The use of Chebyshev series for the evaluation of oscillatory integrals

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Clenshaw and Curtis (1960) have given a scheme for the numerical integration of a well-behaved function $f(x)$, with the interval of integration normalised to $[-1, 1]$, which is based on the approximation of $f(x)$ in a series of Chebyshev polynomials, $T_n(x)$. In this context, the function is said to be well-behaved if the coefficients in the Chebyshev expansion fall off rapidly. This method is extended to integrals of the form

$$\int_a^b f(x) \frac{\cos}{\sin} px\, dx$$

A new algorithm is presented which evaluates the resulting basic integrals directly by a method which is analogous to the automatic generation of quadrature formulae of the Newton-Cotes type, as presented by Alaylioglu, Evans and Hyslop (1975). The stability of the method is discussed and critical comparisons, including numerical tests on several practical examples, are carried out with the related earlier work of Bakhvalov and Vasil'eva (1968) and Piessens and Poleunis (1971).

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1. Introduction

Normalisation of the range of integration leads to consideration of integrals of the form

$$\int_{-1}^{1} f(x) \frac{\cos}{\sin} \omega x\, dx\ . \tag{1}$$

The usual methods of evaluating (1) rely on approximating $f(x)$ by a series

$$f(x) \simeq \sum_{i=0}^{n} a_i A_i(x) \tag{2}$$

so that the integrals

$$\int_{-1}^{1} A_i(x) \frac{\cos}{\sin} \omega x\, dx \tag{3}$$

are obtainable analytically. For instance, the choice

$$A_i(x) = x^i \tag{4}$$

yields the existing quadrature formulae of Clendenin (1966), Filon (1928) and Flinn (1960) corresponding to $n = 1, 2$ and 5 respectively. The automatic generation of the quadrature formula for general order n has been described by Alaylioglu, Evans and Hyslop (1975).

The theory of approximation (Davis, 1963) suggests that a better form for $A_i(x)$ would be the Chebyshev polynomial $T_i(x)$. This process has been widely used for non-oscillatory integrands and gives the well-known formulae of Clenshaw and Curtis (1960). However, the evaluation of the integrals (3) in the oscillatory case seems to present a problem when Chebyshev polynomials are employed.

Bakhvalov and Vasil'eva (1968) have briefly considered this problem (although their main theme was the use of Legendre polynomials $P_i(x)$). They suggest that, if the zeros of the Chebyshev polynomials are used as the interpolatory points in a Lagrange interpolation formula, then orthogonality relations can be used to evaluate the required coefficients. They state that the resulting quadrature formulae are somewhat more complicated than the results they quote for the Legendre polynomial procedure and imply that the effect of round-off in the calculation may therefore be more serious.

Piessens and Poleunis (1971) also consider the use of Chebyshev polynomials for $A_i(x)$ but deviate from the Bakhvalov and Vasil'eva approach in that they effectively evaluate the basic integral (3) by a somewhat indirect method involving a truncated infinite series of Bessel functions, instead of utilising the orthogonality properties of summation over the zeros of the Chebyshev polynomials.

In the present work the Bakhvalov and Vasil'eva approach to the derivation of the Chebyshev based quadrature formula is adopted and comparisons with the earlier papers are noted as the method is developed.

Thus, following Bakhvalov and Vasil'eva, the integral

$$I = \int_{-1}^{1} f(x)\, e^{i\omega x}\, dx \tag{5}$$

is treated by introducing the Lagrangian interpolation polynomial of degree n

$$f(x) \simeq \sum_{i=0}^{n}{}' a_i T_i(x) \tag{6}$$

which collocates with $f(x)$ at the $(n + 1)$ points x_j $(j = 0, 1, 2, \ldots, n)$, the prime denoting that the first term in the summation is to be multiplied by $\frac{1}{2}$. If the points x_j are chosen to be the zeros of $T_{n+1}(x)$, yielding

$$x_j = \cos\left[\frac{2j + 1}{n + 1} \cdot \frac{\pi}{2}\right] j = 0, 1, 2, \ldots, n \tag{7}$$

then the coefficients a_i may be found by using orthogonality relations. The required relations are

$$\begin{aligned} \sum_{j=0}^{n} T_i(x_j)\, T_k(x_j) &= 0 && i \neq k \\ &= \tfrac{1}{2}(n + 1) && i = k \neq 0 \\ &= (n + 1) && i = k = 0 \end{aligned} \tag{8}$$

and result in the well-known expression

$$\begin{aligned} a_i &= \frac{2}{n + 1} \sum_{j=0}^{n} f(x_j)\, T_i(x_j) \\ &= \frac{2}{n + 1} \sum_{j=0}^{n} f(x_j) \cos\left[\frac{(2j + 1)}{(n + 1)} \cdot \frac{\pi}{2}\right] \end{aligned} \tag{9}$$

It is interesting to note the equivalent way of considering equation (6) which utilises the integral orthogonality result

$$\begin{aligned} \int_{-1}^{1} T_i(x)\, T_k(x)\, (1 - x^2)^{-\frac{1}{2}}\, dx &= 0 && i \neq k \\ &= \tfrac{1}{2}\pi && i = k \neq 0 \\ &= \pi && i = k = 0 \end{aligned} \tag{10}$$

which produces the formula

$$a_i = \frac{2}{\pi} \int_{-1}^{1} T_i(x)\, f(x)\, (1 - x^2)^{-\frac{1}{2}}\, dx \tag{11}$$

If $f(x)$ is to be represented by a polynomial of degree n given by (6) then this integral is obtained *exactly* from the Gauss–Chebyshev equal weight quadrature formula of order $(n + 1)$ for the weight function $w(x) = (1 - x^2)^{-\frac{1}{2}}$ on the interval $[-1, 1]$. The required result is

$$\int_{-1}^{1} F(x)(1 - x^2)^{-\frac{1}{2}}\,dx = \frac{\pi}{(n+1)} \sum_{j=0}^{n} F(x_j) + \varepsilon_{n+1} \quad (12)$$

where x_j is given by (7) and the error term by

$$\varepsilon_{n+1} = 2\pi F^{(2n+2)}(\theta)/[2^{2n+2}(2n+2)!] \quad (-1 < \theta < 1) \quad (13)$$

(Abramowitz and Stegun, 1965, p. 889). Equation (9) then follows immediately.

The expansion (6) with a_i given by (9) is now utilised in the integral I and produces

$$I \simeq \sum_{i=0}^{n}{}' a_i N_i(\omega) \quad (14)$$

where

$$N_i(\omega) = \int_{-1}^{1} T_i(x)\, e^{i\omega x}\, dx\ . \quad (15)$$

This may be written in Bakhvalov and Vasil'eva form as

$$I \simeq \sum_{j=0}^{n} D_j f(x_j) \quad (16)$$

where

$$D_j = \frac{2}{(n+1)} \sum_{i=0}^{n}{}' N_i(\omega)\, T_i(x_j) \quad (17)$$

the order of summation having been changed. These results are entirely analogous to the Legendre based prescription of the Bakhvalov and Vasil'eva paper.

The main difficulty associated with this approach is the evaluation of the basic integral $N_i(\omega)$ and it is worthwhile describing, at this stage, the related work of Piessens and Poleunis (1971). These authors attempt to avoid the direct evaluation of $N_i(\omega)$ by using the alternative *infinite* expansion

$$(1 - x^2)^{\frac{1}{2}} f(x) = \sum_{k=0}^{\infty}{}' c_k T_k(x)\ . \quad (18)$$

The resulting integrals

$$\int_{-1}^{1} T_k(x)(1 - x^2)^{-\frac{1}{2}} \begin{matrix}\cos\\ \sin\end{matrix}\ \omega x\, dx \quad (19)$$

may be evaluated *analytically* and yield the results

$$\int_{-1}^{1} f(x) \cos \omega x\, dx = \sum_{k=0}^{\infty}{}' c_{2k}(-1)^k \pi J_{2k}(\omega) \quad (20)$$

and

$$\int_{-1}^{1} f(x) \sin \omega x\, dx = \sum_{k=0}^{\infty}{}' c_{2k+1}(-1)^k \pi J_{2k+1}(\omega) \quad (21)$$

involving infinite series of Bessel functions.

The integral orthogonality result (10) is then used in (18) and gives

$$c_k = \frac{2}{\pi} \int_{-1}^{1} f(x)\, T_k(x)\, dx\ . \quad (22)$$

Piessens and Poleunis then suggest that this integral should be evaluated by using the finite expansion (6) and this leads to the result

$$c_k \simeq \frac{2}{\pi} \sum_{i=0}^{n}{}' a_i \int_{-1}^{1} T_k(x)\, T_i(x)\, dx\ . \quad (23)$$

The integrals in (23) are easily evaluated analytically and the results for even and odd k as required by (20) and (21) are quoted by Piessens and Poleunis.

It should be pointed out that their final results are also obtainable from the direct Bakhvalov and Vasil'eva procedure embodied in equations (14) and (15). In fact, it is easily seen that their approach is merely equivalent to the evaluation of $N_i(\omega)$ by using the expansion of $\exp(i\omega x)$ with $x = \cos\theta$ as an infinite series of Bessel functions (Abramowitz and Stegun, 1965, p. 361).

It is the evaluation of this infinite summation which suggests a defect in the Píessens and Poleunis version of the Bakhvalov and Vasil'eva–Chebyshev approach. Piessens and Poleunis demonstrate that the terms in the series (20) and (21) decrease rapidly for $k > \omega/2$ and suggest that truncation may be effected after M terms where M is 'only a little larger than $\omega/2$'. Clearly, for large ω the method is unsatisfactory especially since the evaluation of the Bessel functions is required in the terms of the series. An alternative procedure which avoids this infinite series is therefore investigated.

The original Bakhvalov and Vasil'eva–Legendre work involved the evaluation of the integral

$$M_i(\omega) = \int_{-1}^{1} P_i(x)\, e^{i\omega x}\, dx \quad (24)$$

which is replaced by (15) in the present investigation, by a three term recurrence relation. This relation proved to be unstable in the forwards direction, particularly for small ω, and it was necessary to use Miller's algorithm (Abramowitz and Stegun, 1965, p. 452). Similarly, in the present case, by writing

$$I_k(\omega) = i^{-k} N_k(\omega) \quad (25)$$

and integrating the appropriate recurrence relations for the Chebyshev polynomials, it is possible to establish the result

$$\omega I_{k+2}(\omega) = (2k+4)\, I_{k+1}(\omega) + (2k-4)\, I_{k-1}(\omega) - 2\omega\, I_k(\omega) - \omega I_{k-2}(\omega)\ . \quad (26)$$

This relation is again unstable in the forwards direction and the use of Miller's algorithm is once more necessitated, a complication here being that (26) is a five term relation. It is also possible to derive three term recurrence relations for the separate integrals in (3), but these are again unstable and have the additional disadvantage of being 'inhomogeneous'.

Indeed, the question of the recurrence relation approach to the Bakhvalov and Vasil'eva–Chebyshev procedure is being currently investigated by Patterson and his co-workers (Patterson, 1974)*. Consequently, in the present paper, a simple alternative approach is proposed for the direct evaluation of $N_i(\omega)$ in (15) by a method which is analogous to the techniques described by the authors in their work on the corresponding Newton–Cotes based formulae.

2. The quadrature formula

The basis of the method is the evaluation of $N_i(\omega)$ by picking out the coefficients, $D_{i,r}$, of x^r in $T_i(x)$ and then making use of the results

$$\int_{-1}^{1} x^r \cos \omega x\, dx = \sum_{l=0}^{r} l! \begin{bmatrix} r \\ l \end{bmatrix} \frac{x^{r-l}}{\omega^{l+1}} \sin(\omega x + \tfrac{1}{2} l\pi) \Big|_{-1}^{1} \quad (27)$$

$$\int_{-1}^{1} x^r \sin \omega x\, dx = -\sum_{l=0}^{r} l! \begin{bmatrix} r \\ l \end{bmatrix} \frac{x^{r-l}}{\omega^{l+1}} \cos(\omega x + \tfrac{1}{2} l\pi) \Big|_{-1}^{1} \quad (28)$$

(Gradshteyn and Ryzhik, 1965).

The Chebyshev polynomials are of the form (Abramowitz and Stegun, 1965)

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \\ &\vdots \end{aligned} \quad (29)$$

*Since the preparation of this paper the work of Littlewood and Zakian (to be published), which also follows the recurrence relation approach, has been brought to the attention of the authors.

and the coefficients, $D_{i,r}$, of x^r in $T_i(x)$ can be easily calculated by means of the recurrence relation

$$D_{i,r} = 2D_{i-1,r-1} - D_{i-2,r}\quad i \geqslant 2, r \leqslant i \qquad (30)$$

yielding a 'Pascal' triangle, which facilitates computation. The basic integrals $N_i(\omega)$ of (15) are then easily obtained. In practice these are usually separated into their real and imaginary parts for the separate calculation of the integrals involving $\cos\omega$ or $\sin\omega$ using, either (27) or (28) for even or odd r respectively.

The formula (14) is then used directly here and embodies the Chebyshev fit (6) at the Gaussian based abscissae (7). In their original work on non-oscillatory integrals Clenshaw and Curtis utilised the alternative approximation

$$f(x) \simeq \sum_{i=0}^{n}{}'' a_i T_i(x) \qquad (31)$$

with collocation at the points x_j $(j = 0, 1, 2, \ldots, n)$ where x_j is now given by

$$x_j = \cos\frac{\pi j}{n}, (j = 0, 1, 2, \ldots, n) \qquad (32)$$

the double primes denoting that the first and the last terms in the summations are to be multiplied by $\frac{1}{2}$. The alternative orthogonality relation

$$\begin{aligned}\sum_{j=0}^{n}{}'' T_i(x_j)\, T_k(x_j) &= 0 \quad i \neq k\\ &= \frac{n}{2} \quad i = k \neq 0 \text{ or } n \qquad (33)\\ &= n \quad i = k = 0 \text{ or } n\end{aligned}$$

(*cf.* Equation (8)) produces the result

$$\begin{aligned}a_i &= \frac{2}{n}\sum_{j=0}^{n}{}'' f(x_j)\, T_i(x_j)\\ &= \frac{2}{n}\sum_{j=0}^{n}{}'' f(x_j)\cos\frac{i\pi j}{n} \qquad (34)\end{aligned}$$

which is then used in the quadrature formula (14) as an alternative to the Gauss-based prescription (9). It will be noticed that equation (31) is a closed formula in that it involves the end points $x = -1$ and $x = 1$, whereas (6) is open. Elliott (1965) has pointed out that, in the evaluation of the non-oscillatory integral, the truncation error involved in the use of the classical or open formula (6) is of the order of $1/n^2$. However, when the practical or closed series (31) is utilised the truncation error is of order $1/n^3$.

In the present work, the use of the Clenshaw and Curtis formula (31) is proposed, although (6) could be adopted if an open formula is specifically required, such as in the case where the integrand has singularities at its end points. Elliott's analysis gives intuitive backing to our method, that collocation at the practical abscissae is better than collocation at the classical Chebyshev zeros, if we are more interested in the integral of $f(x)$ than in approximating $f(x)$ itself.

Piessens and Poleunis demonstrate, that as for Clenshaw–Curtis quadrature in the non-oscillatory case, the *integral* of the finite Chebyshev expansion converges more quickly than the expansion itself. It is hoped therefore to retain the advantages of the Clenshaw and Curtis formulation in the oscillatory case. In particular, by choosing the order of the formulae as $n = 2^i$, $i = 1, 2, \ldots$ the adaptive nature of the procedure could be retained.

The errors involved in the Clenshaw and Curtis formulae have been discussed by many authors such as O'Hara and Smith (1968), Gentleman (1972) and Elliott (1965). The errors are, in fact, less than might be expected. O'Hara and Smith show that the error terms are such that the accuracy may even approach that of the corresponding n-point Gauss formula in certain instances. In general however, more functions evaluations are normally required for the Clenshaw–Curtis case than for the corresponding Gaussian quadrature. Nonetheless, it is considered that the present prescription is worth investigating as a practical alternative in the oscillatory case.

It is important to note that, as pointed out by Bakhvalov and Vasil'eva it may be better not to sub-divide the range of integration but to increase the order of the formula used when Chebyshev (or Legendre) fitting is used for $f(x)$. This is in contrast with the methods of the previous work by the present authors where equally spaced abscissae were used and gave rise to formulae of the Newton–Cotes type. Due to instabilities in the higher order coefficients, it was not possible to proceed to large n there and the recommendation in practice was to limit the order to $n = 5$ or 6 and sub-divide the interval of integration uniformly. This method was also adopted by Bakhvalov and Vasil'eva for comparison purposes in one of their numerical applications, where they used a formula of order $n = 4$ with a large number of sub-divisions, to consider an integral involving $f(x) = \cos\pi u x^2$ with large u. The use of this technique is not entirely satisfactory in general, and great care must be exercised in the highly oscillatory cases when ω is large. Indeed preliminary calculations based on the present method have indicated that uniform sub-division may produce similar instabilities to those mentioned above and that the order of the formulae would have to be similarly curtailed. Hence, uniform sub-division is not adopted here. However it is worth noting that it may be possible to avoid the cancellation effects produced by uniform sub-division by using special techniques appropriate to the particular function considered. As an example good results are obtained for $f(x) = \cos\pi u x^2$ by integrating between the 'peaks' (which occur at the zeros of $\sin = \pi u x^2$) or between the zeros of $\cos\pi u x^2$ using either Newton–Cotes or the present methods. This example is discussed in detail in a later section.

Consequently to return to the derivation of the quadrature formula, when the integral

$$I_c = \int_a^b f(x)\cos px\, dx \qquad (35)$$

is considered, a linear transformation enables the result to be written in the form

$$\begin{aligned}I_c &= \tfrac{1}{2}(b - a)\cos K\int_{-1}^{1} F(t)\cos\omega t\, dt\\ &\quad - \tfrac{1}{2}(b - a)\sin K\int_{-1}^{1} F(t)\sin\omega t\, dt \qquad (36)\end{aligned}$$

where

$$K = \tfrac{1}{2}p(b + a) \qquad (37)$$

$$\omega = \tfrac{1}{2}p(b - a) \qquad (38)$$

and

$$F(t) \equiv f[\tfrac{1}{2}(b + a) + \tfrac{1}{2}(b - a)t]\ . \qquad (39)$$

Approximating $F(t)$ by the polynomial of degree n

$$F(t) \simeq \sum_{i=0}^{n}{}'' a_i T_i(t) \qquad (40)$$

collocating at the $(n + 1)$ points t_j

$$t_j = \cos\frac{\pi j}{n} \quad j = 0\,(1)\,n \qquad (41)$$

yields

$$\begin{aligned}I_c &\simeq \tfrac{1}{2}(b - a)\cos K\sum_{i=0}^{n}{}'' a_i\sum_{r=0}^{i} D_{i,r}\int_{-1}^{1} t^r\cos\omega t\, dt\\ &\quad - \tfrac{1}{2}(b - a)\sin K\sum_{i=0}^{n}{}'' a_i\sum_{r=0}^{i} D_{i,r}\int_{-1}^{1} t^r\sin\omega t\, dt \qquad (42)\end{aligned}$$

where

$$a_i = \frac{2}{n} \sum_{j=0}^{n}{}'' F(t_j) \cos \frac{\pi j i}{n} \tag{43}$$

Similarly

$$I_s = \int_a^b f(x) \sin px \, dx \tag{44}$$

yields

$$I_s \simeq \tfrac{1}{2}(b-a) \sin K \sum_{i=0}^{n}{}'' a_i \sum_{r=0}^{i} D_{i,r} \int_{-1}^{1} t^r \cos \omega t \, dt$$

$$+ \tfrac{1}{2}(b-a) \cos K \sum_{i=0}^{n}{}'' a_i \sum_{r=0}^{i} D_{i,r} \int_{-1}^{1} t^r \sin \omega t \, dt \tag{45}$$

The basic integrals required in (42) and (45) are supplied by (27) and (28).

3. Stability of the algorithm and practical recommendations

It will be noticed that the finite series occurring in equations (27) and (28) converge rapidly when ω is large. This will be emphasised when the function $f(x)$ is sufficiently smooth for accurate fitting to be possible with a formula whose order, n, is reasonably small. On the other hand, if $f(x)$ requires a formula of high order with a large value of n to achieve an accurate fit the coefficients

$$\frac{n!}{(n-l)!} \cdot \frac{1}{\omega^{l+1}} \qquad l = 0\,(1)\,n \tag{46}$$

which appear in (27) and (28) may become very large. (Note that the largest value of r, namely $r = n$, has been taken here to accentuate the effect.) This will be particularly noticeable when ω is small and serious instabilities may arise in this case of small ω and large n. This is clearly due to the generation of very large numbers, with the resulting cancellation when the terms in the alternating series are summed.

An alternative procedure which avoids this instability is to use series expansions for the trigonometric functions in (27) and (28). The expressions

$$\int_{-1}^{1} x^r \cos \omega x \, dx = 2 \sum_{l=0}^{\infty} \frac{(-1)^l \omega^{2l}}{(2l + r + 1)(2l)!} \tag{47}$$

when r is even and

$$\int_{-1}^{1} x^r \sin \omega x \, dx = 2 \sum_{l=0}^{\infty} \frac{(-1)^l \omega^{2l+1}}{(2l + r + 2)(2l + 1)!} \tag{48}$$

when r is odd, are readily obtained and are obviously most useful in precisely those circumstances (small ω, large r) under which the finite series (27) and (28) are least stable. In practice, it is easily demonstrated that the maximum value of l required to yield double precision accuracy (about 22 figures) for the basic integrals is given roughly by

$$l_0 = 2\omega + 10 , \tag{49}$$

round-off to integral values being implied. This estimate for the truncation point of the infinite series is reliable for $\omega \leqslant 10$. For larger values of ω, it tends to be a gross over-estimate. For instance, at $\omega = 100$, $l_0 = 210$ whereas the actual maximum value of l required is only about 158. However, for such large values of ω, it is likely that the finite series (27) and (28) would be used instead and, hence, it is suggested that (49) provides a reasonable estimate of the number of terms required in all practical cases.

Indeed, it is clear that formulae (47) and (48) exhibit instabilities for large ω which are 'complementary' to those shown by the finite series (27) and (28). It is possible to discuss this effect qualitatively by considering the behaviour of the related series for cos ω whose general term is of the form

$$(-1)^l \omega^{2l}/(2l)! \qquad l = 0\ 1\ 2\ \ldots . \tag{50}$$

Thus, the factors such as $(2l + r + 1)^{-1}$ in (27) and (28) which assist convergence in any case have been omitted. In the case of the ω^{-1} series in (27) and (28) the coefficients of $\pm\sin\omega$ and $\pm\cos\omega$ are given by equation (46) and range from $1/\omega$ when $l = 0$ to $n!/\omega^{n+1}$ when $l = n$. A measure of the instability of the series is provided by the ratio of these quantities, namely

$$n!/\omega^n \tag{51}$$

which are the reciprocals of the terms in (50) or the corresponding terms in the series for sin ω, thus demonstrating the 'reciprocal' nature of the instabilities.

An examination of the magnitude of the terms in (50) with $0 \leqslant l \leqslant l_0$ demonstrates that, for a given ω, the maximum value is attained when $2l = [\omega]$ and the required maximum is therefore

$$L = \omega^{[\omega]}/[\omega]! \ . \tag{52}$$

Consequently, when the alternating series for cos ω is summed, this initial build up in size of the terms before the final convergence, results in severe cancellation if L is large and produces a loss of roughly s significant figures, where s is the exponent of L. For example if $\omega = 10$, L is equal to $0.27557\ldots \times 10^4$ and cos 10 is obtained to be -0.8390715112 using 11 figure arithmetic. This is correct to only $(11 - 4) = 7$ significant figures when compared with the accurate value -0.8390715291. The value $\omega = 10$ is, of course, rather large to use in a power series approach and more realistic values produce smaller cancellation effects. Thus, for $\omega = 5$ only two significant figures are lost and for values less than four there is scarcely any diminution in accuracy.

The complementary effect is observed for the original ω^{-1} series (27) and (28) when the inverse ratio (51) is considered. Ultimately this ratio will become very large for a given ω if n is allowed to increase indefinitely and total instability would then arise. However, in practice, the value of n will be restricted by the user and examination of the ratio (51) shows that, for a given $\omega > 1$, no serious build up in magnitude occurs until n reaches values well beyond $[2\omega]$. (It will be recalled from the discussion leading to equation (52) that, as n increases, the ratio (51) actually decreases to a minimum at $n = [\omega]$ before starting to increase). The situation is clearly best for large ω when it is possible to tolerate large values of n before instability arises.

It appears, therefore, that the main ω^{-1} finite series in (27) and (28) will be stable if n is restricted to values less than a critical value, n_c, which is given by

$$n_c = [2\omega] \ . \tag{53}$$

In practice, this is found to be much too stringent and it is possible to replace it by a relation of the form

$$n_c = [2\omega] + \tau \tag{54}$$

where values of τ as large as $\tau = 10$ are tolerable, particularly for large ω. Note that similar practical limits have been proposed by Bakhvalov and Vasil'eva in their implementation of Miller's algorithm.

For values of ω which are less than 1, the ratio (51) increases monotonically with n and the resulting series are completely unstable. However, the alternative series (47) and (48) are then available and are stable for all n.

In practice, it is therefore recommended that for large ω, say $\omega > 4$, the basic ω^{-1} series (27) and (28) should be utilised, bearing in mind the restrictions implied by (54). When ω is smaller, a switch should then be made to the alternative ω series (47) and (48). This point is elaborated in the discussion of the numerical examples presented in the next section.

A further effect arises in observing that in the limit as $\omega \to 0$, the value $2/(r + 1)$ is obtained from series (47) and that the corresponding summation in equation (42) becomes

$$\sum_{r=0}^{i} 2D_{i,r}/(r+1) \quad (55)$$

where the summation extends over even values of r and i is also even in this, the symmetrical cosine case. The exact value of this summation is given by integrating $T_i(x)$ (i even) and the result is

$$\sum_{r=0}^{i} 2D_{i,r}/(r+1) = -2/(i^2-1) \ (i \text{ even}) \quad (55a)$$

When this expression is used in quadrature formula (14), the result is, of course, the Clenshaw–Curtis prescription for the integral

$$\int_{-1}^{1} f(x)\, dx \ .$$

This is compared with the Bakhvalov and Vasil'eva approach which in the limit as $\omega \to 0$ reproduces the Gauss–Legendre formula for this integral.

However, if the numerical evaluation of summation (55) or, indeed, the more general series

$$B_i(\omega) = \sum_{r=0}^{i} D_{i,r} \int_{-1}^{1} x^r \begin{matrix}\cos\\ \sin\end{matrix}\, \omega x\, dx \quad (56)$$

is attempted directly by the integration routine described here, serious cancellation effects are observed when i is large. The cancellation is due to the alternating signs and varying magnitudes of the Chebyshev coefficients (e.g. $D_{i,0} = \pm 1$; $D_{i,i} = 2^{i-1}$). Since the magnitudes of the integrals fall off with increasing ω, the instability effect is therefore most pronounced for small ω. A rough measure of this instability is given as $\omega \to 0$ by

$$i^2 \,.\, 2^i \quad (57)$$

and the number of figures lost by cancellation in the r series (56) is of the order of the exponent of this quantity. Thus, at $i = 12$, about five figures are lost in evaluating $B_i(0)$ whilst at $i = 20$ about eight figures are lost. A rough guide to the number of figures lost is provided by the expression

$$0.3i + 2 \ . \quad (58)$$

For large values of ω, this accuracy loss will be reduced roughly by the exponent of ω.

At first sight, it appears that this is a very serious defect in the method, but it should be recalled that the actual values of the $B_i(\omega)$ are to be used in a quadrature formula of the form

$$\int_{-1}^{1} f(x) \begin{matrix}\cos\\ \sin\end{matrix}\, \omega x\, dx = \sum_{i=0}^{n}{}'' a_i B_i(\omega) \quad (59)$$

in conjunction with the coefficients, a_i, of the Chebyshev series for $f(x)$. Consequently, if $f(x)$ is reasonably smooth, so that accurate fitting is possible for fairly small values of n and the a_i coefficients ($i = 0, 1, 2, \ldots\ n$) fall off rapidly with increasing i, then very accurate values of the integral (59) are obtained. Convergence is aided by the fall off of $B_i(\omega)$ with increasing i. This is particularly true in the case of small ω where cancellation in series (56) is at its worst, since, in this instance, B_i falls off most rapidly (approximately as $1/i^2$).

In practice, because of this effect, it has been found possible to proceed to values of n in formula (59) which are much larger than might be suspected from the restriction (58). This will be illustrated by the examples described in the next section. Even in the case of a badly-behaved function, where it was necessary to use n values around 50 to achieve a modest fit, the contributions from the smaller values of i were substantial. These could be calculated accurately and resulted in reasonable values for the integral in this extreme case.

However, if the function $f(x)$ is such that a large 'tail' exists in its Chebyshev expansion so that the contribution from the a_n end of the series is still large compared with the a_0 end, then errors could occur. An even worse situation would arise for the class of functions which are expressible only in the form

$$f(x) \simeq \sum_{i=N}^{N+n} a_i T_i(x) \quad (60)$$

where N is large. In fact, the integral

$$\int_{-1}^{1} T_N(x) \cos \omega x\, dx \quad (61)$$

itself, corresponding to $a_N = 1$ and $a_i = 0$ ($i \neq N$) provides an extreme example. The accuracy loss, according to (58), would be roughly $(0.3N + 2)$ figures, less a large ω contribution of about $\log(1 + \omega)$ figures. It would be necessary, in such examples, to use double precision (or even, in extreme cases, $N \sim 50$ in (61), multiple precision) arithmetic to carry out the r summation in (56).

In practice, as mentioned above, the functions $f(x)$, arising in most applications are sufficiently smooth for the a_i terms for small i to dominate the series and the errors resulting from the large i instability are, therefore, insignificant, in these cases. It is necessary, of course, to take certain practical precautions in using the algorithms and these were adopted in the treatment of the examples in the next section. Thus, in conducting convergence tests on a given integral with increasing n, it is suggested that the stability at large n should be checked by

Table 1 Absolute errors in the numerical evaluation of

$$\int_0^1 e^x \cos px\, dx$$

Order n	p = 1	10	100	1,000	10,000
1	1·2 (−1)	5·8 (−4)	1·6 (−4)	1·3 (−6)	2·3 (−9)
2	6·1 (−4)	1·7 (−3)	1·9 (−6)	3·5 (−8)	2·7 (−9)
3	1·4 (−4)	1·0 (−4)	2·4 (−6)	2·0 (−8)	2·0 (−11)
4	6·9 (−7)	1·9 (−5)	1·2 (−8)	2·8 (−10)	2·0 (−11)
5	2·6 (−8)	2·2 (−7)	8·8 (−9)	7·0 (−11)	1·0 (−11)
6	1·5 (−10)	2·1 (−8)	4·0 (−11)	1·0 (−11)	1·0 (−11)
7	5·0 (−11)	1·1 (−10)	1·0 (−11)	1·0 (−11)	exact
8	exact	1·0 (−11)	1·0 (−11)	1·0 (−11)	exact
9	exact	2·0 (−11)	1·0 (−11)	1·0 (−11)	exact
10	exact	1·0 (−11)	1·0 (−11)	1·0 (−11)	exact
11	exact	1·0 (−11)	1·0 (−11)	1·0 (−11)	exact
12	exact	1·0 (−11)	1·0 (−11)	exact	exact
exact	1·37802461355	−0·17889960288	−0·01362867977	0·00224821809	−0·00008311049

proceeding beyond the point at which convergence of the successive values of the integral has been established. Again, for the reasonably well behaved functions treated, good results were obtained for values of n less than 20 using single precision arithmetic. However, the computations were then repeated in double precision in order to check the results. This procedure is recommended in practice in selected instances. Double precision arithmetic was utilised in series (56) when values of n in excess of 20 were employed.

4. Computational procedure and numerical applications

The quadrature rules (42) and (45) are generated for any order. It is noted that some of the basic integrals of the form (27) and (28) are not required, as the ith order Chebyshev polynomial involves only $[(i + 2)/2]$ non-zero coefficients. Efficiency is achieved by using the following representations for the sums, where the non-zero Chebyshev coefficients are declared by the array $[1:(i + 2) \div 2]$ *real* d.

$$S1 = \sum_{r=0}^{R-1} d_{i,r+1} \int_{-1}^{1} t^{2r} \cos \omega t \, dt \quad (i \text{ even}) \tag{62}$$

$$S2 = \sum_{r=1}^{R} d_{i,r} \int_{-1}^{1} t^{2r-1} \sin \omega t \, dt \quad (i \text{ odd}) \tag{63}$$

where $R = [(i + 2)/2]$, the non-zero coefficients of the $\{D\}$ being denoted by d. Also, the values of the basic integrals are stored for all the i's considered and then used in the procedure which evaluates I_c or I_s, thus resulting in computational economy.

Furthermore, the number of cosines required in formula (43) has been minimised by taking symmetry into account.

Again, it will be noticed that equations (27) and (28) involve only *two* independent trigonometric functions, namely $\cos \omega$ and $\sin \omega$.

An ALGOL 68 version of the algorithm is presented in the Appendix and incorporates the switch from the ω to the ω^{-1} series at $\omega = 4$. Double precision arithmetic is recommended for values of n much beyond $n = 20$.

The algorithm is applied first of all to the integral considered in the earlier work, namely,

$$\int_0^1 e^x \cos px \, dx = [e(\cos p + p \sin p) - 1](p^2 + 1)^{-1} \tag{64}$$

The absolute errors (defined by $|\text{exact value} - \text{computed value}|$) in the numerical evaluation of the above integral are presented in **Table 1** for $p = 10^i$, $i = 0\,(1)\,4$. The notation $a(-m)$ is used to denote $a \times 10^{-m}$. The calculations were carried out in single precision arithmetic (about 11 figures) to start with and it is seen that machine accuracy is rapidly approached as the order, n, of the formula is increased, particularly for the larger values of p. The function $f(x) = \exp(x)$ is so smooth on $[0, 1]$ that accurate fitting is possible for relatively small values of n (say 8 or 9) and excellent results are obtained as a consequence of the good behaviour of equations (27) and (28). The stability of the ω^{-1} series was tested by extending the order well beyond the limits where the successive values of the integral had converged. Stability was observed for $p \geqslant 10$ for values of n up to at least 25, thus providing a test of the robustness of the algorithm.

In the case $p = 1$, corresponding to $\omega = \frac{1}{2}$, the basic ω^{-1} series (27) and (28) exhibited instability for values of n beyond $n = 12$. Thus, although convergence to the exact result was observed at about $n = 8$, the calculated values of the integral began to diverge from the exact at about $n = 12$. Clearly, this was a case for a switch to be made to the alternative ω series (47) and (48) and it was confirmed that stable results were then obtained for values up to at least 27.

The results presented show considerable improvement over the Filon-type quadrature prescriptions of the earlier work. It is noticed that in the lowest order cases $n = 1$ and $n = 2$, the two algorithms become, in fact, identical. The reason is that the Clenshaw and Curtis abscissae

$$t_j = \cos \frac{\pi j}{n} \qquad j = 0\,(1)\,n \tag{65}$$

which were used in equation (40) degenerate into the Newton–Cotes equally spaced abscissae

$$t_j = \frac{2j}{n} - 1 \tag{66}$$

in the cases $n = 1$ and $n = 2$. This degeneracy does not, of course, occur for $n \geqslant 3$ and considerable improvement in accuracy is obtained in these cases over the earlier calculations.

The present calculations were repeated using double precision arithmetic for checking purposes, one of the main objects being the removal of the inaccuracies associated with the evaluation of $\cos p$ and $\sin p$ when p is very large. For instance, subtraction of large multiples of π from the argument may result in the loss of about four figures in accuracy when $p = 10^4$, when the standard subroutines are employed. The double precision calculations yielded greater accuracy and confirmed the validity of the single precision results.

It is also of interest to use quadrature formula (45) for the test integrals of Piessens and Poleunis. For the purpose of illustration the integral

$$\int_0^{2\pi} x \cos x \sin px \, dx = \begin{cases} -2\pi p(p^2 - 1)^{-1} & (p = 2, 3, 4, \ldots) \\ -\pi/2 & (p = 1) \end{cases} \tag{67}$$

is considered. Numerical results are depicted in **Table 2** for $p = 1, 2, 4, 16, 64$ and 256. To facilitate direct comparison, double precision calculations were carried out. The order of the formula used was increased until the errors were less than those obtained by Piessens and Poleunis for 30 function evaluations. This accuracy was achieved for all values of p from about 19 or 20 function evaluations, thus representing an improvement over the earlier calculations. This is due, presumably, to a

Table 2 Comparison of the absolute errors in the numerical evaluation of

$$\int_0^{2\pi} x \cos x \sin px \, dx \,.$$

		Piessens and Poleunis		*Present method*	
p	*Exact*	*Absolute error*	*No. of function evaluations*	*Absolute error*	*No. of function evaluations*
1	−1·5707963267948966	5 (−15)	30	4 (−16)	19
2	−4·1887902047863910	9 (−15)	30	6 (−16)	19
4	−1·6755160819145564	3 (−15)	30	1 (−15)	20
16	−0·3942390780975427	1 (−14)	30	5 (−15)	20
64	−0·0981987447275930	3 (−15)	30	2 (−16)	20
256	−0·0245440671189132	1 (−15)	30	2 (−16)	19

Table 3 Comparison of the absolute errors in the numerical evaluation of

$$\int_0^{2\pi} x \cos x \sin px \, dx$$

using formulae (6) (open) and (31) (closed)

		n (order)				
p	*formula*	2	10	15	17	22
1	closed	>1	6·1 (−7)	2·4 (−13)	1·0 (−15)	2·2 (−18)
	open	>1	6·1 (−7)	3·3 (−13)	1·7 (−15)	2·2 (−18)
2	closed	1·0	3·0 (−5)	4·4 (−13)	2·4 (−15)	2·0 (−19)
	open	1·3	4·3 (−5)	7·1 (−13)	3·4 (−15)	2·0 (−19)
16	closed	1·5 (−3)	2·8 (−7)	1·9 (−11)	2·9 (−13)	1·4 (−17)
	open	3·6 (−2)	3·5 (−7)	1·2 (−10)	6·5 (−13)	2·7 (−17)
64	closed	2·4 (−5)	1·2 (−7)	2·3 (−12)	2·5 (−14)	8·0 (−19)
	open	8·6 (−3)	1·6 (−6)	5·3 (−12)	1·7 (−14)	1·4 (−18)
256	closed	3·7 (−7)	1·9 (−9)	3·8 (−14)	4·0 (−16)	1·9 (−20)
	open	2·1 (−3)	4·6 (−7)	2·7 (−12)	1·1 (−14)	3·5 (−19)

Table 4 Absolute errors in the numerical evaluation of

$$\int_{-1}^{1} \cos \pi u x^2 \cos \pi q x \, dx$$

			n		
u	*q*	*exact*	9	15	22
	$\frac{5}{4}$	−0·25816237030406	2·3 (−8)	1·5 (−13)	exact*
$\frac{1}{4}$	$\frac{41}{4}$	0·02966470953267	1·4 (−7)	5·7 (−12)	exact
	$\frac{451}{4}$	0·00283575769375	1·1 (−9)	8·9 (−14)	exact
			n		
u	*q*	*exact*	34	40	47
	$\frac{5}{4}$	0·38215576878521	1·7 (−8)	4·5 (−12)	9·1 (−13)
$\frac{23}{4}$	$\frac{41}{4}$	0·09736925629823	1·6 (−5)	2·0 (−8)	6·0 (−12)
	$\frac{451}{4}$	0·00256072719178	8·4 (−8)	1·6 (−10)	2·9 (−12)
			n		
u	*q*	*exact*	34	40	47
	$\frac{5}{4}$	0·24111868127101	4·3 (−5)	8·0 (−6)	1·0 (−7)
$\frac{47}{4}$	$\frac{41}{4}$	0·26746038313496	2·5 (−2)	6·2 (−4)	1·4 (−8)
	$\frac{451}{4}$	0·00233286903630	2·8 (−4)	1·4 (−4)	1·6 (−5)

*indicates accuracy in excess of 16 digits.

decrease in round-off errors generated by the present algorithm compared with the earlier Bessel-function series prescription for the evaluation of the basic integrals. It will also be noticed that the values of ω are $\omega = p\pi$ here and that these are large enough for the stability criterion (54) to be applicable for the values of n used to fit $f(x) = x \cos x$ on $[0, 2\pi]$. The stability of the algorithms was again tested by proceeding to larger values of n and it was possible to go to $n = 25$ even for $p = 1$ and still use the ω^{-1} series.

Numerical tests were also carried out for this integral using the alternative 'open' or Gaussian based Chebyshev zeros of expression (7) as utilised by Piessens and Poleunis. The results are compared with those of the present algorithm which use the Clenshaw–Curtis or 'closed' abscissae (32) and are shown in **Table 3** for increasing n. It will be observed that the closed formula is converging more rapidly particularly for large ω, although for large values of n the accuracy obtained by both methods is substantially the same. In this case, it will be noticed that the adaptive nature of the closed formulae could be taken into account here with advantage, to reduce the number of

Table 5 Absolute errors in the numerical evaluation of

$$\int_{-1}^{1} \cos \pi u x^2 \, dx \cdot$$

		n		
u	exact	34	40	47
$\frac{47}{4}$	0·186880300	6·6 (−5)	1·2 (−5)	4·4 (−7)

Table 6 Absolute errors in the numerical evaluation of

$$\int_{-1}^{1} \cos \pi u x^2 \cos \pi q x \, dx$$

by integrating over separate cycles of $\cos \pi u x^2$.

		No. of function evaluations			
u	q	79	97	121	145
	$\frac{5}{4}$	2·1 (−9)	3·4 (−11)	5·6 (−14)	1·8 (−16)
$\frac{47}{4}$	$\frac{41}{4}$	7·9 (−9)	1·1 (−10)	5·9 (−14)	8·2 (−16)
	$\frac{451}{4}$	9·6 (−8)	1·3 (−9)	4·9 (−12)	2·9 (−14)

function evaluations required. This is not true for the case of the open formula.

Finally, a much more stringent test of the present algorithm is carried out by considering the test integral of Bakhvalov and Vasil'eva which involves the badly-behaved function $f(x) = \cos \pi u x^2$. The integral is denoted by

$$I(u, q) = \int_{-1}^{1} \cos \pi u x^2 \cos \pi q x \, dx \qquad (68)$$

and has the exact value

$$I(u, q) = \frac{1}{\sqrt{2u}} \{\cos A[C(B_1) + C(B_2)] + \sin A[S(B_1) + S(B_2)]\} \qquad (69)$$

where

$$A = \pi q^2/(4u)$$

$$B_1 = (2/u)^{\frac{1}{2}} (u + q/2)$$

and

$$B_2 = (2/u)^{\frac{1}{2}} (u - q/2) \qquad (70)$$

and $C(z)$ and $S(z)$ are the Fresnel integrals. (Abramowitz and Stegun, 1965, p. 304).

Bakhvalov and Vasil'eva have considered a set of 11 values of q ranging from 5/4 to 451/4, coupled with a set of 14 values of u ranging from 1/4 to 45/4 and have tabulated the relative and absolute errors, quoting the maximum errors obtained over the set $\{q\}$. They have also repeated the exercise using a Newton–Cotes type formula with $n = 4$ with up to 90 sub-divisions for comparison purposes. Here, attention is confined to the extreme values of u and q together with one intermediate value in each case to give a smaller, though representative, set of calculations. Thus, the u and q values are taken to be

$$u = \frac{1}{4}, \frac{23}{4}, \frac{47}{4} \qquad (71)$$

and

$$q = \frac{5}{4}, \frac{41}{4}, \frac{451}{4} \qquad (72)$$

the corresponding ω values being given by $q\pi$.

The results obtained for various values of n are shown in **Table 4** in which the absolute errors are presented. It will be seen that when $u = 1/4$ the function $f(x) = \cos \pi u x^2$ is very well-behaved and hence the order of the formula required is small, reasonable results being obtained even for $n = 9$. Consequently, since the lowest ω value is $5\pi/4$ which is nearly 4, the ω^{-1} series may be used confidently here.

However, in the cases $u = 23/4$ and $u = 47/4$ the function $f(x)$ possesses 12 and 24 zeros respectively on the range $[-1, 1]$. Hence it will be necessary to use a high order formula particularly in the latter case. The ω^{-1} series should be stable here for the values $q = 41/4$ and $q = 451/4$ for which the corresponding values of $[2\omega]$ as required by the stability criterion (53) are 64 and 708 respectively. This is confirmed by the entries in Table 4 where accurate results are obtainable in most instances although values of n as large as $n = 47$ are required to fit $f(x)$. In the worst case, $u = 47/4$ and $q = 451/4$, convergence is very slow, due mainly to the badly-behaved nature of the function. Some improvement was observed on extending the calculation to order $n = 55$ where the absolute error was found to be 5.5(−7). However, such values of n are extreme both from the point of view of the stability criterion (58) and also economically, since it is desirable to produce an accurate answer with a minimal number of function evaluations. This particular case was therefore also treated by special techniques as described below.

To return to Table 4 for $q = 5/4$ the value of $[2\omega]$ is only about eight and, clearly, the ω^{-1} series will be completely unstable long before a large enough n value is attained to fit the function accurately. It follows, therefore, that the alternative ω series (47) and (48) must be used here. This is again borne out by the results obtained. The accuracy attained over the u and q ranges is comparable with the results quoted by Bakhvalov and Vasil'eva. Greater accuracy is apparently obtained by the present algorithm in some instances, but it should be pointed out that the present calculation has employed larger orders than the maximum ($n = 36$) used in the earlier work and that double precision arithmetic was necessary in the evaluation of the series (62) and (63) for $n > 20$. In fact it is noted that values of n around 36 are necessary even to begin to fit the function $\cos \pi u x^2$, when u is large. Note also, that good results are obtainable for various u values for both the large q and the small q values.

As a further illustration of the behaviour of the present algorithm for small values of ω, the limiting case $\omega \to 0$ is treated by taking $q = 0$ in the worst behaved instance of the function $f(x)$, namely $u = 47/4$. This is the case where the cancellation effects in series (56) are at their worst and should be a stringent practical test of the stability of the algorithm, since very high orders are required to fit $f(x)$. The results are given in **Table 5** and show that, even though instabilities of this type are present, the contribution from the smaller values of i are relatively large and produce reasonable values of the integral.

The badly-behaved nature of the function $f(x) = \cos \pi u x^2$ when u is large (say $u = 47/4$) prompted also an investigation into the special techniques suggested in Section 2. Thus, the range was sub-divided between the complete cycles of $\cos \pi u x^2$, starting at $x = 0$ and integrating over each cycle separately. It was hoped to reduce the cancellation effects arising on sub-division by this device. Thus, integration is carried out between the points $x = (2m/u)^{\frac{1}{2}}$ where $m = 0, 1\ 2\ \ldots, m_0$ and finally between $(2m_0/u)^{\frac{1}{2}}$ and $x = 1$. The maximum value of m_0 is given by $[u/2]$. The results are shown in **Table 6** and demonstrate high accuracy for an economical number of function evaluations in this, the most badly-behaved case of $f(x)$. The number of sub-divisions used here is $5 + 1 = 6$ and the order of formula employed in each cycle varied from $n = 13$ to $n = 24$. Accurate single precision results are therefore obtainable by this method for an economical number (around 100) function evaluations using maximum order of around 20.

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Appendix

An ALGOL 68 version of the algorithm is presented. Although this program does not use any advanced features of ALGOL 68 (and so could have been programmed in other languages such as FORTRAN or ALGOL 60) the language chosen enables the algorithm to be presented in a neat and efficient form.

```
'PROC' QUADRULE=('PROC'('REAL')'REAL'F,'REAL'A,B,P,'INT'N,
                 'BOOL'TYPE)'REAL':
'BEGIN'
'C' THIS PROCEDURE EVALUATES THE INTEGRAL OF F(X) SIN OR COS P*X
ON [A,B] USING N-TH ORDER QUADRATURE FORMULA OF CLENSHAW-CURTIS
TYPE, NOTATION AS IN THE TEXT, THE BOOLEAN TYPE IS TRUE IF
SIN PX IS THE WEIGHT FUNCTION AND FALSE IF COS PX
IS THE WEIGHT. 'C'

   [0:N]'REF'[]'REAL'D; [0:N]'REAL'CAPF,TCWT,TSWT;
   'REAL'INTEGRAL,S1,S2,W1,WB,CS1,CS2,AI,CAPK,PART1,
        PART2,OMEGA,X1,X2,SINE,COSE;
   'INT'R1,L,K,UPB,N2,J1,NI1;
   X1←(B+A)/2.0; X2←(B-A)/2.0; CAPK←P*X1; OMEGA←P*X2;
   SINE←SIN(OMEGA); COSE←COS(OMEGA);'BOOL'ODD;

  'PROC'INTEG=('INT'N,'REAL'OMEGA,'REF''REAL'SI)'REAL':
  'BEGIN'
  'C' THIS IS A PROCEDURE TO EVALUATE THE INTEGRAL OF
  X↑N SIN OR COS OMEGA*X ON [-1,1] USING EQUATIONS (4.2.1) AND
  (4.2.2), THE INTEGRAL FOR COSINE WEIGHT FUNCTION IS
  DELIVERED WHILE THE INTEGRAL FOR SINE WEIGHT FUNCTION IS
  ASSIGNED TO SI . 'C'
    'REAL'S1,S2,P1,P2,W1,W2,T1,T2,P3;P1←P2←1.0;S1←S2←0.0;P3←1.0;
    'INT'IB,NN; [0:N]'REAL'Y;
    'BOOL'SW←'TRUE',SV←'ODD'N;
    'REAL' EPS←1.0&-60;
     'IF'N=0'THEN'SI←0.0; 2.0*SINE/OMEGA
            'ELSE'
            P3←1.0/OMEGA; Y[0]←P3;
            'FOR'I1'TO'N'WHILE'P3>EPS'DO'
            (P3'TIMES'(N-I1+1)/OMEGA;Y[I1]←P3;NN←I1);
            IB←NN+1-(NN'/'4)*4; W1←SINE; W2←COSE; T1←P3;
            ('ODD'(N-NN)!SW←'FALSE'; T2←-T1|SW←'TRUE';T2←T1);
                   'FOR'I1'FROM'NN'BY'-1'TO'0'DO'
                   'BEGIN'
                       (I1#NN|T1←Y[I1];
                             (SW!T2←-T1;SW←'FALSE'
                             |T2←T1 ; SW←'TRUE'));
                   'IF'SV'THEN'
                       'CASE'IB'IN'(P1←W2;P2←W2;IB←4),
                                   (P1←-W1;P2←W1;IB←1),
                                   (P1←-W2;P2←-W2;IB←2),
                                   (P1←W1;P2←-W1;IB←3)
                       'ESAC'
                          'ELSE'
                       'CASE'IB'IN'
                                   (P1←W1;P2←-W1;IB←4),
                                   (P1←W2;P2←W2; IB←1),
                                   (P1←-W1;P2←W1;IB←2),
                                   (P1←-W2;P2←-W2;IB←3)
                       'ESAC'
                   'FI';
                   S1'PLUS'T1*P1;S2'PLUS'T2*P2
                'END';

                SI←(SV!S2-S1!0.0);
                (SV|0.0|S1-S2)
   'FI'
'END';

 'PROC'INTEG2=('INT'N,'REAL'OMEGA,'REF''REAL'SI)'REAL':
 'BEGIN'
 'C' THE VALUE OF THE INTEGRAL (4.2.21) IS DELIVERED AND
 THAT OF (4.2.22) IS ASSIGNED TO SI . 'C'
    'REAL'S,   SK,T1,T2,W2←OMEGA*OMEGA;
 'INT'M←18; 'BOOL'BOOL←'ODD'N;
    SI←(BOOL!T1←OMEGA/(N+2); S←T1;
      'FOR'I'TO'M'WHILE'('ABS'(T1/S)>1.0&-22)'DO'
    'BEGIN'
      T1'TIMES'-W2*(2*I+N)/(2*I+N+2)/(2*I)/(2*I+1);S'PLUS'T1
    'END';
      S'TIMES'2.0|0.0);
    SK←(BOOL!0.0|T2←1.0/(N+1);S←T2;
    'FOR'I'TO'M'WHILE'('ABS'(T2/S)>1.0&-22)'DO'
    'BEGIN'
     T2'TIMES'-W2*(2*I+N-1)/(2*I+N+1)/(2*I)/(2*I-1);S'PLUS'T2
    'END';
    S'TIMES'2.0);
    SK
 'END';

'PROC'CHEBCOEF=('INT'R)'REF'[ ] 'REAL':
'BEGIN'
'C' THIS IS A PROCEDURE TO CALCULATE THE NON-ZERO COEFFICIENTS
OF THE R-TH ORDER CHEBYSHEV POLYNOMIAL USING THE
RECURRENCE RELATIONSHIP (4.2.4) . 'C'
   'INT'R1←(R+2)'/'2;
   [1:R1]'REAL'DB,DD;
    DD[1]←DB[1]←1;

    'IF'R>1'THEN'
          'FOR'I'FROM'2'TO'R1'DO'
          'FOR'J'FROM'I'BY'-1'TO'1'DO'
          'BEGIN'
             DD[J]←(J=I|2*DB[J-1]|:J=1|-DD[1]
                                  |2*DB[J-1]-DD[J]);
             DB[J]←(J=I|2*DD[I]|2*DD[J]-DB[J])
          'END';
   (R#(R'/'2)*2!'FOR'I'TO'R1'DO'DD[I]←DB[I])
   'FI'; DD
'END';

 'PROC'CAPI=([  ]'REAL'CAPF)'REAL':
 'BEGIN'
   'INT'N2←N'/'2; [0:N]'REAL'CJI;
   INTEGRAL←0.0;
   'C' THE INTEGRAL DEFINED BY (4.2.16) AND (4.2.19) 'C'
   PART1←PART2←0.0;
   'FOR'I'FROM'0'TO'N'DO'
     'BEGIN'
      S1←S2←0.0; R1←(I+2)'/'2; ODD←'ODD'I; AI←0.0;
      'C' S1 AND S2 REPRESENT THE WEIGHTED INTEGRALS OF THE
      CHEBYSHEV POLYNOMIALS DEFINED BY (4.3.1),  (4.3.2)
      FOR USE IN FORMULAE (4.2.16) AND (4.2.19) . 'C'
      'FOR' R 'TO' R1 'DO'
        'BEGIN'
          L←(ODD!2*R-1|2*(R-1)); W1←D[I][R];
          S1'PLUS'W1*TCWT[L]; S2'PLUS'W1*TSWT[L]
        'END';
      CJI[0]←0.5;
      'FOR'J'TO'N2'DO' CJI[J]←(I=0!1.0!COS(PI*J*I/N));

      'C' FOR USE OF THE OPEN FORMULA THE PRECEDING TWO
       LINES ARE REPLACED BY
      'FOR' J'FROM'0'TO'N2'DO'
      CJI[J]←(I=0|1.0!COS(PI*I*(2.0*J+1.0)/(2.0*N+2.0))); 'C'

      'FOR'J'FROM'0'TO'N2'DO' AI'PLUS'CAPF[J]*CJI[J];
      (I=(I'/'2)*2! 'FOR'J'FROM'N2+1'TO'N'DO'
                    AI 'PLUS'CAPF[J]*CJI[N-J]
                  ! 'FOR'J'FROM'N2+1'TO'N'DO'
                    AI'PLUS'CAPF[J]*(-CJI[N-J]));
      AI 'DIV'(I=0'OR'I=N|N|N/2.0);

      'C' FOR USE OF THE OPEN FORMULA (4.1.21) THE PRECEDING
      LINE IS REPLACED BY
      AI'DIV'(I=0|N+1|(N+1)/2.0); 'C'

      PART1'PLUS'AI*S1; PART2'PLUS'AI*S2
    'END';
```

```
    INTEGRAL 'PLUS' 'IF' TYPE 'THEN'
                    'C' THE WEIGHT FUNCTION IS SIN PX 'C'
                    PART1*CS2+PART2*CS1
                    'ELSE'
                    'C' THE WEIGHT FUNCTION IS COS PX 'C'
                    PART1*CS1-PART2*CS2
                    'FI';
    INTEGRAL 'TIMES'X2
'END';


N2←N'/'2; [0:N2]'REAL'XS;
K←0;
L2: UPB←(K+2)'/'2;
D[K]←'LOC'[1:UPB]'REAL';
D[K][1:UPB]←CHEBCOEF(K);
((K'PLUS'1)<=N! 'GOTO' L2);
'C' THE PRECEDING PART GENERATES A TRIANGULAR ARRAY
TO RETAIN ALL THE NON-ZERO COEFFICIENTS OF CHEBYSHEV
POLYNOMIALS OF ORDER 0,1,2,...,N . 'C'
```

```
  'C' XS IS (B-A)T/2 AS IN (4.2.13), ONLY HALF OF THE
  NUMBER OF XS ARE CALCULATED .'C'
  XS[0]←X2; 'FOR'S'TO'N2'DO' XS[S]←X2*COS(PI*S/N);

  'C' FOR USE OF THE OPEN FORMULA (4.1.21) THE PRECEDING LINE
  IS REPLACED BY
  'FOR'S'FROM'0'TO'N2'DO'
  XS[S]←XI*COS(PI*(2.0*S+1.0)/(2.0*N+2.0)); 'C'

  (OMEGA>4.0!
  'FOR'I'FROM'0'TO'N'DO'(TCWT[I]←INTEG(I,OMEGA,WB); TSWT[I]←WB)
  !'FOR'I'FROM'0'TO'N'DO'(TCWT[I]←INTEG2(I,OMEGA,WB); TSWT[I]←WB) );
  'C' TCWT AND TSWT STORE THE VALUES OF INTEGRAL OF T↑I COS OR
  SIN OMEGA*T ON [-1,1] .  FOR OMEGA>4 PROCEDURE INTEG,
  OMEGA<4 INTEG2 IS CALLED . 'C'

  CS1←COS(CAPK); CS2←SIN(CAPK);
  'FOR'J'FROM'0'TO'N2'DO' CAPF[J]←F(X1+XS[J]);
  'FOR'J'FROM'N2+1'TO'N'DO' CAPF[J]←F(X1-XS[N-J]);

  INTEGRAL←CAPI(CAPF);
  INTEGRAL
'END';
```

References

ABRAMOWITZ, M., and STEGUN, I. A. (1965). *Handbook of Mathematical Functions*, New York: Dover Publication Inc.

ALAYLIOGLU, A., EVANS, G. A., and HYSLOP, J. (1975). Automatic generation of quadrature formulae for oscillatory integrals, *The Computer Journal*, Vol. 18, No. 2, pp. 173-176.

BAKHVALOV, N. S., and VASIL'EVA, L. G. (1968). Evaluation of the integrals of oscillating functions by interpolation at nodes of Gaussian quadratures, *Z. Vycisl. mat. i mat. Fiz.*, Vol. 8, pp. 175-181.

CLENDENIN, W. W. (1966). A method for numerical calculation of Fourier Integrals, *Numer. Math.*, Vol. 8, pp. 422-436.

CLENSHAW, C. W., and CURTIS, A. R. (1960). A method for numerical integration on an automatic computer, *Num. Math.*, Vol. 12, pp. 197-205.

DAVIS, P. J. (1963). *Interpolation and Approximation*, Blaisdell.

ELLIOTT, D. (1965). Truncation errors in two Chebyshev Series Approximations, *Math. Comp.*, Vol. 19, pp. 234-248.

FILON, L. N. G. (1928). On a quadrature formula for trigonometric integrals, *Proc. Roy. Soc. Edinburgh*, Vol. 49, pp. 38-47.

FLINN, E. A. (1960). A modification of Filon's method for numerical integration, *JACM*, Vol. 7, pp. 181-184.

FRASER, W., and WILSON, M. W. (1966). Remarks on the Clenshaw-Curtis quadrature scheme, *SIAM Review*, Vol. 8, No. 3, pp. 322-327.

GENTLEMAN, W. M. (1972). Implementing Clenshaw-Curtis quadrature, I Methodology and Experience, *CACM*, Vol. 15, pp. 337-342.

GRADSHTEYN, I. S., and RYZHIK, I. M. (1965). *Table of Integrals, Series and Products*, New York: Academic Press.

LITTLEWOOD, R. K., and ZAKIAN, V. (to be published). *J. Inst. Maths. Applics.*

O'HARA, H., and SMITH, F. J. (1968). Error estimation in the Clenshaw-Curtis quadrature formula, *The Computer Journal*, Vol. 11, pp. 213-219.

PATTERSON, T. N. L. (1974). Private communication.

PIESSENS, R., and POLEUNIS, F. (1971). A numerical method for the integration of oscillatory functions, *BIT*, Vol. 11, pp. 317-327.

Book review

Queueing Systems, Vol 1: Theory by Leonard Kleinrock, 1975; 417 pages. (*John Wiley*, £9·75).

This first volume of a two volume work is aimed at providing an acceptable basis, neither excessively theoretical nor sloppily intuitive, for its successor.

Volume 1 is divided into four parts: (*a*) preliminaries; (*b*) elementary queueing theory; (*c*) intermediate queueing theory; (*d*) advanced material. None the worse for a tendency towards avuncular jocularity and an occasional idiosyncracy (Thusly (sic) on page 17) this is a competent presentation of a range of topics in queueing theory. The bias is towards the computer applications promised for Vol. 2, which confers a distinctive character on the book. Indeed it contains much material and discussion which has probably not been collected together before. A balance has indeed been achieved in the level of treatment between the intensely pure mathematical and the over-simplified, and this should appeal to a wide audience of under graduates and some graduates. This reviewer would not choose it as a teaching companion, but it is assured a place on his reference shelf.

The second part of the book discusses the equilibrium theory of M/M systems with one or more servers, with and without finite storage capacity (waiting room), with and without a closed population of customers. Use is made of simple general results for state dependent demand and service of the 'λ_n and μ_n' kind. Attention is then devoted to generalisation with Erlangian service or demand, the familiar application to group arrival and bulk service being noted. Finally, there is discussion of networks of Markov queues.

Thie third part of the book incorporates first a treatment of M/G/1. This is acceptable except for the busy period where the author would have done better to describe immediately Prabhu's beautiful time domain analysis leading to joint probability and probability density function of number served and duration in time. There is no mention of output. We are then led to the multiserver G/M/m whose treatment is relatively cursory. In particular there is no discussion of busy period.

In the fourth part of Volume 1 we find a more extended discussion of Lindley's treatment of waiting time, of Kingmann's algebraic formalism, and of duality relations.

The volume is concluded with an Appendix on the Laplace transform and on generating functions (here called *z*-transforms, which is most irritating and rates a definite minus mark) and their use in the solution of differential-difference equations arising in *Queueing Theory*. A second appendix is an aide mémoire on probability.

The glossary of notation and summary list of 'important results' constitutes a definite plus mark.

Other features which merit plus marks are the following. There is a painstaking discussion of traffic intensity and utilisation factor very early. Little's formula is also introduced at a much earlier stage than is common. It is right to emphasise such widely applicable general results.

The reviewer awards negative marks for a poor selection of references, poorly displayed. It is more useful to collect a list at the end.

But the overall impression has a positive balance. Volume 1 and its promised successor deserve success.

B. W. CONOLLY (London)