Seek times for disc file processing: some results from probability theory

V. Siskind* and J. Rosenhead

London School of Economics, Houghton Street, London WC2A 2AE

This paper considers the duration of seek times for a skip-serially processed file on a moveable head disc pack. It is shown that under certain assumptions about the generation of requests for records, the distributions of both the number of records to be processed and the number of distinct cylinders to be accessed per run are binomial. Probabilistic arguments from the theory of order statistics are used to derive the distribution of the number of cylinders separating consecutive cylinders accessed. A general functional form is introduced for the relation between inter-cylinder distance and duration of seek time, and used to derive a general expression for the expectation of total seek time for a scan of the file. Results are computed for a numerical example. These show, first, the advantage of software which ensures that successive scans operate in opposite directions; and second, the danger of serious inaccuracy (especially at low hit-ratios) if simple approximations are used based on the value of the seek time function at the mean inter-cylinder distance.

(Received December 1975)

1. Introduction

In many data processing systems, files are stored on disc as secondary memory and blocks of data are called into main memory when particular records require processing. Once a request for a record is activated, there follows first, a *seek time* while the single read/write head is moved to the appropriate cylinder; second, a *rotational delay* while the disc revolves until the desired block reaches the head; and finally a *transfer time*, while the block is actually transferred to main memory (and perhaps updated and written back onto disc). Seek times can constitute a major, and sometimes a dominant, component of total computer runtime. When computer runtime is not dominated by other factors, mis-estimates of the seek time contribution may have serious economic implications, either at the system design stage or through operational inadequacies (Waters, 1975). In this paper we conduct a probabilistic analysis to estimate the average seek time for skip-serially processing requests for a particular file, which may be of use in estimating the total running time of programs, or in devising more efficient software. It will be demonstrated that simple approximations to seek time based on the average number of cylinders traversed by the head in a single movement can be quite inaccurate.

For definiteness, and in the interests of mathematical tractability, we have made a number of simplifying assumptions. We consider a situation where requests are for access to a single file, which is organised sequentially on a single disc pack, and there is no multiprogramming interference. Requests are batched and sorted for skip-serial processing, and the batch cannot be added to during a scan of the file. We neglect effects due to overflow or chaining. These assumptions correspond well with those made in other analyses of seek time (see, for example, Coffman *et al.*, 1972). While they limit the situations to which the results can be applied directly, the results should at least be suggestive for a wider range of problems. For simplicity we also assume that there is exactly one record per block of data, though the approach of this paper can easily be extended to the case of several records per block. Other assumptions, concerning the mode of generation of requests, are specified in Section 2.

Some elementary theory of seek times for disc systems is provided by Martin (1967, Chapter 28), while several papers in recent years (e.g. Nielsen, 1971; Hess, 1963; Lowe, 1968) have analysed other aspects of these or similar systems. Denning (1967) has addressed himself to the problem of seek times when the requested records are, effectively, sequentially ordered: he considers the effect of various strategies on the length of time a request spends in the file system—of which the mechanical positioning delay of the movable arm of the disc unit is a major component. He concludes that (for the somewhat imprecisely specified way in which incoming requests enter the queue) scanning the arm forward and back across the disc provides the 'best' overall service. These ideas are further developed by Coffman *et al* (1972) and Gotlieb and MacEwen (1973). The aims of these papers are not quite ours, and the approach is very different to the one to be presented here. Waters (1975) examines the validity of some common approximations to average seek time under a variety of circumstances. He incorporates very few of the probabilistic aspects discussed in this paper, but describes skip-serial file processing and its applications in greater detail. Waters' paper provides the practical background to many of the assumptions made in the present paper.

We commence in the next section with an explicit statement of the precise assumptions underlying the model to be studied. It is then shown how certain additional assumptions about the way in which record requests arrive lead to binomial distributions for both the number of records to be processed and the number of distinct cylinders to be accessed per run. To establish some of these results we make use of elements of the mathematical theory of occupancy (see Barton and David, 1962, Chapter 14). Many of the probabilistic concepts we employ, for instance the term 'at random', are set out in Barton and David's book.

When the number of cylinders to be accessed is given, their positions (ordinal numbers) constitute under the assumptions of the model an ordered sample drawn randomly and without replacement from the integers $\{1, 2, \ldots, K\}$, where K is the number of cylinders on a disc pack. The standard theory of order statistics (see e.g. Kendall and Stuart, 1963, Chapter 14) is easily modified to yield the probability distribution of the distance between successive cylinders accessed. If the head does not move between scans, we will also need the distribution of the distance between the final position of the head at the end of one scan and its starting position for the next. These topics are dealt with in the third section.

The time dimension is introduced in the fourth section. The function mapping distance travelled by the head onto time is a characteristic of the particular disc pack in use and is not normally linear over its entire range. The main quantity with which we are concerned, namely the total seek time, is the sum

*Now at: University of Queensland, Brisbane, Australia.

of $m - 1$ values of this function, where m is the number of cylinders accessed in a given scan, plus the value of the function for the initial, positioning, head movement. Its mean and variance are obtained by averaging, or squaring and averaging, with respect to the distributions of number of cylinders accessed and distances between them. A numerical example is given.

2. Basic assumptions; the number of requests and of cylinders

Regarding the file structure it is assumed that:

1. Each cylinder holds the same number, L, of records, and no record extends over more than one cylinder. The number of cylinders is K, so that the number of records is KL.

The model for request generation includes the assumptions that

2. The positions of records requested are statistically independent.
3. The hit-ratio, ρ, where applicable, is the same for each record; moreover
4. The process is memoryless in the sense that the probability of any record being requested remains constant regardless of when the record was last requested.

The number of records accumulated before a scan is initiated may be fixed or random. If the latter, the appropriate distributional form seems to us to be the binomial; this can be made plausible by considering three apparently different mechanisms by which batches of requests are formed; in fact a little thought shows all three to be versions of the same process.

Firstly assume that a batch is formed more or less instantaneously, as for instance when a customer sends in an order requesting a number of different products, each having its own record. Assumptions 2, 3 and 4 imply that the number of distinct records, say N, requested at any such epoch will be binomially distributed, i.e.

$$\Pr(N = n) = \binom{KL}{n} \rho^n (1 - \rho)^{KL-n} . \qquad (2.1)$$

Secondly, let requests arrive at random, waiting until, after a fixed time, T, the access run commences. At first sight this would seem to lead to a Poisson distribution if the arrival rate, λ, were constant. However, this would imply an infinite file; relying on Assumption 4 we allow records to be requested more than once during the accumulation interval. In Appendix 1 it is shown that this gives us (2.1) again with

$$\rho = 1 - e^{-\lambda T/KL} . \qquad (2.2)$$

As the final variant, we may allow the requests arriving at random to be for distinct records, but the rate at which they arrive to be proportional to the number of unrequested reeords in the file, i.e. to drop to $\lambda(KL - j)/KL$ after j requests have been received. This situation is now exactly equivalent to a simple death process (Cox and Miller, 1968, Chapter 4) in which each individual has, independently of the other, a chance $1 - e^{-\lambda T}$ of dying in the interval $[0, T]$; the number dying in the interval will clearly have once more the distribution (2.1) with parameter given by (2.2).

A reasonable modification might be to allow the scan to start before time T if a threshold number of requests had accumulated; we will assume that this value is large enough for this event to have small probability. We will in addition not concern ourselves with the case of fixed N—the approach used in this paper gives results which though computationally simple are analytically unwieldy.

When the number of distinct records requested has the binomial distribution (2.1), the number of cylinders on which these records lie, M, say, has itself a binomial distribution, i.e.

$$\Pr(M = m) = \binom{K}{m} \eta^m (1 - \eta)^{K-m} \qquad (2.3)$$

where

$$\eta = 1 - (1 - \rho)^L \qquad (2.4)$$

can be regarded as the 'cylinder hit-ratio'.

The proof is outlined in Appendix 2. It should be pointed out that (2.3) can be derived by direct argument from the assumptions, most easily in the case of the first mechanism—'instantaneous' batch generation—presented above.

3. Distance distributions

Let $Y_1 < Y_2 < \ldots < Y_m$ be the ordinal numbers (positions) of the m cylinders to be accessed (counting from an arbitrary edge of the disc). Then our assumptions ensure that the Y_j are order-statistics of a sample of m integers drawn randomly and without replacement from the set $\{1, 2, \ldots, K\}$. Define $D_r = Y_{r+1} - Y_r,\ r = 1, 2, \ldots, m - 1$.

Suppose that on the previous scan m' cylinders were accessed and that their positions, in order, were $\tilde{Y}_1, \ldots, \tilde{Y}_{m'}$. In the procedure we are considering the head moves either (a) always in one direction, starting at Y_1 or $\tilde{Y}_1$ and ending at Y_m or $\tilde{Y}_{m'}$, respectively, or (b) in alternate directions. At the commencement of the run under consideration the head, if undisturbed between runs, must travel in case (a) from $\tilde{Y}_{m'}$ to Y_1, whereas in the situation (b) it will pass either from $\tilde{Y}_{m'}$ to Y_m, or, probabilistically equivalent, from $\tilde{Y}_1$ to Y_1. The distribution of both $U = |\tilde{Y}_{m'} - Y_1|$ and $W = |\tilde{Y}_{m'} - Y_m|$ will be needed.

Relegating an outline of the very tedious derivations to Appendix 3, we present here only a few important results. For $r, s = 1, 2, \ldots, m - 1$,

(a) $\Pr(D_r = d|M = m) = m(K - d)^{(m-1)}/K^{(m)} =$

$$\binom{K-d}{m-1} \Big/ \binom{K}{m} ,\ d = 1, 2, \ldots, K - m + 1 \qquad (3.1)$$

where for non-negative integers,

$$H, n, H^{(n)} = H!/(H - n)!,\ n \le H ,$$
$$= 0, \qquad n > H .$$

Note that the right hand side of (3.1) is independent of r. Moreover, if one defines $Y_0 = 0$, $Y_{m+1} = K + 1$, then (see Appendix 3) D_0 and D_m have this distribution as well. Thus on average the Y_r divide the interval, $Y_{m+1} - Y_0 = K + 1$, into $m + 1$ equal segments, i.e. $E(D_r) = (K + 1)/(m + 1), r = 1, 2, \ldots, m - 1$, as can of course also be shown by multiplying (3.1) by d and summing.

(b) $\Pr(U = u|M = m) = mm'(1 - \tfrac{1}{2}\delta_{u,0})$

$$\times\{\sum_{j=u+1}^{K} (j - 1)^{(m'-1)} (K - j + u)^{(m-1)} + \sum_{j=1}^{K-u} (j - 1)^{(m'-1)} (K - j + u)^{(m-1)}\}/K^{(m)} K^{(m')} \qquad (3.2)$$

$\delta_{u,0}$ being the Kronecker delta. Unless m and m' are both quite small, the second term in (3.2), which is associated with the unlikely event, $Y_1 > \tilde{Y}_{m'}$, will be negligible. There is an analogous formula of much the same form for the probability distribution of W.

If the second moment of overall seek time per scan is to be computed, one will also need various joint distributions: that for D_r and D_s, $s \ne r$, is

(c) $\Pr(D_r = d_1, D_s = d_2|M = m) =$

$$m(m - 1)(K - d_1 - d_2)^{(m-2)}/K^{(m)} =$$

$$\binom{K - d_1 - d_2}{m-2} \Big/ \binom{K}{m},\ 2 \le d_1 + d_2 \le K - m + 2 . \qquad (3.3)$$

The joint distributions of D_r and U, and of D_r and W, are of a somewhat more complicated form than this; however, as pointed out in the next section, when one takes expectations of these expressions with respect to the distribution of m, a considerable simplification occurs. For one thing the summations in (3.2) become far more tractable.

Note that (3.1) has no relevant interpretation unless $m > 1$;

(3.2) requires also $m' > 1$, and (3.3), $m > 2$. It can be shown that these requirements do not affect the calculations in the next section: only zero terms are neglected. We have chosen not to add complexity by making the conditions on m explicit in any of the equations.

4. The time dimension

We can now introduce a function, $\tau(d)$, representing the time taken for the head to traverse d tracks; $\tau(0) = 0$ and $\tau(d) > 0$ for $d > 0$. If m cylinders are to be accessed on a given scan, and m' were accessed on the previous run, the overall seek time for that run is a random variable

$$S_m(D, V) = \sum_{r=1}^{m-1} \tau(D_r) + \tau(V) = T_m(D) + \tau(V), \text{ say,} \quad (4.1)$$

where V is the initial distance travelled by the head, i.e. either U or W. The average overall seek time may be written

$$E\{S_m(D, V)\} = E_m[(m - 1)E_D\{\tau(D)|m\}] + E_{m,m'}[E_V\{\tau(V)|m, m'\}]$$
$$= \sum_d \tau(d)\, E_m\{(m - 1) \Pr(D = d|m)\} + \sum_v \tau(v)\, E_{m,m'}\{\Pr(V = v|m, m')\} \ . \quad (4.2)$$

Subscripts to the expectation symbol denote that the function is being averaged only with respect to the designated variable(s). The expression for $E\, S_m^2(D, V)$ will involve also the expectations with respect to the distribution of m and m' (assumed independent and of the form (2.3)) of the joint distributions mentioned in the previous section.

As shown in Appendix IV,

$$E\{(m - 1) \Pr(D = d|m)\} = (K - d)\eta^2(1 - \eta)^{d-1},\quad d = 1, 2, \ldots, K - 1 \ . \quad (4.3)$$

where η is as defined in (2.4).

Similarly

$$E\{\Pr(U = u|m, m')\} = (1 - \delta_{u,0})(K - u)\eta^2\{(1 - \eta)^{K-u-1} + (1 - \eta)^{K+u-1}\},\ u = 0, 1, \ldots, K - 1 \ . \quad (4.4)$$

while

$$E\{\Pr(W = w|m, m')\} = (2 - \delta_{w,0})\, \eta(1 - \eta)^w \times \{1 - (1 - \eta)^{2K-2w}\}/(2 - \eta)\ ,\quad w = 0, 1, \ldots, K - 1 \ . \quad (4.5)$$

These three formulae are enough to evaluate (4.2) for any function $\tau(.)$. Second moments involve far lengthier expressions but introduce no new concepts.

5. Numerical example

To illustrate the application of the formulae, and to compare its results with a simpler approach, we will take $K = 200$ and use the continuous, piecewise-linear time function given by Abate *et al* (1968) as an approximation to that for the IBM2314 disc pack sketched in Martin (1967, p. 442), viz.

$$\begin{aligned} \tau(x) &= 0, & x &= 0 \\ &= 25 + 2x & 0 &< x \le 20, \\ &= (370 + x)/6 & 20 &< x \le 80, \\ &= 35 + 0{\cdot}5x, & 80 &< x \le 199. \end{aligned} \quad (5.1)$$

(Times in milliseconds.)

A further parameter needed is η, defined in (2.2). On average, $K\eta$ cylinders are accessed per run, and from (2.2), the mean number of records in a batch, $KL\rho$, will lie between $K\eta$ and $-K \ln(1 - \eta)$. In this example, we take $\eta = 0{\cdot}05$; $0{\cdot}1$; $0{\cdot}2$; $0{\cdot}4$; whence $E(m) = \bar{m}$, say, $= 10, 20, 40, 80$, respectively.

In the case of a piecewise linear function, τ, it is possible after much tedious algebra to exhibit the expectations occurring in (4.2) in a reduced but still rather cumbersome form. This is hardly practicable in the case of the second moments, so that all sums were evaluated directly by computer: the results are tabulated below.

A common practical procedure in such situations (see Waters, 1975) is to calculate the value of a required function at the mean(s) of the variable(s) entering into it, rather than to compute the expected value of the function with respect to these variables. Thus for given $\bar{m}$ let $\bar{d} = E(D|M = \bar{m})$, $\bar{u} = E(U|M = \bar{m})$, etc. Plausible approximations in the present case might then be

$$E\{T_m(D)\} \doteqdot (\bar{m} - 1)\tau(\bar{d}),\ E\{\tau(U)\} \doteqdot \tau(\bar{u}),\ E\{\tau(W)\} \doteqdot \tau(\bar{w}) \ . \quad (5.2)$$

These formulae are compared with the exact computations in **Table 1.** (Values for $E\{S_m(D, U)\}$ and $E\{S_m(D, W)\}$ can readily be obtained from the tabulated quantities.) Standard errors are presented in **Table 2.**

In most of the remainder of this section, where no confusion can occur, $\bar{m}$ will for reasons of typographic convenience be written as m, which is to be taken to be a fixed integer quantity. We have already pointed out that $\bar{d} = (K + 1)/(m + 1)$; using (3.2), and the analogous expression for $\Pr(W = w)$, one can readily show that

$$\bar{u} = (m - 1)(K + 1)/(m + 1) + 0(2^{-2m}),$$

$$\bar{w} = 2(K + 1)/(m + 1) - \sum_{j=0}^{K-m} (K - m)^{(j)}/K^{(j)},$$

(useful for moderate m/K)

$$= 2(K + 1)/(m + 1)\, [1 + (K + 1)/(m + 1) \times \sum_{j=1}^{m+1} (-1)^j \left\{\binom{m+1}{j}\right\}^2 \Big/ \binom{K - m + j}{j}\binom{2m + 1}{j}\Big] ,$$

(for m/K small).

For the parameters here employed, $\bar{d} = 18{\cdot}27, 9{\cdot}57, 4{\cdot}90, 2{\cdot}48$; $\bar{u} = 164{\cdot}5, 181{\cdot}9, 191{\cdot}0, 196{\cdot}0$; $\bar{w} = 16{\cdot}91, 8{\cdot}82, 4{\cdot}29, 1{\cdot}84$ respectively.

Table 1 Expected value of seek times, and approximations

η	m	$T_m(\bar{d})$	$E\{T_m(D)\}$	$\tau(\bar{u})$	$E\{\tau(U)\}$	$\tau(\bar{w})$	$E\{\tau(W)\}$
0·05	10	553·9	454·8	117·2	115·6	58·8	50·6
0·1	20	838·7	801·1	125·9	125·5	42·6	40·5
0·2	40	1357·2	1353·4	130·5	130·5	33·6	31·0
0·4	80	2366·8	2367·0	133·0	133·0	28·7	22·5

Table 2 Standard deviations of seek times

	Standard deviations of				
η	$T_m(D)$	$\tau(U)$	$S_m(D, U)$	$\tau(W)$	$S_m(D, W)$
0·05	126·8	13·42	131·2	17·47	125·2
0·1	133·2	6·71	134·9	16·01	132·1
0·2	147·4	3·16	147·9	13·56	146·8
0·4	174·2	1·37	174·3	13·42	174·0

A consideration of Table 1 reveals that the advantage, $E\{\tau(U) - \tau(W)\}$, of a back-and-forth head movement over the procedure in which scanning is in one direction only is positive in this example. The advantage increases absolutely as the cylinder hit-rate, η, becomes larger, but decreases relative to the average total seek time per run. For $\eta = 0{\cdot}05$ the saving is about 11 per cent, dropping to 4·4 per cent when $\eta = 0{\cdot}4$.

Secondly, it is clear that when η is not too small, the approximations to the mean values are, with the apparent exception of $E\{\tau(W)\}$, adequate, particularly as η increases. On the other hand, $T_m(\bar{d})$ overestimates $E\{T_m(D)\}$ by 22 per cent when $\eta = 0{\cdot}05$.

It is instructive to examine the reasons for this. One factor is that the mean of a linear function of a random variable is that linear function of the mean of the variable. Another is that the variances of the random variables, D, U and W decrease with increasing η or m, as the case may be, and their distributions tend to concentrate around their means. For the function given by (5.1), most head movements would then occur in the first linear segment in the case of D and W, and in the last segment in the case of U, provided the means of these variables are not near a point at which the slope changes.

The function $\tau(.)$ has a discontinuity at zero, which is an attainable value of W but not of D. Thus, writing α, β for the intercept and slope, respectively, of $\tau(.\}$ in its first segment, $E\{T_m(D)\}$ is nearly

$$\begin{aligned} E_m\{(m-1)E(\alpha+\beta D|m)\} &= \alpha(\bar{m}-1)+\beta E_m\{(K+1)(m-1)/(m+1)\} \\ &= \alpha(\bar{m}-1)+\beta(K+1)(\bar{m}-1)/(\bar{m}+1) \\ &= T_m(\bar{d}) \end{aligned}$$

since when m is binomial,

$$E\{(m-1)/(m+1)\} = (\bar{m}-1)/(\bar{m}+1) + 0(\bar{m}^{-2}) \; .$$

However,

$$\begin{aligned} E(\tau(W)) &\doteqdot \alpha + \beta E_m\{E(W|m)\} - \tau(0^+)\Pr(W=0) \\ &\doteqdot \alpha + \beta\bar{W} - \alpha\eta/(2-\eta) \; . \end{aligned}$$

With this correction, the approximations to $E\{\tau(W)\}$ are now 58·2, 41·3, 30·8, 22·4 for η=0·05, 0·1, 0·2, 0·4, respectively, the latter three being appreciably closer to the exact mean value than the entries on Table 1.

There is no discontinuity in $\tau(.)$ at K, near which the distribution of U tends to concentrate; moreover the final segment is a long one. It follows from the previous arguments, that, even when the cylinder hit ratio is fairly small,

$$E\{\tau(U)\} \doteqdot \tau(\bar{u}) \; .$$

Of course, when the head movements tend to fall into more than one segment, linear approximations will break down. More generally, if the time function is not even roughly linear over the relevant portion of the range, the sort of approximations we have been discussing cannot be relied on.

Finally, the variability of these head movement times is difficult to assess by asymptotic methods. The usual asymptotic formula for the variance of a function of a random variable (Kendall and Stuart, 1963, p. 232) when applied to $\tau(U)$ gives in our example

$$\mathrm{Var}\,\{\tau(U)/m\} \simeq (\tfrac{1}{2})^2\{2m(K-m)(K+1)\}/\{m+1)^2(m+2)\},$$

from the first term in (3.2) and the formula for $\bar{u}$. This works tolerably well for $m \geq 40$, less adequately for smaller m and for the other variables of interest.

Our main concern in this paper has been to demonstrate the feasibility of an analytic approach to some probabilistic aspects of skip-serial file processing which are commonly ignored. However certain conclusions may be drawn from the analysis in this section. Thus for the particular form of the time function given in (5.1) the relative advantage of back and forth (rather than uni-directional) scanning has been demonstrated. We have also shown that commonly used approximations to the average seek time (based on the mean distance between successive cylinders) can be seriously inaccurate, at least for low cylinder hit-ratios. The nature of the analysis in this section is however quite general. Any function of time can be inserted into the formulae; for some of these it may well be found that the approximate methods are at least as unsatisfactory as in the above example.

Acknowledgement

We wish to acknowledge the continuing assistance of S. J. Waters, who first drew the problem to our attention.

Appendix 1

Assume that requests for records arrive at random at constant rate λ, and that any given record has a constant probability of being requested, irrespective of past events. Suppose that in $[O, T]$ h requests are received; the distribution of the number n of distinct records among these is, from a standard result in occupancy theory (v. Barton and David, 1962, p. 242)

$$\binom{KL}{n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}(j/KL)^h \; . \qquad \text{(A1.1)}$$

The overall number of requests is clearly a Poisson variable with mean λT; averaging (A1.1) with respect to this distribution gives the binomial result (2.1) and (2.2).

Appendix 2

Suppose n distinct records are to be found; to obtain the distribution of the number of distinct cylinders on which the records lie, one may again use an occupancy theory formulation. Thus consider an array of K rows (cylinders) each with L cells (records); at most one ball can be placed in any cell; n balls are thrown at random into the array. If M is the number of rows with at least one full cell, standard techniques yield

$$\Pr(M=m) = \binom{K}{m}\sum_{m \geqslant j \geqslant n/L}\binom{m}{j}(-1)^{m-j}(jL)^{(n)}/(KL)^{(n)}. \text{(A2.1)}$$

If the number of records is itself a binomial random variable with parameters ρ and KL, averaging (A2.1) with respect to this distribution is easily shown to give the binomial result (2.3).

Appendix 3

The direct probabilistic arguments used to derive the (joint) distributions of order statistics of a sample of independent, identically distributed random variables can easily be modified to give for the $\{Y_r\}$ of Section 3, with m, m' fixed and less than K,

$$\begin{aligned} \Pr(Y_r=j) &= m!/\{(r-1)!(m-r)!\} \\ &\quad \times(j-1)^{(r-1)}(K-j)^{(m-r)}/K^{(m)}, \\ &\qquad j=r, r+1, \ldots, K-m+r \quad \text{(A3.1)} \end{aligned}$$

$$\begin{aligned} \Pr(Y_s=i, Y_r=j) &= m!/\{(r-1)!\,(s-r-1)!\,(m-s)!\} \\ &\quad \times(j-1)^{(r-1)}(i-j-1)^{(s-r-1)}(k-i)^{(m-s)}/K^{(m)}, \\ i=s, s+1, \ldots, s+K-m;\; &j=r, r+1, \ldots, r+i-s \; . \end{aligned}$$

Now $D_r = Y_{r+1} - Y_r$, and $\Pr(D_r=d) = \sum \Pr(Y_{r+1}=i, Y_r=j)$, the sum being taken over the set $\{i, j;\, i-j=d\}$, i.e. $\Pr(D_r=d) = m!/(r-1)!\,(m-r-1)!$

$$\begin{aligned} &\times \sum_{j=r}^{K-m+r-d-1}(j-1)^{(r-1)}(K-d-j)^{(m-r-1)}/\,K^{(m)} \\ &= m(K-d)^{(m-1)}/K^{(m)} \; , \end{aligned}$$

after repeated summation by parts.

Consider next the distribution of $U = |\tilde{Y}_{m'} - Y_1|$, $\tilde{Y}_{m'}$ and Y_1 being independent. It will be necessary to sum $\Pr(\tilde{Y}_{m'}=i, Y_1=j)$ over the set $\{i, j;\, i=j\}$, for $\Pr(U=0)$, and otherwise over the sets $\{i, j;\, i=j+u\}$ and $\{i, j;\, j=i+u\}$; e.g.

$$\Pr(U=0) = mm'\sum_{j=1}^{K}(j-1)^{(m'-1)}(K-j)^{(m-1)}/K^{(m)}K^{(m')} \; .$$

Note that in fact $m' \leq j \leq K-m$, but that the additional terms in the sum are all identically zero, being retained for later convenience and to indicate the similarity to (3.2).

To derive the distribution of $W = |\tilde{Y}_{m'} - Y_m|$, and the various joint distributions, one would proceed on similar lines.

Appendix 4

The results, (4.3), (4.4), etc., are all direct or indirect consequences of the following simple lemma:

Let m have the distribution (2.3), i.e. be binomial with parameters η and K, and let s, t and j be non-negative integers; then

$$E\{m^{(s)}(K-j)^{(m-t)}/K^{(m)}\} = \eta^s(1-\eta)^{j-t}(K-j)! \times (K-j+t-s)! \quad \text{(A4.1)}$$

Proof: First note that $\binom{K}{m} = K^{(m)}/m!$; thus

$$\sum_{m=0}^{K} \binom{K}{m} \eta^m(1-\eta)^{K-m}m^{(s)}(K-j)^{(m-t)} =$$

$$\{(K-j)!\,(1-\eta)^{j-t}/(K-j+t)!\} \times \sum_{m=0}^{K-j+t} m^{(s)} \binom{K-j+t}{m} \eta^m(1-\eta)^{K-j+t-m}$$

$$= \{(K-j)(1-\eta)^{j-t}/(K-j+t)!\}\, E\{m^{(s)}\}$$

(where now m is binomially distributed with parameters η and $K-j+t$)

$$= (K-j)!\,(1-\eta)^{j-t}/(K-j+t)!\,\eta^s(K-j+t)^{(s)},$$

which is the same as (A4.1).

For example, from (3.1)

$$E\{(m-1)\Pr(D=d|m)\} = E\{m(m-1)(K-d)^{(m-1)}/K^{(m)}\}$$
$$= (K-d)!\,\eta^2(1-\eta)^{d-1}/(K-d+1-2)!$$

which reduces to (4.3).

References

ABATE, J., DUBNER, H., and WEINBERG, S. (1968). Queueing analysis of the IBM 2314 disk storage facility, *JACM*, Vol. 15, pp. 577-589.

BARTON, D. E., and DAVID, F. N. (1962). *Combinatorial Chance*, Griffin, London.

COFFMAN, E. G., KLIMKO, L. A., and RYAN, B. (1972). Analysis of scanning policies for reducing disk seek times, *SIAM Journal on Computing*, Vol. 1, No. 3, pp. 269-279.

COX, D. R., and MILLER, H. (1968). *The Theory of Stochastic Processes*, Wiley, New York.

DENNING, P. J. (1967). Effects of scheduling on file memory operations, *Proc. AFIPS SJCC*, Vol. 30, AFIPS Press, Montvale N. J. pp. 9-21.

GOTLIEB, C. C., and MACEWEN, G. H. (1973). Performance of movable-head disc storage devices, *JACM*, Vol. 20, pp. 604-623.

HESS, H. (1963). A comparison of disks and tapes, *CACM*, Vol. 6, No. 10, pp. 634-638.

KENDALL, M. G., and STUART, A. (1963). *The Advanced Theory of Statistics*, Vol. 1, Griffin, London.

LOWE, T. C. (1968). The influence of data base characteristics on direct access file organisation, *JACM*, Vol. 15, No. 4, pp. 535-548.

MARTIN, J. (1969). *Design of Real-Time Computer Systems*, Prentice-Hall, Englewood Cliffs.

NIELSON, N. R. (1971). An analysis of some time-sharing systems, *CACM*, Vol. 14, No. 2, pp. 79-90.

WATERS, S. J. (1975). Estimating magnetic disc seeks, *The Computer Journal*, Vol. 18, No. 1, pp. 12-17.

Book review

Introduction to Decision Science, by S. M. Lee and L. J. Moore; 589 pages. (*Petrocelli/Charter, Input Two-Nine*, £9·00)

Business applications of decision sciences, by S. Paranka; 156 pages. (*Petrocelli/Charter, Input Two-Nine*, £6·00)

It is natural that those who have had experiences of teaching classes at universities or similar institutions should find it convenient to collect their lecture notes, to add to them and to edit them, and then to offer them, in book form, to a wider readership. Both these books appear to have had this origin. Such texts are certainly of use to the author's students. What we are interested in is if there is some more extensive public which could profit from them.

Although the titles of these two volumes might give a different impression, they are addressed to the same audience. The much larger one, entitled *Introduction to Decision Science*, is according to its preface 'intended primarily for undergraduate students of business, administration, social sciences and engineering', while the smaller one 'should be useful as a text for an advanced undergraduate course or a graduate level course in business decision making'.

Both deal, at different lengths, with modelling, Bayesian decision making (Paranka on 14 pages, Lee and Moore in a short section with the final remark: 'there is no unanimous opinion among scholars and practising managers about the Bayesian decision rule's superiority over other decision analysis techniques under uncertainty'), linear programming, queueing theory, simulation, Markov analysis, and inventory control.

Lee and Moore have also one chapter each on network models: PERT-CPM, and on game theory (but not on bidding), Paranka has three chapters concerned with computers (hardware as well as software), and one on regression and correlation. In the Lee-Moore book reference to computers appears in the appendix, which contains programs for linear programming and for goal programming, in FORTRAN.

Both books have, of course, references, and Lee and Moore have also, after their chapters, 'Questions' and 'Problems' (roughly: theory and practice). To compare the treatment of selected topics, covered in both books, we choose linear programming, and queueing.

Paranka has a chapter 'Linear Programming Model' of 18 pages, with a very brief description of the concept, describing well the graphical approach with two variables, and somewhat clumsily the 'algebraic method', about which he states: 'although the algebraic solution can handle more than three variables, it is not an efficient method. The Simplex Method is the easiest and quickest approach to finding the optimal solution (page 56). The latter method is presented 'conceptually' on a few pages. There follow applications to capital budgeting, and to media allocation.

Lee-Moore have four chapters on the same topic: Linear programming: Introduction and graphical solutions; Simplex method of linear programming; Goal programming; Transportation and assignment methods. These cover 212 pages. Topics not dealt with by Paranka are sensitivity analysis, details of the simplex method, and those of the last two titles mentioned. Goal programming, in particular, attempts to minimize deviations from desired goals, and the treatment leans heavily on the publications and joint-publications of the first mentioned author.

The chapter on Queueing Theory in Paranka's book (16 pages) relies mainly on simulation. That on 'Waiting line analysis—Queueing theory' (42 pages) in the larger book deals more extensively with basic theory. (We might mention here that we have used the spelling 'queu(e)ing as it is done in the two books.)

The general impression of the *Introduction to Decision Science* is that of a text from which a manager can get a reasonably clear idea of what techniques of this science are about. Paranka's book might serve a similar purpose for those who are less interested. But will they want to read such a book at all? Both books are typical products of the trans-Atlantic climate in business education, and they are not the worse for it. But their wide dissemination into the British market must be doubtful.

S. VAJDA (Sussex)