in that they appeal to an external human informer; in the first instance Von Mises requires 'formulae for omission', and in the second instance, Kolmogorov requires 'minimal length programs'.

Effective definitions of an arbitrarily long random sequence are not possible because it has been shown that there will always exist a computable (and therefore *non-random*) sequence which is labelled as *random* by the definition (Levin, Minsky and Silver, 1962).

An evolutionary estimate of relative complexity

Structure is detected in a sequence when it becomes possible to predict terms in the sequence according to some rule. It might seem safe to say that, in general, a sequence is more complex than another if it is more difficult to think up a prediction rule. Although this has the right spirit, it is far too vague to be useful in a rigorous definition.

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‡For a discussion on effective computability see Minsky (1967), Rogers (1967).

Nevertheless, suppose an evolutionary type of process was conceived which had as its goal the correct prediction of binary sequences. It would be possible to get an idea of the complexity of the current binary sequence by monitoring the error rate. It would not be acceptable to use these ideas in an absolute definition of randomness because an evolutionary process *itself* requires a source of random changes. However, it is possible to give an estimate of the complexity of one sequence relative to another.

These concepts can now be expressed more formally

Definition

Suppose $s_1, s_2, \ldots, s_N$ is a finite binary sequence then a *predictive evolutionary procedure* Φ is a sequence of functions $\phi_1, \phi_2, \ldots, \phi_M$ $(M < N)$ which generates a finite binary prediction sequence $s'_1, s'_2, \ldots, s'_N$ where

$$s'_r = \phi_i\{s_1, s_2, \ldots, s_{r-1}\} \quad r > 1, i > 1$$

and $s'_1 = \phi_1$

with $\phi_1 = 1$, say.

If $s'_r \neq s_r$ then ϕ_{i+1} is produced from ϕ_i by a random change with the constraint that

$$s_r = \phi_{i+1}\{s_1, s_2, \ldots, s_{r-1}\}$$

Definition

The finite binary sequence $a_1, a_2, \ldots, a_N$ is *more complex* (in terms of this definition) with respect to Φ than the finite binary sequence $b_1, b_2, \ldots, b_N$ if the following condition can be satisfied:

Let $a'_1, a'_2, \ldots, a'_N$ and $b'_1, b'_2, \ldots, b'_N$ be prediction sequences generated by a predictive evolutionary procedure

$$\Phi = \{\phi_1, \phi_2, \ldots, \phi_M\}$$

where

$$a'_r = \phi_i\{a_1, a_2, \ldots, a_{r-1}\}$$
$$b'_r = \phi_j\{b_1, b_2, \ldots, b_{r-1}\}$$

Then there exists a positive integer $R < N$ such that $g(r, \Phi, a_1, a_2, \ldots, a_N) < g(r, \Phi, b_1, b_2, \ldots, b_N)$ for all $R < r \leqslant N$ where $g(r, \Phi, s_1, s_2, \ldots, s_N)$, the *predictability score*, is the number of correct predictions minus the number of incorrect predictions in the first r terms of the prediction sequence $s'_1, s'_2, \ldots, s'_N$.*

It must be emphasised that the value of N is crucial to this definition. N is the length of the binary sequence under investigation and must be sufficiently large to see a stable trend in $g(r, \Phi, s_1, s_2, \ldots, s_N)$ as $r \to N$. N should certainly be much greater than the length of any periodicity that is known to be present in the binary sequences. In general it is felt that values of N should be determined empirically, some experimental results are given in the next section which give some indication of the relationship between N and $g(N, \Phi, s_1, s_2, \ldots, s_N)$.

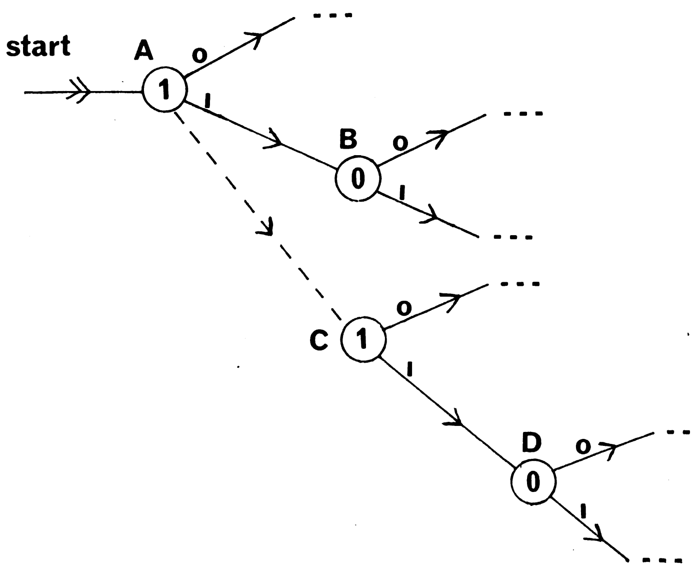
The definition enables finite binary sequences to be ordered in terms of their predictability scores. This ordering is by no means an absolute indication of the randomness of a sequence because other attempts at the same ordering would not necessarily give identical results.

We observe that Φ is not an effective procedure in the accepted sense (Minsky, 1967; Rogers, 1967). This is because the process employs random changes and the computation is not therefore carried forward deterministically; there is no guarantee that an evolutionary experiment would give the same results if it was carried out at two different times†. This means that it would not be possible to design a computable sequence which would be guaranteed to baffle an estimate of complexity based on an evolutionary procedure.

The definition of 'more complex' is not mathematically satisfactory to a formalist because it relies heavily on the way Φ is specified. However, it does provide a practical technique for the investigation of redundancy in binary sequences. In the next section a particular Φ is defined and it is shown how the predictability scores relate to certain simple sequences.

Experimental results

The particular Φ chosen for the experiment was a 32-state Moore machine which was continuously evolved to predict binary input sequences. Thirty-two states were chosen only for programming convenience. Sixteen of the states were associated with the output of a 1 and the remaining sixteen with 0. Initially the interconnections between states were random and an arbitrary start state was chosen. This meant that an arbitrary binary output was given as the first prediction. If the next binary input agreed with this output, no change was made to the machine. On the other hand, if the input disagreed with the prediction, then the last state was changed (randomly) to one which *would* have given a correct output had the process been repeated.



input	1	1	0	---
output	1	0	0	---

Fig. 2. Part of finite-state machine model

For example, consider the input 11 0 ... to the Moore machine a part of which is illustrated by a state diagram in **Fig. 2.** In this diagram each machine state is represented by a circle and the output associated with that state is given within the circle. The input associated with a transition from one state to another is given alongside a directed line (edge) joining those two states. The first output 1 is a correct prediction and control is passed to state B which gives a 0 output. The second input is a 1 and this means that the output by state B was in fact wrong. The edge connecting states A and B is then randomly altered so that the edge now connects A to some state C which would have given the correct output of a 1. The current input 1 is then applied to C and the next prediction 0 is output by state D.

In the experiment this process was continued and small blocks of the input sequence which occurred frequently were found to correspond to one or more connected groups of states in the evolved machine. If the sequence pattern was very common

*This measure of prediction ability was used by Levin, Minsky and Silver (1962).

†Of course, if evolution is simulated on a computer and a pseudo-random number generator is used, then results can be repeated.

Table 1

SEQUENCE	PREDICTABILITY SCORES 100	1000	10000
	66	842	8680
1010101010101010101010101010101	68	840	8678
	66	840	8662
	56	706	7306
00000000000000011111111111111	56	702	7302
	66	702	7306
	68	690	6998
1001001001001001001001001001001	48	676	7006
	50	654	6770
	56	548	6372
1011010110101101011010110101101	42	542	6316
	52	570	5934
	44	472	5712
10011001100110011001100110011	50	538	5452
	42	558	5694
	48	328	3952
1011001011001011001011001011001	−2	396	3786
	34	322	3772
	10	274	3176
1011011101111011111011111110	20	302	3284
	12	388	3294
	46	206	2402
101100011011000110110001101100011	16	190	2570
	18	326	2616
	4	48	1148
1001110100100111010010011101001	4	100	1052
	18	134	1080
	12	32	768
010011000111000011110000011111	20	60	566
	14	64	566
	2	−104	−626
1110100010010101100001110011011	−4	− 88	−618
	2	− 84	−564

there were several groups of states describing the structure of the pattern. In effect this meant that the sequences which occurred frequently were easily predicted by Φ. In this simulation the predictability score was taken over 100, 1,000, and 10,000 inputs.

Discussion of results

Several binary sequences were presented to the system and are illustrated in **Table 1.** Each of these sequences was cycled as many times as were necessary to supply 100, 1,000 and 10,000 inputs to the machine. Each predictability score was obtained from a different random start machine and three sets of such scores were obtained for each sequence.

It was seen that in most cases the percentage variation in scores decreased as the length of input sequence was increased.

Table 1 is arranged with the most predictable sequences at the top. It is felt that this ordering is to a certain extent consistent with subjective estimates of the relative complexities of the sequences.

One sequence gave a significantly negative predictability. This might be considered surprising in view of the fact that a purely random sequence would be expected to have a *zero* predictability score. However, this sequence is a form of pseudo-noise sequence (Golomb, 1967) which has minimal correlation with shifted versions of itself. This means that this type of sequence is the 'theoretical worst' in terms of predictions which are based on short patterns discovered earlier in the process. In fact the pseudo-noise sequence displayed a distinctive structure by its *inability* to be predicted by this evolutionary procedure. It is also significant to point out that pseudo-noise binary sequences satisfy the following three intuitively acceptable criteria for randomness

(*a*) A balance of 0 and 1 terms
(*b*) Two runs of length n for each run of length $n + 1$
(*c*) A two-level auto-correlation function.

Simple sequences composed of cycles which were short $\left(\leqslant \frac{N}{2}\right)$ compared with the size of the machine, sometimes gave quite variable scores. This was because on occasion the machine was able to lock precisely into the correct cycle and thereby achieve a considerably higher score than the average. The sequences in Table 1 were chosen to be of length ~ 30 although their internal structure was often quite simple.

Conclusions

This paper has highlighted some of the difficulties to be encountered when accepted definitions of randomness are applied to real binary sequence. It was observed that these definitions were not constructible in the practical sense and therefore conveyed little or no information about particular sequences.

A practical estimate of the relative complexity of binary sequences was then defined. This definition is based on the concept of an evolutionary procedure which is capable of computer implementation. Several sequences were processed using this scheme and their predictability scores obtained. These scores provided a plausible estimate of the relative complexity of the sequences. Further work is necessary on the effect of the machine size on the predictability scores. The very simple Φ described in this paper possesses only a few states and is therefore limited in its ability to distinguish structured sequences.

The non-deterministic search technique set out in this paper has already been applied to related areas in the design process. This includes the reduction of finite-state machines (Stentiford and Lewin, 1971) and the design of features for Optical Character Recognition (Stentiford, 1972). It is felt that evolutionary searches will provide useful tools in many areas of information processing where conventional methods have been unsuccessful.

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