# A constructive geometry for computer graphics

A. Ricci
*CNEN Centro di Calcolo, Via Mazzini 2, Bologna, Italy*

**In the present paper a general approach to the definition of complex 3D objects from simpler ones is illustrated. Intersection and union operations are defined which can be approximated to obtain a smooth joining of object volumes with one another.**

(Received February 1972)

The representation of the shape of a 3D object in terms of numerical information stored in the computer memory, generally by means of a suitable data structure, is still an important problem in computer graphics.

In the techniques for object representation till now developed (see references), the information stored in the data structure generally relates to the definition of the object surface, often subdivided in surface patches, thus requiring, unless the object shape is simple, a large amount of data to define surface points, continuity conditions, etc. This gives rise to a certain degree of uneasiness in the modification of the object shape, particularly when extensive changes are required, as frequently happens in the early stages in the design process.

The approach to the representation and manipulation of 3D objects by means of their global definition as solids seems to be more natural and promising. The technique of the definition of complex objects in terms of simpler ones has been attempted (for example, Goldstein and Nagel, 1971) but, while less information needs to be handled, the component objects retain their individuality in the final shape by reason of the lack of a smooth joining of object volumes with one another.

A certain degree of smoothing has been obtained in a particular technique for the detection of intersections of 3D objects (Comba, 1968), but this method apparently does not apply to non-convex objects.

In the present paper, a general approach to the solution of the problem, through what can be called a constructive geometry, is presented.

For any solid, connected or disconnected, in the 3D space a set of associated functions is defined. Functions relating to different objects can be combined to obtain a new function representing a new solid, allowing the designer to define it by means of a small amount of information. The combination of solids can be realised by applying a suitable sequence of intersection and union operations. The operations in the sequence can be approximated, namely substituted for by operations which give a slightly different result, thus giving rise to a controlled smoothing of matching volumes and surfaces. By suitably regulating the smoothing parameter, in the final solid the component ones may not even be recognisable.

Since solids in the geometry illustrated here can be disconnected, a function defining a collection of separate objects can also be used. In addition, solid defining functions lend themselves to a simple solution of the hidden points problem and no serious difficulty arises from the implicit, i.e. non-parametric, form of the equation satisfied by the points of the object surface, provided that efficient contour mapping techniques are used to compute paths on the surface.

## 1. Preliminary definitions

In the present paper, when a solid $S$ in the 3D Euclidean space $E^3$ is considered, it is intended that it can be connected or disconnected, that is to say it can comprise one object or more objects separated from one another. All the results obtained here apply to this general definition of a solid.

For any solid $S$, the set of its interior points will be denoted by $I$, the set of its boundary points by $B$ and the set of its exterior points by $T$, with

$$I \cup B \cup T = E^3 \qquad (1)$$
$$I \cap B = B \cap T = I \cap T = \phi$$

A continuous function $f(P)$, non-negative for every $P$ in $E^3$, will be called a defining function for a solid $S$ if $f(P) < 1$ when $P$ belongs to $I$, $f(P) = 1$ when $P$ belongs to $B$ and $f(P) > 1$ when $P$ belongs to $T$. For any given solid, many different defining functions can be found, for example if $f(P)$ is a defining function for the solid $S$ also $(f(P))^p$, being $p$ a positive real number, is a defining function for $S$.

An interesting property of defining functions, as they have been introduced above, is that if $f(P)$ is a defining function for the solid $S$, $(f(P))^{-1}$ is a defining function for the solid complement $S^C$ as defined by $I^C = T$, $B^C = B$, and $T^C = I$.

Another useful definition is that of the surface equation for a solid $S$, namely the equation that is satisfied by the points belonging to $B$. For any solid $S$ with $f(P)$ as a defining function, the surface equation is

$$f(P) = 1 \qquad (2)$$

As an example, for a sphere having the radius $r$ and its centre at the origin of the reference system, a possible defining function is

$$f(P) = (x/r)^2 + (y/r)^2 + (z/r)^2 \qquad (3)$$

and $f(P) = 1$ will define the surface of the sphere.

## 2. Intersection and union operations

To establish a really useful constructive geometry in computer graphics, we need operations, allowing simple objects to be suitably combined into more complex ones, which will be easy and natural. Most conveniently, the said combination of solids can be realised by applying a sequence of intersection and union operations. They can be defined in terms of defining functions and the following two statements will show how the defining function for the resulting solid can be derived from those for the component solids.

*Statement 1*—Let $n$ solids $S_1, \ldots, S_n$ respectively have defining functions $f_1(P), \ldots, f_n(P)$. Then a defining function of their intersection is given by

$$f^I(P) = \max(f_1(P), \ldots, f_n(P)) \qquad (4)$$

To prove the statement, we firstly note that

$$T^I = T_1 \cup \ldots \cup T_n \qquad (5)$$
$$I^I = I_1 \cap \ldots \cap I_n \qquad (6)$$
$$B^I = \text{complement of } I^I \cup T^I.$$

Then $f^I(P) > 1$ implies that at least one $f_i(P)$ is greater than unit, $P$ belongs to $T_i$ and, for (5), $P$ also belongs to $T^I$. If $f^I(P) < 1$, all $f_i(P)$ are lower than unit and $P$ belongs to every $I_i$ and, for (6), $P$ belongs to $I^I$. In the remaining case, if

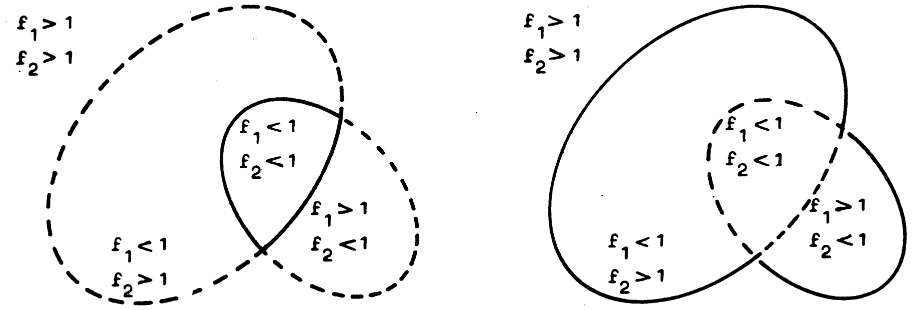



**Fig. 1. Schematic illustration of the surface equations $\max(f_1, f_2) = 1$ and $\min(f_1, f_2) = 1$**

$f^I(P) = 1$, no $f_i(P)$ can be greater than unit and $P$ cannot belong to $T^I$, at least one $f_i(P)$ is not lower than unit and $P$ cannot belong to $I^I$, then $P$ belongs to the complement of the union of $T^I$ and $I^I$, namely to $B^I$.

As a corollary, the surface equation for the solid $S^I$ is

$$\max(f_1(P), \ldots, f_n(P)) = 1 \quad (7)$$

as illustrated in **Fig. 1**.

As an example, the intersection of the three infinite slabs with defining functions

$$f_1 = (x/r)^2 \quad f_2 = (y/r)^2 \quad f_3 = (z/r)^2 \quad (8)$$

has the following surface equation

$$\max((x/r)^2, (y/r)^2, (z/r)^2) = 1 \quad (9)$$

which defines the surface of a cube centred at the origin of the reference system.

*Statement 2*—Let $n$ solids $S_1, \ldots, S_n$ respectively have defining functions $f_1(P), \ldots, f_n(P)$. Then a defining function of their union $S^U$ is given by

$$f^U(P) = \min(f_1(P), \ldots, f_n(P)) \quad (10)$$

For the solid $S^U$ we have

$$I^U = I_1 \cup \ldots \cup I_n \quad (11)$$
$$T^U = T_1 \cap \ldots \cap T_n \quad (12)$$
$$B^U = \text{complement of } I^U \cup T^U$$

Now, if $f^U(P) < 1$ at least one $f_i(P)$ is lower than unit and $P$ belongs to $I_i$ and then, for (11), to $I^U$. If $f^U(P) > 1$, all $f_i(P)$ are greater than unit and $P$ belongs to every $T_i$ and, for (12), $P$ also belongs to $T^U$. When $f^U(P) = 1$, no $f_i(P)$ can be lower than unit and $P$ cannot belong to $I^U$, at least one $f_i(P)$ is not greater than unit and $P$ cannot belong to $T^U$, then $P$ belongs to the complement of the union of $I^U$ and $T^U$, namely to $B^U$.

As a corollary, the surface equation for the solid $S^U$ is

$$\min(f_1(P), \ldots, f_n(P)) = 1 \quad (13)$$

as shown in Fig. 1.

As an example, the union of the two infinite slabs with defining functions

$$f_1 = ((x - a)/3a)^2 \quad f_2 = ((x + a)/3a)^2 \quad (14)$$

has the surface equation

$$\min(((x - a)/3a)^2, ((x + a)/3a)^2) = 1 \quad (15)$$

which defines the surface of an infinite slab centred at the origin of the reference system and having a half-thickness of $4a$.

## 3. Smoothing approximation of intersection and union operations

To realise a smooth joining of component solids into a final one, max and min functions must be approximated by means of suitable functions depending on a parameter which can be used to control the degree of smoothing. In addition, differentiable approximating functions can be used to avoid possible difficulties in computation due to the undifferentiability of max and min functions, provided that defining functions involved in the intersection and union operations are themselves differentiable. In both cases, an approximation may require that defining functions be everywhere positive in $E^3$.

Meeting the last condition is by no means a real difficulty. In fact, a small positive quantity $\varepsilon$ can be added to a defining function to remove zeroes. The quantity $\varepsilon$ can be chosen so as not to alter significantly the solid defined by the function. For example, adding $10^{-5}$ to the defining function (3) will give rise to a modification of the sphere radius of about 0·05 per cent.

A large variety of sequences of approximating functions can be used, but only one way of approximating max and min functions will be illustrated in the present paper. The approximating functions chosen to be substituted for max and min functions here are respectively

$$I_p(f_1, \ldots, f_n) = (f_1^p + \ldots + f_n^p)^{1/p} \quad (16)$$
$$U_p(f_1, \ldots, f_n) = (f_1^{-p} + \ldots + f_n^{-p})^{-1/p}$$

where $p$ is a positive real number.

To prove that $I_p$ and $U_p$ can be used as $p$-approximations of respectively max and min functions, the following statements must be shown to be true.

*Statement 3*—For any point $P \in E^3$,

$$\lim_{p\to\infty} I_p(f_1, \ldots, f_n) = \max(f_1, \ldots, f_n) = I_\infty \quad (17)$$

To prove the statement, we observe that, for any $P \in E^3$, the uniform norm $\|f\|_\infty$ of Cartesian space $R^n$ of elements $f = (f_1, f_2, \ldots, f_n)$ is given by

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty \quad (18)$$

where $\|f\|_p$ is the space $p$ norm (Davis, 1963). Since $f_i \geqslant 0$ $(i = 1, 2, \ldots, n)$, $\|f\|_p = I_p(f_1, f_2, \ldots, f_n)$ and $\|f\|_\infty = \max(f_1, f_2, \ldots, f_n)$ and (17) is equivalent to (18) and thus proved.

*Statement 4*—For any point $P \in E^3$,

$$\lim_{p\to\infty} U_p(f_1, \ldots, f_n) = \min(f_1, \ldots, f_n) = U_\infty \quad (19)$$

To prove the statement, it is now sufficient to observe that, letting $\bar{f}_i$ be the solid complement defining functions $\bar{f}_i = f_i^{-1}$,

$$U_p(f_1, f_2, \ldots, f_n) = [I_p(\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n)]^{-1} \quad (20)$$
$$U_\infty = \min(f_1, f_2, \ldots, f_n) = [\max(\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n)]^{-1}$$

Then statement 4 is proved by reason of statement 3.

As an example of application of the above statements, a sphere centred at the origin of the reference system and having a radius $r$ can be obtained from the defining functions (8) by means of their approximated intersection $I_1$. Generally, an intersection solid approximated according to a given value of the parameter $p$ is interior to the intersection solid approxim-

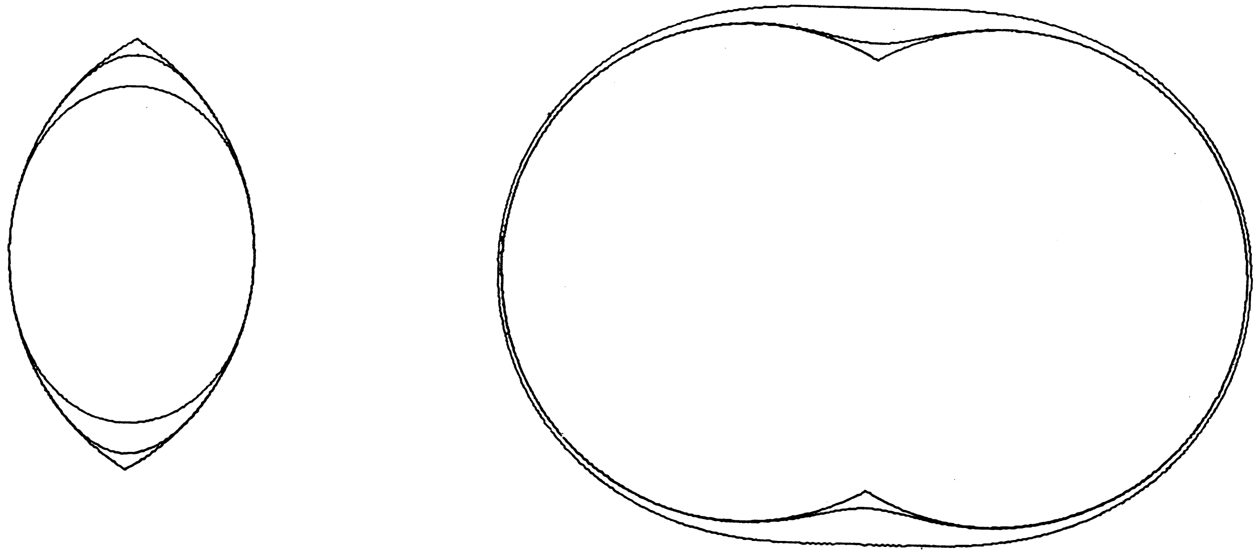


Fig. 2. $I_\infty$, $I_4$ and $I_2$ intersections and $U_2$, $U_4$ and $U_\infty$ unions of two spheres

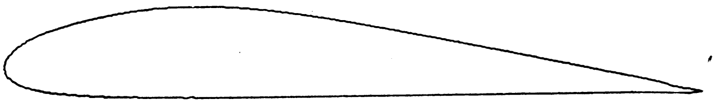


Fig. 3. An example of constructed bi-dimensional profile

ated according to a greater value of the parameter, while for the union operation an opposite behaviour is experienced (**Fig. 2**).

As an additional example, in this case a bi-dimensional one, the profile illustrated in **Fig. 3** has been obtained through approximated intersection and union operations applied to defining functions of the general form exp $(ax + by + c)$, each of them defining a half-plane determined by the numerical values of the constants $a$, $b$ and $c$.

It is worth noting that a solid defining function need not be expressed analytically, but can be furnished to a computer program in the form of a table or a subprogram, thus permitting the designer a wide possibility of definition of a whatever shape. This is important when the solid shape must satisfy particular technical constraints, the effects of which could be directly computed in the function sub-program.

Nevertheless only global control over the shape of the constructed object is apparently available, sharply defined and localised modifications can be obtained with a suitable choice of component solids and approximated intersection and union parameters.

## 4. The representation of the solid surface and the removal of hidden points

Starting from the surface equation (2), a large number of different techniques can be used to graphically represent the surface of the solid.

For example, it is possible to cut sections of the solid on parallel planes and draw curves on them satisfying the surface equation. On a section plane, the solid reduces to a planar shape, which can be connected or disconnected, and a reduced defining function is obtained which has the same characteristics as the corresponding 3D one. All of the results here obtained for $E^3$ are also valid for $E^2$. In **Fig. 4**, a few sections of an aircraft model are shown as drawn by a low precision plotter.

The fact that the above proved statements are valid for $E^2$ suggests a simple way to remove hidden points in the representation of the solid by means of bi-dimensional line drawing.

If sections $C_1, C_2, \ldots, C_i, \ldots$ are cut perpendicularly to the view-line, with $C_i$ nearer to the view-point than $C_{i+1}$, a set of bi-dimensional defining functions $h_i(P)$ is generated, with $P \epsilon E^2$. By plotting lines satisfying the boundary equations

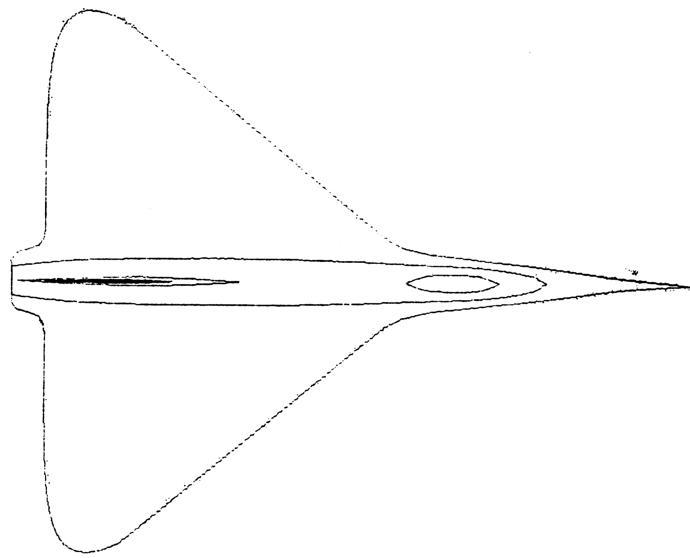
$$H_i(P) = 1 \tag{21}$$

with

$$H_1(P) = h_1(P) \tag{22}$$
$$H_{i+1}(P) = U_\infty(H_i(P), h_{i+1}(P))$$

a view with hidden points removed is obtained, as exemplified by **Fig. 5**.

A perspective representation can be easily realised by suitably mapping the original $E^3$ space into a new one (Ricci, 1971) with no change needed in the structure of the solid defining function.



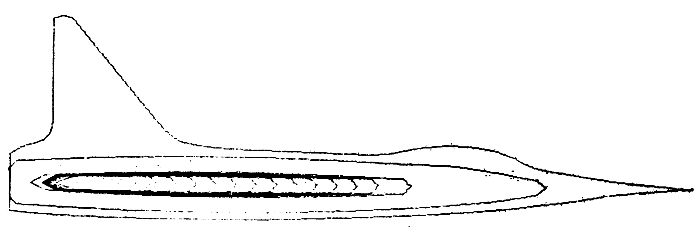


Fig. 4. Sections of a constructed aircraft model

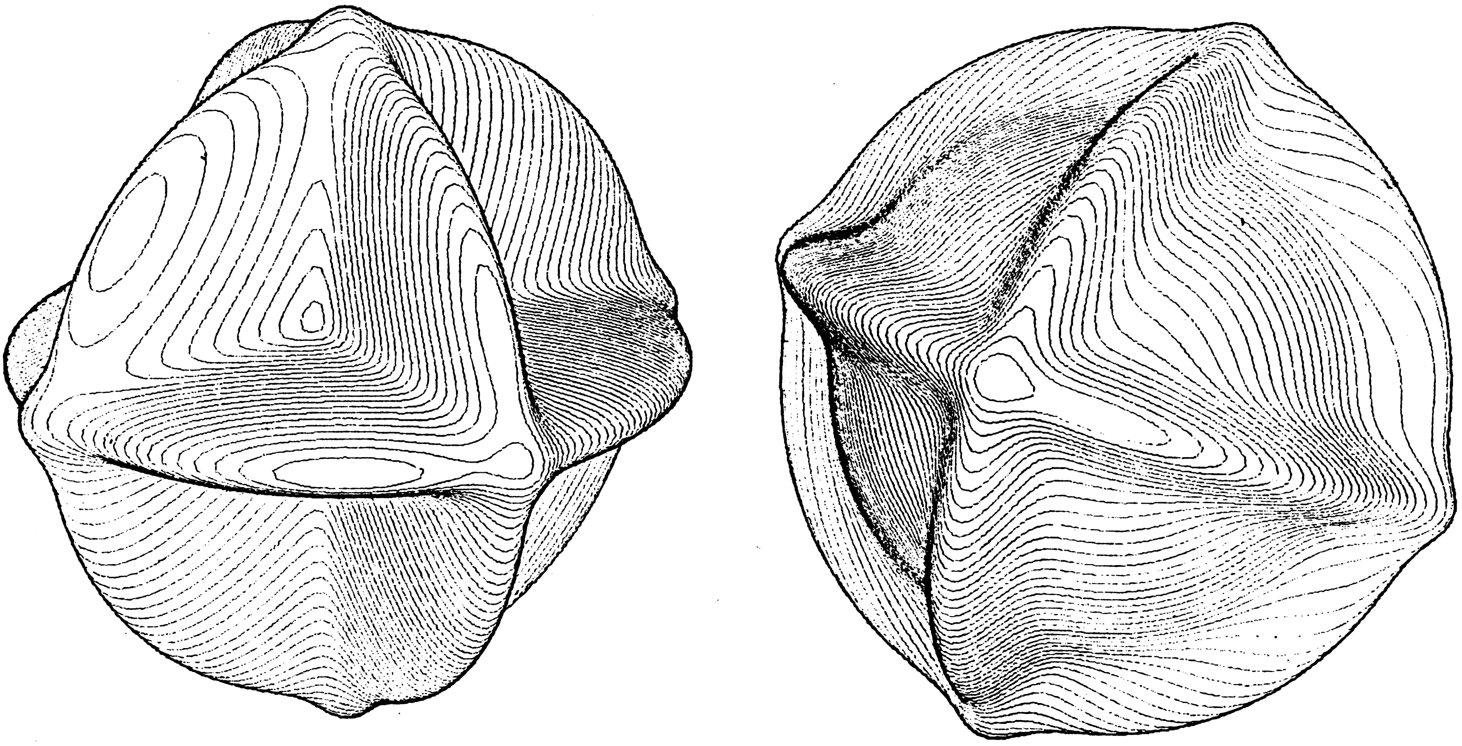


**Fig. 5. Examples of hidden points removal in constructed solids**

## 5. The implementation of the geometry

Any implementation of the constructive geometry illustrated here should be suitably realised in the form of an interactive graphics system, to permit a real-time design of solid objects. Since most of the information needed to define solids is carried by analytical relations among few numerical parameters, constructive geometry is a challenge to the development of interactive languages for symbolic formula manipulation and evaluation.

The work illustrated in the present paper (Figs. 2 to 5 are computer-drawn) has been implemented in the form of an interactive program for an IBM 2250 display unit supported by an IBM 360/75 computer. In this program, solid defining functions and related intersection and union operations are described within FORTRAN subprograms which can be displayed, modified and compiled on-line. Most conventional 3D interactive graphics operations are available and the modification of numerical parameters can be directly effected and does not require any compilation.

Since the IBM 2250 display unit is of the image refreshing type, the program is limited by the unit buffer size to display only one section of the constructed object at a time, the section being interactively chosen by the operator and computed by a contour-mapping routine. When a representation of the constructed solid as a line drawing with hidden points removed is wanted, the program produces it through a digital plotter.

As far as computation times are concerned, the time required to compute a section of even a complicated object is within few, rarely more than five, seconds, which does not give rise to an intolerable delay in the display response.

**Acknowledgement**

The author is grateful to G. Casadei and A. Teolis for helpful discussions on the contents of the present paper.

**References**

Comba, P. G. (1968). A procedure for detecting intersections of three-dimensional objects, *JACM*, Vol. 15, pp. 354-366.
Coons, S. A. (1967). *Surfaces for computer-aided design of space forms*, M.I.T., MAC-TR-41.
Davis, P. J. (1963). *Interpolation and approximation*, Blaisdell Publishing Co., New York.
Forrest, A. R. (1968). *Curves and surfaces for computer aided design*, Cambridge Computer-Aided Design Group, Thesis, July 1968.
Goldstein, R. A. and Nagel, P. (1971). 3-D Visual simulation, *Simulation*, January 1971, pp. 25-31.
Lee, T. M. P. (1969). A class of surfaces for computer display, *AFIPS Proc. SJCC*, 1969.
Ricci, A. (1971). An algorithm for the removal of hidden lines in 3D scenes, *The Computer Journal*, Vol. 14, pp. 375-377.