The application of Chow parameters and Rademacher-Walsh matrices in the synthesis of binary functions

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This paper surveys and brings together separate existing information from the field of (*a*) threshold logic synthesis, and (*b*) general binary synthesis using Rademacher-Walsh coefficients. The correlation between the Chow parameters widely used in the former and the latter Rademacher-Walsh coefficients is pursued. With a better understanding of the common features of both fields, inspiration towards further profitable developments may be hastened.

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Index terms
Boolean functions, Chow parameters, linear-separability, Rademacher-Walsh matrices, threshold functions, spectra.

List of symbols used

a_i, $i = 0$ to n, = minimum-integer weighting in a threshold realisation.
b_i, $i = 0$ to n, = threshold function characterising parameter, (Chow parameter).
$f(x)$ = binary function, value 0 (false) or 1 (true)
$f(y)$ = ditto, value -1 or $+1$
$f(z)$ = ditto, value $+1$ or -1
n = number of binary input variables per system.
R_i, $i = 0$ to n, = Rademacher variables or functions, $+1$, -1 valuation.
R_i, $i = 0, 1, \cdot\cdot, 12, \cdot\cdot$, = Rademacher-Walsh functions, $+1$, -1 valuation.
$\hat{R}_i$, $i = 0, 1, \cdot\cdot, 12, \cdot\cdot$, = modified Rademacher-Walsh functions, 0, 1 valuation.
$[R] = 2^n \times 2^n$ Rademacher-Walsh matrix.
$[\hat{R}] = 2^n \times 2^n$ modified Rademacher-Walsh matrix.
p = minterm, e.g. $p = 1$ is minterm $\cdots 001$
W_i, $i = 0, 1, \cdots 2^n$, = Walsh function, $+1$, -1 valuation
W_i', $i = 0, 1, \cdots 2^n$, = ditto, but taken in different matrix row order from W_i
$[W]$ and $[W'] = 2^n \times 2^n$ Walsh matrix
w_i = real-number weight associated with R_i, W_i, or W_i', $i = 0$, 1, etc.
$\hat{w}_i$ = real-number weight associated with $\hat{R}_i$, $i = 0$, 1 etc.
(note $w_i = \hat{w}_i$, except for $i = 0$)
x_i, $i = 1, 2, \cdots n$, = binary input variable, value 0 or 1
y_i, $i = 1, 2, \cdots n$, = ditto, value -1 or $+1$
z_i, $i = 1, 2, \cdots n$, = ditto, value $+1$ or -1

Introduction

A considerable volume of work has been covered in recent years in investigating the nature of linear-separability in binary functions. A binary function is said to be linearly-separable if, when represented on a n-dimensional hypercube such as that shown in **Fig. 1,** a separating plane can be drawn which precisely divides all true minterms of the function from all false minterms (Hurst, 1971). Fig. 1 illustrates the 3-variable linearly-separable function $f(x) = [x_1 + x_2x_3]$.

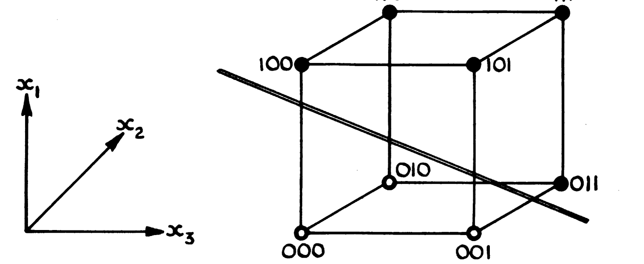

Fig. 1. Hypercube illustration of linearly-separable function $f(x) = [x_1 + x_2x_3]$, threshold realisation $\langle 2x_1 + x_2 + x_3 \rangle_{2:1}$

All such linearly-separable functions are realisable by a single threshold logic gate of appropriate weights and thresholds (Hurst, 1971; Lewis, 1967; Dertouzos, 1965). The example given above is realisable by a gate which weights the x_1 input by two units and the x_2 and x_3 inputs each by one unit, a nominal input summation ≥ 2 being required to operate the gate, ≤ 1 being insufficient to operate the gate. This threshold gate realisation therefore may be expressed arithmetically as $f(x) = \langle 2x_1 + x_2 + x_3 \rangle_{2:1}$.

Associated with this work has been a classification and tabulation procedure for all possible linearly-separable functions, involving the characteristic vectors of such functions and canonic positive Chow parameter classifications (Dertouzos, 1965; Winder, 1964; Winder, 1965; Muroga, Tsuboi and Baugh, 1967).

In the field of general Boolean synthesis, not restricted to the above linearly-separable class of functions, work using orthogonal matrices has been pursued. The matrices involved are normally the $2^n \times 2^n$ Rademacher-Walsh matrices, which lend themselves directly to the specification of any Boolean function of n variables (Coleman, 1961; Lockheed, 1964; Cooper, 1963; Liedl and Pichler, 1971). To date, however, direct circuit realisation using this approach has not been an economic practical proposition.

In the following sections of this paper, therefore, the existing state of the art of binary circuit synthesis in both the above fields is examined and compared. Strong common features will be found, which it is hoped may assist in furthering the necessary continuing development in this field.

1. Characteristic vectors, Chow parameters

The 'characteristic vector' b_i of any linearly-separable (threshold) function uniquely defines the given function (Dertouzos, 1965). The coefficients of a characteristic vector for an n-variable Boolean function $f(x_1, \cdots, x_n)$ are $(n + 1)$ in number, and are defined by:

b_0 = [(number of true minterms) − (number of false minterms)]

b_i, $i = 1$ to n, = [(number of agreements between the value of x_i and $f(x)$ taken over all minterms) − (number of disagreements between x_i and $f(x)$)] .

Chow (1961) formally proved that $(n + 1)$ coefficients as above are sufficient to uniquely define any linearly-separable function. Also, if the magnitudes $|b_i|$ of the coefficients of all possible threshold functions are taken and written in lexicographical order, (e.g. −2, −6, +2, +10, −6 becomes 10, 6, 6, 2, 2) then such invariance operations produce the most compact form of classifying and listing all the possible threshold functions of n variables, giving the canonic positive CHOW PARAMETER or CHOW COEFFICIENT tabulations.

2. +1, −1 valuations

If instead of the false/true valuations 0, 1 for $f(x)$ and $x_1, x_2, \ldots x_n$, we adopt the alternative false/true valuations −1/+1, identified hereafter by $f(y)$ and $y_1, y_2, \cdots y_n$, we have that:

$$f(y) = [2f(x) - 1]$$

and

$$y_i, i = 1 \text{ to } n, = [2x_i - 1]$$

Thence:

$$b_0 = \sum_{p=0}^{2^n-1} \{f(y)\},$$

the summation being taken over all the 2^n minterms from $p = 0$ to $2^n - 1$,

and

$$b_i, i = 1 \text{ to } n, = \sum_{p=0}^{2^n-1} \{f(y) . y_i\}.$$

(NOTE: $\sum$ indicates normal arithmetic summation, *not* modulo-two)

If now an additional 'variable' y_0 is added to the system, where y_0 by definition is always equal to +1, then b_0 may be redefined as:

$$b_0 = \sum_{p=0}^{2^n-1} \{f(y) . y_0\}$$

Hence all the b_i may now be defined by the one relationship:

$$b_i, i = 0 \text{ to } n, = \sum_{p=0}^{2^n-1} \{f(y) . y_i\}$$

Should yet another alternative terminology be adopted, with +1/−1 representing the 0/1 false/true values, identified hereafter by $f(z)$ and $z_1, z_2, \cdots z_n$, we have that:

$$f(z), = -f(y), = [1 - 2f(x)],$$

and

$$z_i, i = 1 \text{ to } n, = -y_i, = [1 - 2x_i]$$

Then:

$$b_0 = -\sum_{p=0}^{2^n-1} \{f(z)\},$$

$$b_i, i = 1 \text{ to } n, = \sum_{p=0}^{2^n-1} \{f(z) . z_i\}.$$

In a similar manner to $f(y)$ above, if an additional parameter z_0 is added, where $z_0 \triangleq -1$, b_0 may be redefined as:

$$b_0 = \sum_{p=0}^{2^n-1} \{f(z) . z_0\},$$

and hence all the b_i may be jointly defined by:

$$b_i, i = 0 \text{ to } n, = \sum_{p=0}^{2^n-1} \{f(z) . z_i\}$$

Given the b_i values for any threshold function, the corresponding minimum-integer realising weights a_i required for the threshold realisation have been fully tabulated. Thus the optimum threshold-gate realisation of

$$f(x) = \langle a_1x_1 + a_2x_2 + \cdots a_nx_n \rangle_{t_1:t_2},$$

where t_1 and t_2 are the upper and lower gate thresholds, therefore is immediately available.

3. Signs of $|b_i|$

As published, each $|b_i|$ Chow parameter listing defines one of several possible threshold functions, with the same coefficient values $|b_i|$. To uniquely define any specific threshold function, reinstatement of the signs of the b_i, $i = 0$ to n, is necessary. The b_0 term with its appropriate sign is essential and cannot be omitted.

For example consider the three linearly-separable functions plotted in **Fig. 2.** If the b_i values for these functions are computed, we obtain:

	b_0	b_1	b_2	b_3
Function (*a*):	−2	−6	+2	+2
(*b*):	−2	+6	+2	+2
(*c*):	+2	+6	+2	+2

Thus only the sign of b_1 differentiates between the two quite distinct functions (*a*) and (*b*), and, similarly, only the sign of b_0 differentiates between functions (*b*) and (*c*).

This therefore illustrates the absolute necessity to take into account the signs of all the b_i when specific functions are being considered. The three functions illustrated in Fig. 2 are of course three of many with the same canonic $|b_i|$ classification of 6, 2, 2, 2.

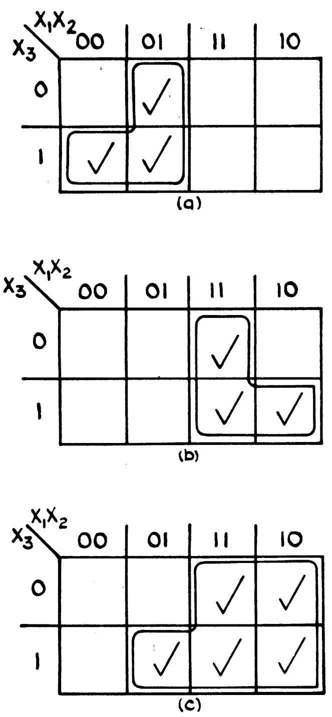

Fig. 2. Linearly-separable functions of three variables
(*a*) $f(x) = [\bar{x}_1(x_2 + x_3)], = \langle 2\bar{x}_1 + x_2 + x_3 \rangle_{3:2}$
(*b*) $f(x) = [x_1(x_2 + x_3)], = \langle 2x_1 + x_2 + x_3 \rangle_{3:2}$
(*c*) $f(x) = [x_1 + x_2x_3], = \langle 2x_1 + x_2 + x_3 \rangle_{2:1}$

4. Non-linearly-separable functions, Rademacher-Walsh coefficients

The $(n + 1)$ Chow parameters are not adequate to define all binary functions, i.e. the non-linearly-separable class.

Consider for example the two dissimilar functions plotted in **Fig. 3,** neither of which are linearly-separable functions (Hurst, 1970).

If the b_i values for each of these functions are computed, both will be found to yield:

b_0	b_1	b_2	b_3
0	0	0	0

Clearly, therefore, from this extreme example, the b_i's, $i = 0$ to n, are not sufficient to explicitly define functions which are not linearly-separable. Thus to define or classify such non-

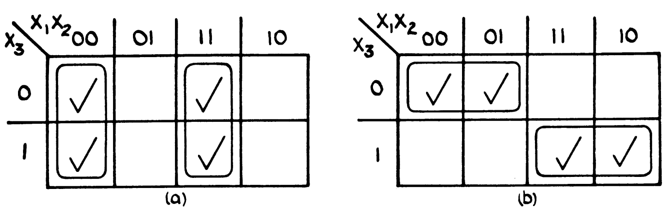

Fig. 3. Simple non-linearly-separable functions
(*a*) $f(x) = [\bar{x}_1\bar{x}_2 + x_1x_2]$
(*b*) $f(x) = [\bar{x}_1\bar{x}_3 + x_1x_3]$

linearly-separable binary functions, we must either augment the $(n + 1)$ Chow parameter coefficients with additional information, or derive an alternative classification more powerful than these basic Chow coefficients.

One algebraic method of defining and also classifying all binary functions is by means of the Rademacher-Walsh coefficients or 'spectra' (Dertouzos, 1965; Henderson, 1964; Golomb, 1959; Liedl and Pichler, 1971). As will be developed below, the Rademacher-Walsh parameters form an augmented form of the Chow parameters, being 2^n in number compared with the $(n + 1)$ total of the Chow classification.

The 2^n Rademacher-Walsh functions or variables—the terminology 'variable' is frequently used in the present context of binary synthesis, although 'function' is possibly more correct —consist of two parts, namely:

(*a*) the Rademacher functions, which constitute the first $(n + 1)$ of the 2^n total,
(*b*) the remaining $2^n - (n + 1)$ Walsh functions, which may be formed from the previous $(n + 1)$ Rademacher set.

The former may be termed the 'primary set', and the latter the 'secondary set' (Dertouzos, 1965).

When these Rademacher-Walsh functions are appropriately multiplied by the minterm values of any given binary function $f(x)$, the result is the Rademacher-Walsh coefficients or 'spectra' of the given function $f(x)$ (Dertouzos, 1965; Lockheed, 1964; Cooper, 1963; Liedl and Pichler, 1971). These 2^n coefficients uniquely define $f(x)$ in a similar way that the previous $(n + 1)$ Chow coefficients uniquely define any given threshold function $f(x)$.

(Note: although outside our present interests, it is of interest to note that these Rademacher-Walsh coefficients will uniquely define *any* function which exhibits discrete values in its output spectra; a binary function with two discrete output values 0 and 1 is therefore merely a particular case of the more general possible application of these Rademacher-Walsh coefficients.)

The Rademacher functions (or variables) may be found defined in several dissimilar but equivalent ways (Dertouzos,

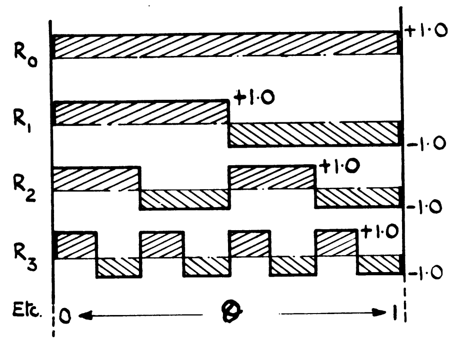

Fig. 4. Rademacher functions in the range 0 to +1.

1965; Cooper, 1963; Henderson, 1964; Barrett and Gordon, 1971). All definitions define a series of square waves with two discrete values, normally -1 and $+1$, which form a complete set over a unit 'range', or 'interval', normally taken as either from $-\frac{1}{2}$ to $+\frac{1}{2}$, or from 0 to 1.

One definition for the Rademacher functions $R_n(\theta)$, $n = 0, 1, 2, \cdots$, over the range of 0 to 1 is:

$$R_n(\theta) = \text{sign}\,\{\sin(2^n\,\pi\,\theta)\}\,.\,1{\cdot}0, \text{ where } 0 \leqslant \theta \leqslant 1\,.$$

An alternative definition, also over the range 0 to 1 is:

$$R_n(\theta) = +1 \text{ if } \frac{m}{2^n} \leqslant \theta \leqslant \frac{m+1}{2^n}\,,\ m = \text{even integer}\,,$$

$$= -1 \text{ if } \frac{m}{2^n} \leqslant \theta \leqslant \frac{m+1}{2^n}\,,\ m = \text{odd integer}\,.$$

Both these definitions are as illustrated in **Fig. 4.** Alternative definitions over the range $-\frac{1}{2}$ to $+\frac{1}{2}$ are similar, but displaced so as to be symmetrical about zero.

It will be apparent from the above that in a binary system the Rademacher functions R_1, R_2, etc. correspond to the binary variables x_1, x_2, etc. of a n-variable binary function, where the full 2^n minterms of the binary function are considered as contained in the range $\theta = 0$ to 1.

Hence a further definition for Rademacher functions (variables) in an n-variable binary system $f(x_1, \cdots x_n)$ is:

$$R_0 \triangleq +1$$
$$R_i,\ i = 1 \text{ to } n,\ = [1 - 2x_i]\,,$$
$$\left.\begin{aligned} &= [-y_i] \\ &= [z_i] \end{aligned}\right\} \text{ see section 2}$$

Thus for example, for 3 binary variables x_1, x_2, x_3, we have the following correlation between the variously defined binary variables and the Rademacher variables:

Mintern	Binary x_i			Binary y_i			Binary z_i			Rademacher R_i			
p	x_1	x_2	x_3	y_1	y_2	y_3	z_1	z_2	z_3	R_0	R_1	R_2	R_3
0	0	0	0	−1	−1	−1	+1	+1	+1	+1	+1	+1	+1
1	0	0	1	−1	−1	+1	+1	+1	−1	+1	+1	+1	−1
2	0	1	0	−1	+1	−1	+1	−1	+1	+1	+1	−1	+1
3	0	1	1	−1	+1	+1	+1	−1	−1	+1	+1	−1	−1
4	1	0	0	+1	−1	−1	−1	+1	+1	+1	−1	+1	+1
5	1	0	1	+1	−1	+1	−1	+1	−1	+1	−1	+1	−1
6	1	1	0	+1	+1	−1	−1	−1	+1	+1	−1	−1	+1
7	1	1	1	+1	+1	+1	−1	−1	−1	+1	−1	−1	−1

The additional $2^n - (n + 1)$ Walsh functions may now be formed by obtaining *all possible different products* of these Rademacher valuations at each minterm p, taken two-at-a-time, three-at-a-time, up to n-at-a-time. (This is the simplest definition of the Walsh functions, although they may be defined in their own right without reference to the Rademacher set (Walsh, 1923; Lackey and Meltzer, 1971)). Thus, denoting the product of $R_1 \times R_2$ by R_{12}, etc. and noting that:

$$R_0 \times R_0 = R_0\,,$$
$$R_i \times R_i,\ i = 1 \text{ to } n,\ = R_0\,,$$
$$R_0 \times R_i,\ i = 1 \text{ to } n,\ = R_i\,,$$
$$R_{ij} \times R_i,\ i, j = 1 \text{ to } n, i \neq j,\ = R_j\,,$$

we have that all the possible Rademacher-Walsh valuations for a 3-variable binary system are as follows:

Rademacher-Walsh R_i: Minterms p:	0	1	2	3	4	5	6	7		
R_0	+1	+1	+1	+1	+1	+1	+1	+1	Prmry Set	Cmplt Set
R_1	+1	+1	+1	+1	−1	−1	−1	−1		
R_2	+1	+1	−1	−1	+1	+1	−1	−1		
R_3	+1	−1	+1	−1	+1	−1	+1	−1		
R_1R_2	+1	+1	−1	−1	−1	−1	+1	+1	Scndry Set	
R_1R_3	+1	−1	+1	−1	−1	+1	−1	+1		
R_2R_3	+1	−1	−1	+1	+1	−1	−1	+1		
$R_1R_2R_3$	+1	−1	−1	+1	−1	+1	+1	−1		

Note: because of the relationships $R_i \times R_i = R_0$ etc. there will always be exactly 2^n rows in the complete set for any n.

5. Alternative matrix row order

The vertical order of the Rademacher-Walsh functions as tabulated above follows from using the primary set to generate the secondary set. If Walsh's original definition and format is followed, the tabulation is made in order of *increasing number of zero crossings* of the +1, −1 values in the given interval. This is analogous to increasing frequency of the Fourier series in sinusoidal working. If frequency is defined as one half the number of zero crossings per unit time, then the corresponding 'frequency' of the Walsh functions, which is termed the 'sequency', is similarly defined as:

sequency = $\frac{1}{2}$ (average number of zero crossings per unit time).

The rows of the previous tabulation for $n = 3$ therefore may be arranged in vertical order of Walsh function sequency as follows:

$$\begin{aligned} W_0 &= R_0 \text{ (zero sequency)} \\ W_1 &= R_1 \\ W_2 &= R_1R_2 \\ W_3 &= R_2 \\ W_4 &= R_2R_3 \\ W_5 &= R_1R_2R_3 \\ W_6 &= R_1R_3 \\ W_7 &= R_3 \text{ (max. sequency)} \end{aligned}$$

A yet further alternative ordering of the rows of the matrix has been proposed by Lechner, Colman and possibly others (Henderson, 1964). For $n = 3$, the equivalence of the modified set W_i' is:

$$\begin{aligned} W_0' &= W_0 = R_0 \\ W_1' &= W_7 = R_3 \\ W_2' &= W_3 = R_2 \\ W_3' &= W_4 = R_2R_3 \\ W_4' &= W_1 = R_1 \\ W_5' &= W_6 = R_1R_3 \\ W_6' &= W_2 = R_1R_2 \\ W_7' &= W_5 = R_1R_2R_3 \end{aligned}$$

which gives a matrix row ordering of:

Minterms p:	0	1	2	3	4	5	6	7
Walsh W_i': W_0'	+1	+1	+1	+1	+1	+1	+1	+1
W_1'	+1	−1	+1	−1	+1	−1	+1	−1
W_2'	+1	+1	−1	−1	+1	+1	−1	−1
W_3'	+1	−1	−1	+1	+1	−1	−1	+1
W_4'	+1	+1	+1	+1	−1	−1	−1	−1
W_5'	+1	−1	+1	−1	−1	+1	−1	+1
W_6'	+1	+1	−1	−1	−1	−1	+1	+1
W_7'	+1	−1	−1	+1	−1	+1	+1	−

The interesting feature of this re-ordering of the rows of the matrix is that is has a simple recursive structure, such that a matrix of order 2^{n+1} is formed by:

$$[W'_{2^{n+1}}] = \begin{bmatrix} [W'_{2^n}] & [W'_{2^n}] \\ [W'_{2^n}] & [-W'_{2^n}] \end{bmatrix}$$

Also the binary 0, 1 equivalent matrix (see next section) may be obtained by a simple relationship, when required.

6. Rademacher-Walsh matrix, 0, 1 valuation

If we convert the R_i Rademacher-Walsh values of +1, −1 into x_i 0, 1 values, $i = 1, 2$, etc., by the direct conversion of +1 to 0, −1 to 1, the first 3-variable Rademacher-Walsh matrix given above becomes as follows:

Minterms:	0	1	2	3	4	5	6	7
$\hat{R}_0$	1	1	1	1	1	1	1	1
$\hat{R}_1 (= x_1)$	0	0	0	0	1	1	1	1
$\hat{R}_2 (= x_2)$	0	0	1	1	0	0	1	1
$\hat{R}_3 (= x_3)$	0	1	0	1	0	1	0	1
$\hat{R}_1\hat{R}_2$	0	0	1	1	1	1	0	0
$\hat{R}_1\hat{R}_3$	0	1	0	1	1	0	1	0
$\hat{R}_2\hat{R}_3$	0	1	1	0	0	1	1	0
$\hat{R}_1\hat{R}_2\hat{R}_3$	0	1	1	0	1	0	0	1

Note: the designation $\hat{R}_0$, $\hat{R}_1$, etc. will be used to differentiate between an 0, 1 Rademacher-Walsh matrix, and a +1, −1 matrix. $\hat{R}_0$, the first row of the matrix, by definition remains +1.

If we now examine the secondary $\hat{R}_1\hat{R}_2$ etc. Rademacher-Walsh products, we see that in this converted 0, 1 set they will be generated by modulo-two addition of the appropriate primary Rademacher values, with any carry of the modulo-two additions discarded.

$$\begin{aligned} \text{e.g. } \hat{R}_1\hat{R}_2 &= \hat{R}_1 \oplus \hat{R}_2, = x_1 \oplus x_2 . \\ \hat{R}_1\hat{R}_3 &= \hat{R}_1 \oplus \hat{R}_3, = x_1 \oplus x_3 . \\ \hat{R}_2\hat{R}_3 &= \hat{R}_2 \oplus \hat{R}_3, = x_2 \oplus x_3 . \\ \hat{R}_1\hat{R}_2\hat{R}_3 &= \hat{R}_1 \oplus \hat{R}_2 \oplus \hat{R}_3, = x_1 \oplus x_2 \oplus x_3 . \end{aligned}$$

To summarise therefore, if these modified 0, 1 Rademacher-Walsh variables $\hat{R}_i$ are considered for binary synthesis and/or analysis, the first $(n + 1)$ *R-W* variables R_0 to $\hat{R}_n$ constitute the actual binary inputs x_i of the system, with $\hat{R}_0 \triangleq 1$, whilst the remaining *R-W* variables are modulo-two additions of these actual inputs. A binary system may therefore be considered as shown schematically in **Fig. 5.**

7. Orthogonality

The complete −1, +1 Rademacher-Walsh matrix as detailed in Sections 4 and 5 will be seen to be *orthogonal* and *normal*, i.e. 'orthonormal', as the integral of the product of any two of the 2^n function R_i, R_j taken over the full interval $\theta = 0$ to 1 is:

$$\int_{\theta=0}^{1} R_i\, R_j . d\theta = 0,\ i \neq j \text{ (orthogonal)}$$
$$= 1,\ i = j \text{ (normal).}$$

As the *R-W* functions are discrete-valued, the above

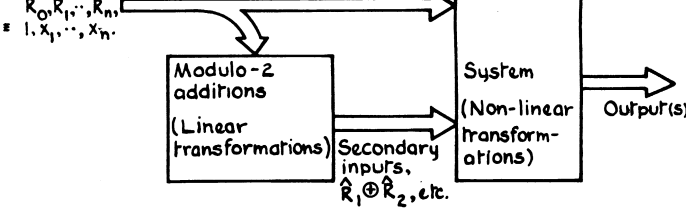

Fig. 5. Rademacher-Walsh synthesis procedure

integration may be replaced by the summation:

$$\frac{1}{2^n}\left\{\sum_{0}^{2^n-1}\{R_i . R_j\}\right\} \begin{array}{l} = 0, i \neq j\,, \\ = 1, i = j\,, \end{array}$$

the summation being taken at each discrete interval 0 to $2^n - 1$ in the full interval of $\theta = 0$ to 1.

Note:

(*a*) without the $\frac{1}{2^n}$ in the latter summation, the resultant for $\sum R_i . R_j$ when $i = j$ will sum to 2^n rather than to unity. The $\frac{1}{2^n}$ is thus a normalising factor;

(*b*) irrespective of the order of the rows of the matrix, for example the Lechner/Colman order given in Section 5, the matrix will remain orthogonal;

(*c*) when, however, the $+1$, -1 values are converted into binary 0, 1 values, see above, the resulting 0, 1 matrix $[\hat{R}_{2^n}]$ is no longer orthogonal for $\hat{R}_i . \hat{R}_j$. Orthogonality now holds for $\hat{R}_i \circledast \hat{R}_j$, as

$$\frac{1}{2^n}\left\{\sum_{0}^{2^n-1}\{\hat{R}_i \circledast \hat{R}_j\}\right\} \begin{array}{l} = 0, i \neq j\,, \\ = 1, i = j\,, \end{array}$$

where $\{\hat{R}_i \circledast \hat{R}_j) \triangleq [1 - 2\{\hat{R}_i \oplus \hat{R}_j\}]$.

Note: $\left\{\sum_{0}^{2^n-1}\{\hat{R}_i \circledast \hat{R}_j\}\right\}$ = [(agreements between $\hat{R}_i$ and $\hat{R}_j$) − (disagreements between $\hat{R}_i$ and $\hat{R}_j$)].

8. Rademacher-Walsh coefficients or spectra, +1, −1 configuration

Any arbitrary switching function of *n* variable can be described by the arithmetic sum of its 2^n Rademacher-Walsh variables R_i, each individually weighted by an appropriate real-number w_i, i.e.

$f(z)_p \triangleq +1$ or -1 at any minterm *p*,

$= (\sum w_i R_i)_p$, where w_i is the weight associated with each respective R_i, $i = 0, 1, 2, \cdots, 12, \cdots$.

These weights w_i of the Rademacher-Walsh variables may now be termed the Rademacher-Walsh 'coefficients' or 'spectra' of the switching function.

The required weights may be expressed by the matrix relationship:

$$w_i] = \frac{1}{2^n}[R] . f_z]\,,$$

where $w_i]$ = column matrix whose elements are the required w_i,

$[R]$ = the $2^n \times 2^n$ square matrix whose rows are the R_i's, and $f_z]$ = column matrix whose elements are the output values +1 or −1 of the function $f(z)$.

Proof of the above follows from the orthogonality of the matrix, see Dertouzos (1965), and others.

For example, given the 0, 1 function

$$f(x) = [x_1\bar{x}_2 + x_2\bar{x}_3 + \bar{x}_1x_3]\,.$$

Converting from 0, 1 to +1, −1, we have:

p	x_1	x_2	x_3	$f(x)$	z_1	z_2	z_3	$f(z)$
0	0	0	0	0	+1	+1	+1	+1
1	0	0	1	1	+1	+1	−1	−1
2	0	1	0	1	+1	−1	+1	−1
3	0	1	1	1	+1	−1	−1	−1
4	1	0	0	1	−1	+1	+1	−1
5	1	0	1	1	−1	+1	−1	−1
6	1	1	0	1	−1	−1	+1	−1
7	1	1	1	0	−1	−1	−1	+1

Hence:

$$\begin{array}{c} p = \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \qquad f(z) \end{array}$$

$$w_i] = \frac{1}{2^3}\begin{bmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 \\ +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 \\ +1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 \\ +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 \\ +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 \end{bmatrix} \begin{bmatrix} +1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ +1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \\ +4 \\ +4 \\ +4 \\ 0 \end{bmatrix}$$

or:

$$\begin{array}{ll} w_0 & = -\tfrac{1}{2}\,, \\ w_1 & = \;\; 0\,, \\ w_2 & = \;\; 0\,, \\ w_3 & = \;\; 0\,, \\ w_1w_2 & = +\tfrac{1}{2}\,, \\ w_1w_3 & = +\tfrac{1}{2}\,, \\ w_2w_3 & = +\tfrac{1}{2}\,, \\ w_1w_2w_3 & = \;\; 0\,. \end{array}$$

∴ The normalised Rademacher-Walsh spectra for the given function is:

$$-\tfrac{1}{2}, 0, 0, 0, +\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2}, 0,$$

giving the realisation for $f(z)$ of:

$$f(z)_p = (\sum\{-\tfrac{1}{2}R_0 + 0 + 0 + 0 + \tfrac{1}{2}(R_1R_2) + \tfrac{1}{2}(R_1R_3) + \tfrac{1}{2}(R_2R_3) + 0\})_p\,,$$

or, equally:

$$(\tfrac{1}{8}\sum\{-4R_0 + 4(R_1R_2) + 4(R_1R_3) + 4(R_2R_3)\})_p\,,$$

which can be quickly shown to correctly realise $f(z) = -1$ or $+1$ over all the 2^n minterms *p*.

Thus if we generate the +1, −1 Rademacher-Walsh variables, it would be possible to synthesise the given function to give the required output $f(z) = +1$ or -1, as shown in **Fig. 6.** The +1, −1 output $f(z)$ could subsequently be converted to an 0, 1 output $f(x)$, if required.

If the above matrix definition for the required weights w_i is

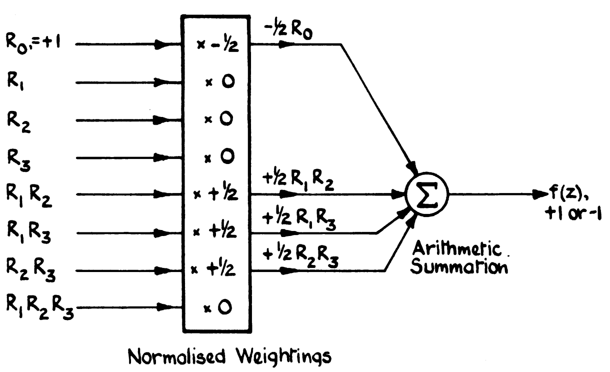

Fig. 6. Rademacher-Walsh realisation for $f(x) = [x_1\bar{x}_2 + x_2\bar{x}_3 + \bar{x}_1x_3]$

re-examined, it will be seen that they may be re-defined as follows:

$$w_0 = -\frac{1}{2^n}\,[(\text{number of true minterms}) - (\text{number of false minterms})],$$

$$= \frac{1}{2^n}\left[\sum_{p=0}^{2^n-1}\{f(z)\}\right],$$

and

$w_i,\ i = 1, 2, \cdots, 12, \cdots, =$

$\frac{1}{2^n}$ [(number of agreements between the value of R_i and $f(z)$ taken over all minterms p) − (number of disagreements between R_i and $f(z)$)].

Both these results may be jointly expressed as:

$$w_i = \frac{1}{2^n}\left[\sum_{p=0}^{2^n-1}\{f(z).R_i\}\right],$$

$$i = 0, 1, 2, \cdots, 12, \cdots.$$

It is now interesting to compare the above, which apply to *all* Boolean functions expressed in the +1, −1 notation, with the threshold Chow parameters b_i, $i = 0$ to n, given in Sections 1 and 2. A basic 'sameness' will be noted, except that the threshold b_i's are not normalised by dividing by 2^n.

9. Rademacher-Walsh spectra, 0, 1 configuration

To redefine the above procedure in terms of the normal 0, 1 binary notation and the resultant modified Rademacher-Walsh variables $\hat{R}_i$, it can be shown (Cooper, 1963) that:

$f(x)_p$, $\triangleq$ 0 or 1 at any minterm p,

$$= \left(\hat{w}_0\hat{R}_0 + \sum \hat{w}_i\hat{R}_i\right)_p,$$

where $\hat{w}_0$ = weight associated with $\hat{R}_0$,
$\hat{w}_i$ = weight associated with each $\hat{R}_i$, $i = 1, 2, \cdots, 12, \cdots$.

These weights $\hat{w}_i$ of the modified Rademacher-Walsh variables may now be termed the modified or binary Rademacher-Walsh 'coefficients' or 'spectra' of the given switching function.

The value of the weights is given by:

$$\hat{w}_0 = f(x)_{p=0},$$

i.e. the value of the function $f(x)$ at minterm $p = 0$, and

$$\hat{w}_i,\ i = 1, 2, \cdots, 12, \cdots, =$$

$$\frac{1}{2^n}\left[\sum_{p=0}^{2^n-1}\{(f(x)\oplus\bar{\hat{R}}_i) - (f(x)\oplus\hat{R}_i)\}\right].$$

Note:

As $\hat{R}_0$ is always 1, $\hat{w}_0\hat{R}_0$ is therefore always $f(x)_{p=0}$.

Also the above expression for $\hat{w}_i$, $i \neq 0$, will on examination be seen to define precisely the same summation as for the +1, −1 case, namely:

$\hat{w}_i,\ i = 1, 2, \cdots, 12, \cdots, =$

$\frac{1}{2^n}$ [(number of agreements between the value of $\hat{R}_i$ and $f(x)$ taken over all minterms p) − (number of disagreements between $\hat{R}_i$ and $f(x)$)].

Considering our previous example

$$f(x) = [x_1\bar{x}_2 + x_2\bar{x}_3 + \bar{x}_1x_3]$$

again, we now have the following development:

p:	0	1	2	3	4	5	6	7
$f(x)$:	0	1	1	1	1	1	1	0
$\hat{R}_0$	1	1	1	1	1	1	1	1
$\hat{R}_1$	0	0	0	0	1	1	1	1
$\hat{R}_2$	0	0	1	1	0	0	1	1
$\hat{R}_3$	0	1	0	1	0	1	0	1
$\hat{R}_1\hat{R}_2$	0	0	1	1	1	1	0	0
$\hat{R}_1\hat{R}_3$	0	1	0	1	1	0	1	0
$\hat{R}_2\hat{R}_3$	0	1	1	0	0	1	1	0
$\hat{R}_1\hat{R}_2\hat{R}_3$	0	1	1	0	1	0	0	1

whence:

$$\begin{aligned}
\hat{w}_0 &= 0\\
\hat{w}_1 &= 0\\
\hat{w}_2 &= 0\\
\hat{w}_3 &= 0\\
\hat{w}_1\hat{w}_2 &= +\tfrac{1}{2}\\
\hat{w}_1\hat{w}_3 &= +\tfrac{1}{2}\\
\hat{w}_2\hat{w}_3 &= +\tfrac{1}{2}\\
\hat{w}_1\hat{w}_2\hat{w}_3 &= 0
\end{aligned}$$

The normalised modified binary Rademacher-Walsh spectra for the given function is therefore:

$$0, 0, 0, 0, +\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2}, 0,$$

giving the realisation for $f(x)$ of:

$$\begin{aligned}
f(x)_p &=\\
&(\Sigma\{0 + 0 + 0 + 0 + \tfrac{1}{2}(\hat{R}_1\hat{R}_2) + \tfrac{1}{2}(\hat{R}_1\hat{R}_3) + \tfrac{1}{2}(\hat{R}_2\hat{R}_3) + 0\})_p,\\
&= (\Sigma\{\tfrac{1}{2}(\hat{R}_1 \oplus \hat{R}_2) + \tfrac{1}{2}(\hat{R}_1 + \hat{R}_3) + \tfrac{1}{2}(\hat{R}_2 + \hat{R}_3)\})_p,\\
&\equiv (\Sigma\{\tfrac{1}{2}(x_1 \oplus x_2) + \tfrac{1}{2}(x_1 \oplus x_3) + \tfrac{1}{2}(x_2 + x_3)\})_p.
\end{aligned}$$

Thus the function may be realised as shown in **Fig. 7.**

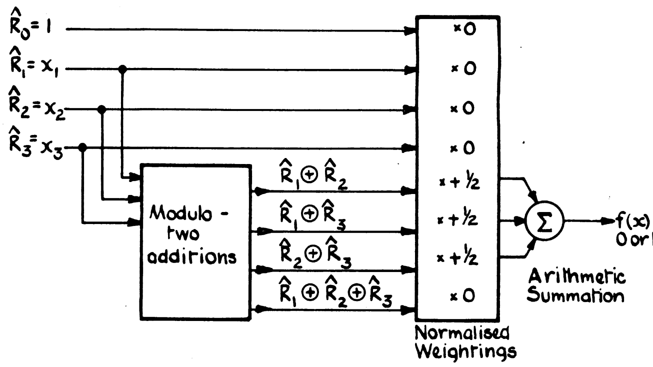

Fig. 7. Modified 0, 1 Rademacher-Walsh realisation for $f(x) = [x_1\bar{x}_2 + x_2\bar{x}_3 + \bar{x}_1x_3]$

10. Alternative definitions for w_i and $\hat{w}_i$

In both the +1, −1 case, see Section 8, and the 0, 1 case, see Section 9, it was shown that the respective required weights w_i and $\hat{w}_i$ for the Rademacher-Walsh variables, $i \neq 0$, can be defined as:

w_i or $\hat{w}_i$, $i = 1, 2, \cdots, 12, \cdots, =$

$$\frac{1}{2^n}\,[(\text{No. of agreements ——}) - (\text{No. of disagreements ——})].$$

(NOTE: the w_0, $\hat{w}_0$ weight is *not* the same in the +1, −1 and the 0, 1 cases).

This expression may be algebraically re-arranged as follows:

Let a = number of minterms where $f(x) = 1$ and $x_i = 1$,
b = number of minterms where $f(x) = 1$ and $x_i = 0$,
c = number of minterms where $f(x) = 0$ and $x_i = 1$,
d = number of minterms where $f(x) = 0$ and $x_i = 0$,

for example see **Fig. 8**(*a*), where a $\bar{x}_1$, x_1 map division is illustrated. Then w_i and $\hat{w}_i$, $i \neq 0$, are given by:

$$w_i,\ \hat{w}_i = \frac{1}{2^n}\,[(a + d) - (b + c)]$$

Considering just the expression within the brackets, re-arranging we have:

$$\begin{aligned}
&[(a - b) - (c - d)],\\
&= [(a - b) - ((2^{n-1} - a) - (2^{n-1} - b))],\\
&= [(a - b) - (-a + b)],\ = 2[(a - b)].
\end{aligned}$$

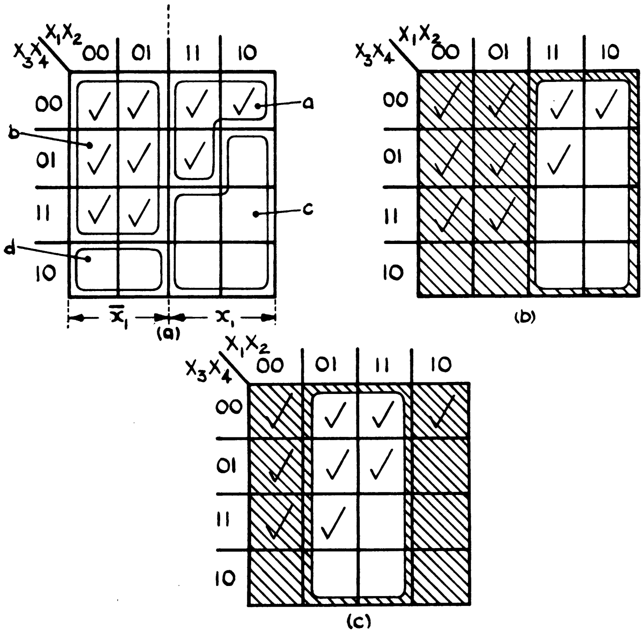

Fig. 8. Karnaugh-map plots of $f(x) = [\bar{x}_1x_4 + x_2\bar{x}_3 + \bar{x}_3x_4]$
(a) $\bar{x}_1$, x_1 division
(b) x_1 'window'
(c) x_2 'window'

Now $(a + b)$ = number of true minterms, = say m.
∴ Substituting we obtain the expression:

$$2\,[a - (m - a)]\,,$$
$$= [4a - 2m]\,.$$

∴ The normalised equation for the required w_i or $\hat{w}_i$, $i \neq 0$, may be written as

$$\frac{1}{2^n}\,[4a - 2m]\,,$$

$$= \frac{1}{2^n}\,[4(\text{number of true minterms of the function when } x_i \text{ is true}) - 2(\text{total number of true minterms})]\,.$$

This alternative definition lends itself to reading off the required w_i or $\hat{w}_i$, $i \neq 0$, from Karnaugh maps by merely counting the number of true minterms in the appropriate part of the map. For example the function shown in Fig. 8(a) is further illustrated in **Fig. 8**(b) and (c), from which the following results may be read:

$$w_1 \text{ or } \hat{w}_1 = \frac{1}{2^n}\,[(4 \times 3) - (2 \times 9)]$$
$$= \frac{1}{2^n}\,[-6]\,,$$
$$w_2 \text{ or } \hat{w}_2 = \frac{1}{2^n}\,[(4 \times 5) - (2 \times 9)]$$
$$= \frac{1}{2^n}\,[+2]\,,$$

and so on for w_3, etc.
NOTE: for the $+1$, -1 situation w_0 may be rewritten as:

$$w_0 = -\frac{1}{2^n}\,[m - (2^n - m)]$$
$$= -\frac{1}{2^n}\,[2m - 2^n]\,,$$
$$= \frac{1}{2^n}\,[2^n - 2m]\,.$$

For the 0, 1 situation, $\hat{w}_0$ remains as the function value $f(x)$ at $p = 0$, see Section 9.

11. Karnaugh-map identification of all 2^n Rademacher-Walsh functions

It is of interest to further consider the areas or parts of a Karnaugh-map layout that all the 2^n Rademacher-Walsh functions (variables) define. We have previously seen that the Rademacher variables, the 'primary set', are equivalent to the binary input variables of the system, R_0 always being $+1$, the 'secondary set' being all different modulo-two additions of this primary set in 0, 1 working.

Thus for $n = 3$, we have the full situation shown in **Fig. 9.** Similar maps for $n = 4$ etc. may be drawn.

It is now of great interest to note that although the primary set $R_0, \cdots, R_n$ are all linearly-separable (threshold) functions, that is each may be realised by one threshold logic gate, NONE OF THE MODULO-2 SECONDARY SET ARE LINEARLY SEPARABLE. In fact, the final Rademacher-Walsh variable $R_{12\cdots n}$, ($\hat{R}_{12\cdots n}$), is as far removed from a threshold function as is possible—it may be considered as 'opposite' in some sense to a threshold function.

This may help to non-mathematically indicate why a combination of all these variables, appropriately employed, is able to realise *any* given binary function, whereas threshold functions, which form a more restricted class of binary functions, require only the appropriately-weighted primary set $R_0, \cdots, R_n$ for their realisation, see **Fig. 10** (a) and (b).

It may also help to explain the interest that one or two researchers have shown, in attempting to realise any given binary function by the use of some minimum combination of threshold gates plus modulo-two gates. By definition a linearly-separable function requires only one threshold gate; if a given function is not linearly-separable, then the minterms which cannot be included by a threshold gate may be included by some modulo-two gate, such as to give some overall minimum network realisation as indicated in **Fig. 11.**

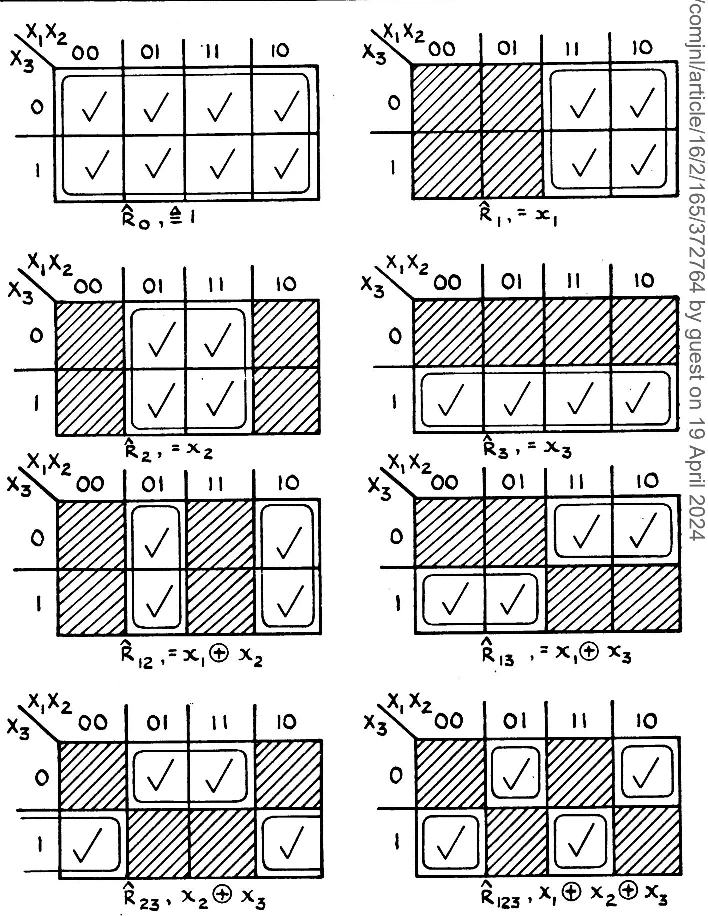

Fig. 9. Karnaugh-map identification of modified Rademacher-Walsh variables

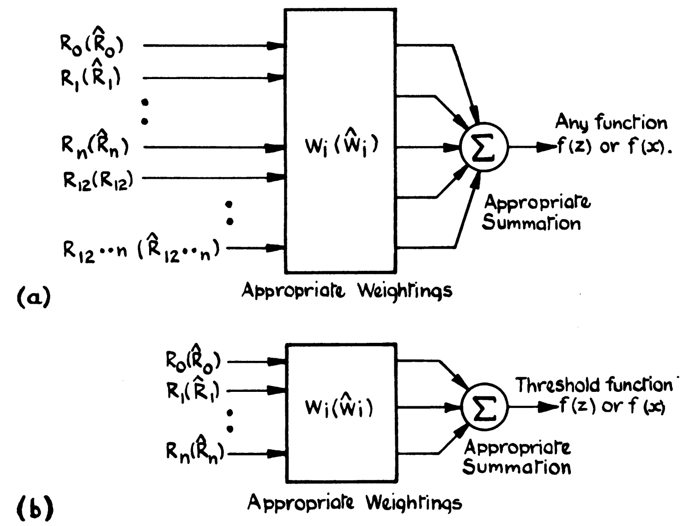

Fig. 10. **Comparison of Rademacher-Walsh and threshold synthesis**
(*a*) **Rademacher-Walsh**
(*b*) **threshold**

It should be noted that the weighting of all the 2^n Rademacher-Walsh variables and their subsequent appropriate summation, whilst mathematically elegant and universal, does not normally provide an economic realisation if a direct one-to-one translation into hardware is contemplated.

Considerable work in this field of minimum network synthesis remains to be investigated.

12. Classification of Boolean functions

The magnitude of the Chow parameters $|b_i|$ taken in a lexicographical order forms the most compact form of classifying and listing all the possible linearly-separable (threshold) functions (Dertouzos, 1965; Winder, 1965), see Section 1. This form of classification is a numerical form of SD ('self-dualised' or 'hyperfunction') classification.

It may therefore be of interest to consider the Rademacher-Walsh w_i to see if they form a classification for all binary functions, in a manner similar to the $|b_i|$'s for threshold functions.

As an exercise, take as a starting point the Chow parameter $|b_i|$ classification 6, 2, 2, 2. This classification covers a series of threshold functions of three variables $f(x_1, x_2, x_3)$, characterised by having either 7, 5, 3 or 1 true minterms. (The 7 and the 1, and the 5 and the 3, are complements of each other, respectively, these two groups being related by the SD classification.)

Let us therefore list a few linearly-separable and non-linearly-separable functions with the above minterm count. The Rademacher-Walsh weights will be listed un-normalised, that is not divided through by $\frac{1}{2^n}$, so as to make the w_0 to w_3 numerically identically to the b_0 to b_3, respectively. **Fig. 12** illustrates the functions chosen.

From these arbitrary examples, the following features will be seen:

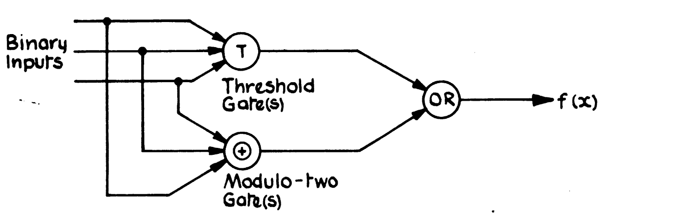

Fig. 11. **Threshold and modulo-two synthesis**

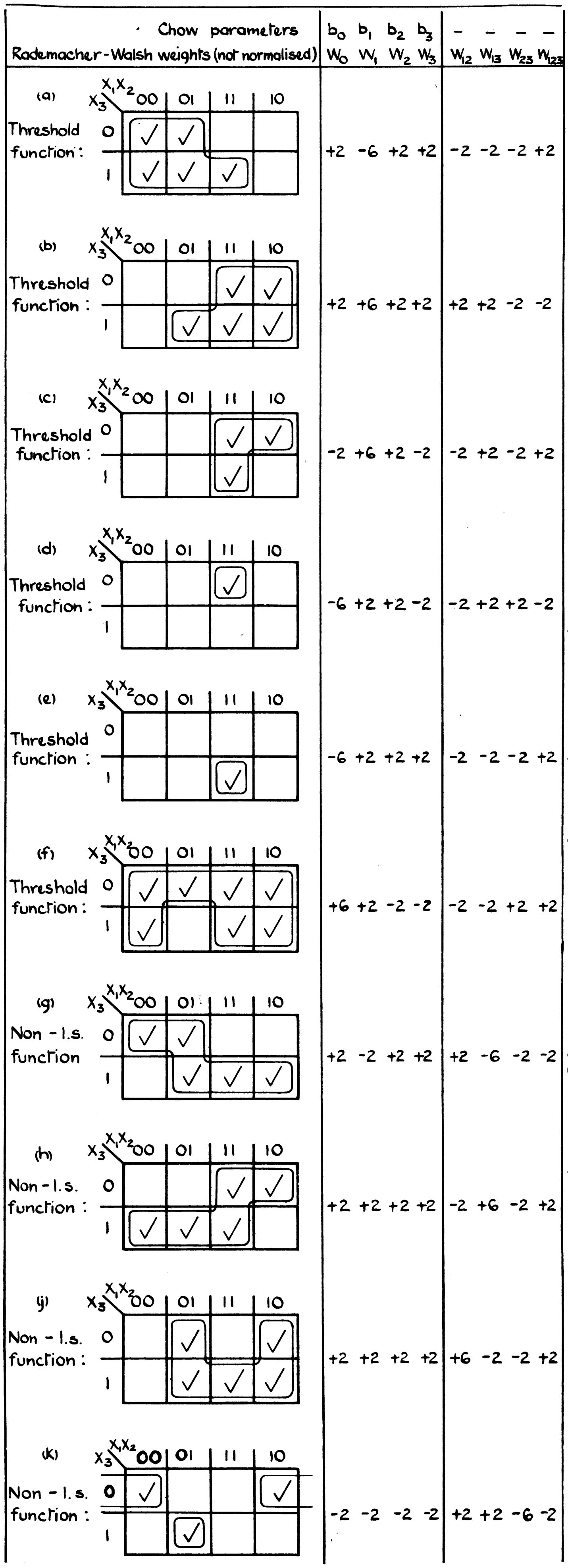

Fig. 12. **Classification of example binary functions**

(i) all given functions are characterised by un-normalised Rademacher-Walsh weights $|w_i|$ of

$$6, 2, 2, 2, 2, 2, 2, 2$$

when listed in descending magnitude;

(ii) if however the '6' appears in the left-hand 'primary set', then the function must be a threshold function; conversely if the '6' appears in the right-hand 'secondary set', leaving $|b_0|$ to $|b_3|$ as 2, 2, 2, 2, then such a function cannot be a threshold function;

(iii) with the threshold functions, the values of the 'primary set' w_0 to w_3 with their appropriate signs, uniquely define the function, as these primary sets are the Chow parameters, see Section 3. Hence the remaining w_i values, $i = 12$, 13 etc., do not add any further information about the function—their values must therefore be determinable from the w_i of the primary set.

The precise correlation between w_i, $i = 0$ to n, and w_i, $i = 12$, 13, etc. of these and all other threshold functions, however, is not directly obvious;

(iv) when the function is not a threshold function, the values of the primary set do not define the function, see Figs. 12(*h*) and (*j*) covering functions with the same w_0 to w_3 values. Equally, however, the secondary set w_i values do not explicitly define the function, see further examples illustrated in **Fig. 13.**

Thus for a non-linearly separable function, all 2^n Rademacher-Walsh weights with their appropriate signs are essential to define the function.

To attempt to summarise the above, it would appear that all binary functions could be classified by their $|w_i|$ values, in the same way that all threshold functions are classified by their $|b_i|$ values. At this stage it is not quite clear just what invariance operations are involved per classification. (The invariance operations involved in the threshold function $|b_i|$ classifications is well understood (Dertouzos, 1965).)

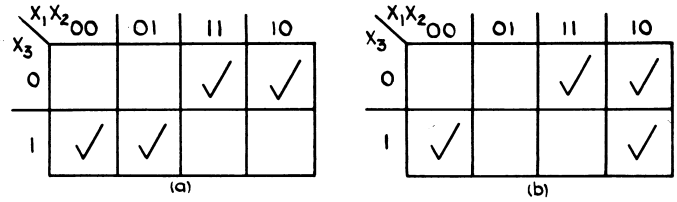

Fig. 13. Classification of dissimilar functions
(a) $f(x) = [x_2\bar{x}_3 + \bar{x}_1x_3]$,
$w_i = 0, -4, 4, 0; 0, 4, 4, 0$
(b) $f(x) = [x_1x_3 + x_2x_3]$,
$w_i = 0, 4, -4, 0; 0, 4, 4, 0$

The precise correlation between the w_i values, $i = 0$ to n, and the w_i values, $i = 12$, 13, etc. for the threshold functions also requires some further thought.

Conclusions

A considerable area of research and consolidation is still open in the field of synthesis of Boolean switching functions. With the increasing practicality of economically generating complex switching functions in integrated-circuit form, such as threshold gates and modulo-two functions, etc., a new approach to the most economical practical realisation of given switching problems may arise. The present-day NAND/NOR gate realisations may well be superceded by more compact realisations, with a smaller gate population, in future generations of equipment.

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References

BARRETT, R. and GORDON, J. A. (1971). *Walsh functions—an introduction*, Proc. Symposium. Theory and Application of Walsh Functions, Hatfield Polytechnic, England, June, 1971.

CHOW, C. K. (1961). *On the characterization of threshold functions*, IEEE Special Publication S.134, August 1961, pp. 34-38.

COLEMAN, R. P. (1961). Orthogonal functions for the logical design of switching circuits, *Trans. I.R.E.*, EC-10, pp. 379-383.

COOPER, J. A. (1963). *Orthogonal expansion applied to the design of threshold-element networks*, Technical Report No. 6204-1, U.S. Office of Naval Research, Contract Nonr. 225(24), NR373360, December 1963.

DERTOUZOS, M. L. (1965). *Threshold logic: a synthesis approach*, Research Monograph No. 32, M.I.T. Press.

GOLOMB, S. B. (1959). On the classification of Boolean Functions, *Trans. I.R.E.*, CT6, (Special Supplement), pp. 176-186.

HENDERSON, K. W. (1964). Some notes on the Walsh functions, *Trans. I.E.E.E.*, EC13, pp. 50-52.

HENDERSON, K. W. (1970). Previous investigation of realization of an arbitrary switching function with a network of threshold and parity elements, *Trans. I.E.E.E.*, C19, pp. 941-942.

HURST, S. L. (1970). Correlation between Karnaugh-map patterns and Chow parameters for 3-variable linearly-separable functions, *Electronic Letters*, Vol. 6, No. 11, pp. 324-326.

HURST, S. L. (1971). *Threshold Logic*, Mills and Boon, London.

LACKEY, R. B. and MELTZER, D. (1971). A simplified definition of Walsh functions, *Trans. I.E.E.E.*, C20, pp. 211-213.

LEWIS, P. M. (1967). A practical guide to threshold logic, *Electronic Design*, Vol. 22, pp. 66-88.

LIEDL, R. and PICHLER, F. (1971). *On harmonic analysis of switching functions*, Proc. Symposium Theory and Application of Walsh Functions, Hatfield Polytechnic, England, June 1971.

MUROGA, S., TSUBOI, T., and BAUGH, C. R. (1967). *Enumeration of threshold functions of eight variables*, Report No. 245, University of Illinois, August 1967.

OSTAIKO, D. L. and YAU, S. S. (1970). Realization of an arbitrary switching function with a two-level network of threshold and parity elements, *Trans. I.E.E.E.*, C19, pp. 262-269.

Research on Automatic computer electronics, Vol. II, Logical Design Research, Lockheed Missile and Space Co., California, Tech. Document Report No. RTD-TDR-63-4173, February 1964.

WALSH, J. L. (1923). A closed set of normal orthogonal functions, *Am. Journal Math.*, Vol. 45, pp. 5-24.

WINDER, R. O. (1964). *Threshold functions through n = 7*, Scientific Report No. 7, AFCRL Contract AF19(604)-8423, October 1964.

WINDER, R. O. (1965). Enumeration of seven-argument threshold functions, *Trans. I.E.E.E.*, EC14, pp. 315-325.