An engineering consideration of spectral transforms for ternary logic synthesis

S. L. Hurst

School of Electrical Engineering, University of Bath, Claverton Down, Bath BA2 7AY

This mainly tutorial paper attempts to look into the problems and possibilities of spectral transformations which may be useful for ternary network synthesis, corresponding to the Walsh and Rademacher-Walsh transforms which are being exploited in the binary area. In order to provide a common basis for many readers, existing binary transforms and their meanings are briefly covered in a largely non-mathematical form, from which starting point the desirable features of corresponding ternary transforms are considered.

It will be seen that the higher valued radix of ternary logic raises complexities of interpretation which are absent in the binary area, and hence the extension of certain existing design techniques using spectral data from binary to ternary may prove difficult. Further extension to radices greater than three, while raising no further fundamental mathematical problems, will compound the difficulties of application.

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List of symbols used

$x_i, i = 1$ to n	independent binary input variables of a network or system, $x_i \in \{0, 1\}$.
$f(x)$	function of the x_i binary input variables, $f(x) \in \{0, 1\}$.
$\oplus$	the Boolean Exclusive-OR operator, = modulo-2 addition in the $\{0, 1\}$ domain.
H	Hadamard matrix, with row and column entries $\in \{+1, -1\}$.
[T]	complete orthogonal $2^n \times 2^n$ square transform matrix, row and column entries $\in \{+1, -1\}$.
F]	2^n column vector representing $f(x)$, column entries $\in \{+1, -1\}$.
S]	2^n column vector, representing the spectrum of $f(x)$, column entries even integers $\in \{-2^n, \ldots, 0, \ldots, +2^n\}$.
$r_i, i = 0$ to $12.n$	the individual spectral coefficients of the spectrum S].
$X_i, i = 1$ to n	independent ternary input variables of a network or system; $X_i \in \{0, 1, 2\}$.
$f(X)$	function of the X_i ternary input variables, $f(X) \in \{0, 1, 2\}$.
$[T_t]$	$3^n \times 3^n$ ternary transform matrix, row and column entries $\in \{1, a, a^2\}$ unless otherwise stated.
$F_t]$	3^n column vector representing the ternary function $f(X)$, column entries $\in \{1, a^2, a\}$ unless otherwise stated.
$S_t]$	3^n column vector representing the spectrum of $f(X)$.
$R_i, i = 0$ to $1122 \ldots nn$	the individual spectral coefficients of the ternary spectrum $S_t]$.
a	$+120°$ rotational operator, $= (-\frac{1}{2} + j0{\cdot}866) = \sqrt[3]{1}$.
j	$+90°$ rotational operator, $= (0 + j1) = \sqrt{-1}$.
$\circledast$	Mod_3 addition in $\{0, 1, 2\}$ domain.
$f(X)$	Mod_3 addition of 1 to $f(X)$, see Section 2.1.
$f(X)$	Mod_3 addition of 2 to $f(X)$, see Section 2.1.

1. Existing binary transforms

Considerable literature now exists and applications have been pursued in the field of binary transforms (Lechner, 1971; Coleman, 1971; Dertouzos, 1965; Karpovsky, 1976; Hurst, 1978; Edwards, 1975a; 1977; Hurst, 1977; University of Bath, 1977). These generally involve Walsh or Rademacher-Walsh or related matrix transformations to convert conventional binary data from the Boolean domain into an alternative domain, the 'spectral' domain. The information content in the two domains is identical, no information being lost in transforming from one to the other and vice versa. This general situation is illustrated in **Fig. 1.**

Within the alternative spectral domain a wider set of numbers is involved, ranging in even-integer values from -2^n to $+2^n$ for any n-variable situation. Due to this set of numbers being $\{-2^n, -(2^n - 2), \ldots, 0, \ldots, +(2^n - 2), +2^n\}$ rather than $\{0, 1\}$, the information content of each number ('coefficient') is considerably higher than that of the conventional $\{0, 1\}$ Boolean numbers. The logical meaning of the mathematical transform and the values of the resulting coefficients in the spectral domain will first be reviewed.

1.1 *Orthogonal transforms*

The transform which may be employed to transform binary data into the alternative spectral domain, or vice versa, is one of many possible complete orthogonal square matrix transforms (Lechner, 1971; Coleman, 1971; Dertouzos, 1965; Karpovsky, 1976; Hurst, 1978; University of Bath, 1977; Harr, 1910; Radamacher, 1922; Walsh, 1923; Paley, 1932). These many possibilities are in general variations of each other, with different row or column orderings, as will be noted later. The mathematical terminology 'complete, orthogonal' effec-

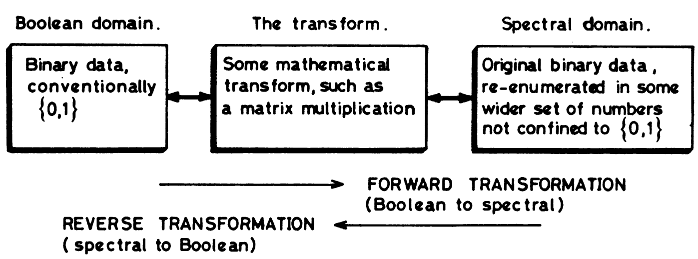

Fig. 1 The spectral transformation of binary data, the same information content being maintained in the two domains

tively means that it is always possible to reconstruct the given data from the transformed data, see Fig. 1, i.e. there is no information loss in applying the transformation from one domain to the other.

The build-up of complete orthogonal binary transform matrices is most readily illustrated by considering the 2-valued Hadamard matrix ordering, namely

$$\mathsf{H}_0 \triangleq [1], \mathsf{H}_{n+1} \triangleq \begin{bmatrix} \mathsf{H}_n & \mathsf{H}_n \\ \mathsf{H}_n & -\mathsf{H}_n \end{bmatrix} \tag{1}$$

Expressing this for the three lowest-order useful cases we have the matrices detailed in **Table 1.** Note, each matrix H_n is a $2^n \times 2^n$ square matrix. Also if we examine, say, the H_3 matrix, we will see that the number of changes from $+1$ to -1 and vice versa along each row, and down each column, of the matrix is dissimilar. The term 'sequency' has been applied to this property.

Looking further at the $2^3 \times 2^3$ Hadamard matrix, for example, we may identify the row sequency as in **Table 2**(*a*). If we rearrange the row ordering the resulting matrices remain complete and orthogonal (Hurst, 1978; Paley, 1932; Hatfield Polytechnic, 1973), but now with dissimilar sequency ordering, as shown by the two further examples given in Table 2.

It is apparent that these matrices, whilst being binary in nature, do not employ the more conventional binary numbers of {0, 1}. However, if we recode $+1 \rightarrow 0$ and $-1 \rightarrow 1$ the matrices detailed in Table 2 translate into the {0, 1} format shown in **Table 3**. However we should not undertake mathematical matrix multiplication with these {0, 1} matrices, as the zeros in the matrices are 'destructive'—multiplication of any number by zero always gives zero, and hence loss of information is inherently present. However what this recoding from $\{+1, -1\}$ to {0, 1} shows is that the rows of the orthogonal transforms in fact correspond to binary inputs x_1 to x_n and all possible Exclusive-OR's of these inputs, together with a constant input which is normally termed x_0. This function interpretation is tabulated alongside the renumbered transforms in Table 3. We will refer to this interpretation in Section 1.2.

Whilst we need not consider here the full mathematical subtleties of these matrices, it may be observed that:

(*a*) except for the first row there is always an equal number of

Table 1 Three low-order Hadamard matrices

$$\mathsf{H}_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \mathsf{H}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\mathsf{H}_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

Table 2 Three possible sequency orderings of the rows of $n = 3$ complete orthogonal matrices. (*a*) Hadamard, (*b*) Walsh, (*c*) Rademacher-Walsh

								Sequency
1	1	1	1	1	1	1	1	0
1	−1	1	−1	1	−1	1	−1	7
1	1	−1	−1	1	1	−1	−1	3
1	−1	−1	1	1	−1	−1	1	4
1	1	1	1	−1	−1	−1	−1	1
1	−1	1	−1	−1	1	−1	1	6
1	1	−1	−1	−1	−1	1	1	2
1	−1	−1	1	−1	1	1	−1	5

(*a*)

								Sequency
1	1	1	1	1	1	1	1	0
1	1	1	1	−1	−1	−1	−1	1
1	1	−1	−1	−1	−1	1	1	2
1	1	−1	−1	1	1	−1	−1	3
1	−1	−1	1	1	−1	−1	1	4
1	−1	−1	1	−1	1	1	−1	5
1	−1	1	−1	−1	1	−1	1	6
1	−1	1	−1	1	−1	1	−1	7

(*b*)

								Sequency
1	1	1	1	1	1	1	1	0
1	1	1	1	−1	−1	−1	−1	1
1	1	−1	−1	1	1	−1	−1	3
1	−1	1	−1	1	−1	1	−1	7
1	1	−1	−1	−1	−1	1	1	2
1	−1	1	−1	−1	1	−1	1	6
1	−1	−1	1	1	−1	−1	1	4
1	−1	−1	1	−1	1	1	−1	5

(*c*)

Table 3 Recoding of the $\{+1, -1\}$ orthogonal matrices of Table 2 into {0, 1} matrices, by the direct substitution of $+1 \rightarrow 0$, $-1 \rightarrow 1$

								Function
0	0	0	0	0	0	0	0	x_0
0	1	0	1	0	1	0	1	x_3
0	0	1	1	0	0	1	1	x_2
0	1	1	0	0	1	1	0	$x_2 \oplus x_3$
0	0	0	0	1	1	1	1	x_1
0	1	0	1	1	0	1	0	$x_1 \oplus x_3$
0	0	1	1	1	1	0	0	$x_1 \oplus x_2$
0	1	1	0	1	0	0	1	$x_1 \oplus x_2 \oplus x_3$

(*a*)

								Function
0	0	0	0	0	0	0	0	x_0
0	0	0	0	1	1	1	1	x_1
0	0	1	1	1	1	0	0	$x_1 \oplus x_2$
0	0	1	1	0	0	1	1	x_2
0	1	1	0	0	1	1	0	$x_2 \oplus x_3$
0	1	1	0	1	0	0	1	$x_1 \oplus x_2 \oplus x_3$
0	1	0	1	1	0	1	0	$x_1 \oplus x_3$
0	1	0	1	0	1	0	1	x_3

(*b*)

								Function
0	0	0	0	0	0	0	0	x_0
0	0	0	0	1	1	1	1	x_1
0	0	1	1	0	0	1	1	x_2
0	1	0	1	0	1	0	1	x_3
0	0	1	1	1	1	0	0	$x_1 \oplus x_2$
0	1	0	1	1	0	1	0	$x_1 \oplus x_3$
0	1	1	0	0	1	1	0	$x_2 \oplus x_3$
0	1	1	0	1	0	0	1	$x_1 \oplus x_2 \oplus x_3$

(*c*)

+1's and −1's in each row of the {+1, −1} matrices—each row except the first therefore sums to zero

(b) the product of the corresponding entries in any two rows of a {+1, −1} matrix always yields the entries of a valid row in the matrix—indeed all rows of complete orthogonal {+1, −1} matrices may be generated by the product of two other appropriate rows

(c) if the corresponding entries in any two dissimilar rows of a {+1, −1} matrix are multiplied together, the products always sum to zero—this is a fundamental property of all orthogonal matrices

(d) similarly, in the recoded {0, 1} matrices modulo-2 addition, which is equivalent to the Exclusive-OR operation, on the row entries always yields a valid row—all rows may therefore be generated by Mod_2 addition of two other appropriate rows. The correspondence of mathematical multiplication in a {+1, −1} domain and Mod_2 addition (Exclusive-OR) in a {0, 1} domain is thus demonstrated.

With these brief comments let us continue to review the results of using these matrices as domain transforms.

1.2 *Boolean-domain to spectral-domain transformation*

Application of the above transforms is as follows:

$$[T]F] = S] \quad (2)$$

where [T] = the appropriate $2^n \times 2^n$ Walsh or Rademacher-Walsh or other chosen transform,
F] = 2^n column vector defining the given function $f(x)$,
S] = resultant 2^n column vector, giving the spectral coefficients for $f(x)$.

The column vector F] is the normal minterm output truth-table for $f(x)$, but recoded into {+1, −1} from the more usual {0, 1} coding.

The inverse transformation is equally available, namely

$$F] = [T]^{-1}S] \quad (3)$$

where $[T]^{-1}$ is the inverse of [T] *

Applying the forward transformation procedure to a simple 3-variable function, say $f(x) = [\bar{x}_1x_2 + x_1\bar{x}_2 + x_2x_3]$, we have the following. The output vector of $f(x)$ in {0, 1} form and recoded in {+1, −1} form is as follows:—

$\bar{x}_1\bar{x}_2\bar{x}_3$	0	1
$\bar{x}_1\bar{x}_2x_3$	0	1
$\bar{x}_1x_2\bar{x}_3$	1	−1
$\bar{x}_1x_2x_3$	1	−1
$x_1\bar{x}_2\bar{x}_3$	1	−1
$x_1\bar{x}_2x_3$	1	−1
$x_1x_2\bar{x}_3$	0	1
$x_1x_2x_3$	1	−1

Hence using, say, the Rademacher-Walsh form of the transform we have the matrix multiplication:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}}_{[T]} \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ -1 \end{bmatrix}}_{F]} = \underbrace{\begin{bmatrix} -2 \\ +2 \\ +2 \\ +2 \\ +6 \\ -2 \\ -2 \\ +2 \end{bmatrix}}_{S]}$$

*Note, the inverse transform $[T]^{-1}$ of any orthogonal binary matrix [T] is equal to $\frac{1}{k}[T]^t$, where $[T]^t$ is the transpose of [T] and k is a scaling factor, equal to 2^n in our case. Should the forward transform [T] be symmetric then $[T]^t \equiv [T]$, and hence the forward and inverse transforms are identical apart from the scaling factor k.

The resultant spectrum S] would conventionally be identified and recorded as:

r_0	r_1	r_2	r_3	r_{12}	r_{13}	r_{23}	r_{123}
−2	2	2	2	6	−2	−2	2

Note if we had used the Hadamard or any other transform with alternative row orderings then we should have produced exactly the same spectral coefficients for S], but in a different order corresponding to the different row ordering of [T].

The logical significance of each of these coefficient values is briefly as follows.

(a) The r_0 term is numerically equal to the number of 0's (+1's) in the function output $f(x)$ minus the number of 1's (−1's); should $f(x)$ be stuck-at-0, say, then r_0 would be $+2^n$, and all other coefficients would be zero.

(b) The r_1 term is numerically equal to {(the number of agreements between input x_1 and output $f(x)$) − (the number of disagreements between x_1 and $f(x)$)}, i.e. it is a correlation coefficient between input x_1 and the function output; should $f(x)$ be merely x_1 then r_1 would be $+2^n$ with all other coefficients zero; should $f(x)$ be merely $\bar{x}_1$ then r_1 would be -2^n, all other coefficients zero.

(c) Similarly the following spectral coefficients $r_2, r_3, \ldots, r_n$ each represent a numerical correlation between an input $x_2, x_3, \ldots, x_n$ and the function output $f(x)$.

(d) The remaining coefficient values each represent {(the number of agreements between an Exclusive-OR combination of the inputs and output $f(x)$) − (the number of disagreements between the Exclusive-OR combination and $f(x)$)}, that is correlation coefficients between all possible Exclusive-OR's of the x_i's and the function output; for example r_{12} represents the correlation between $[x_1 \oplus x_2]$ and $f(x)$, r_{13} the correlation between $[x_1 \oplus x_3]$ and $f(x)$, and so on up to the correlation between $[x_1 \oplus x_2 \oplus \ldots \oplus x_n]$ and $f(x)$. Should $f(x)$ be, say, $[x_2 \oplus x_3]$, then coefficient r_{23} will be maximum-valued, with all other coefficients zero.

This meaning of the spectral coefficients can be emphasised if we rewrite the above {+1, −1} transform in {0, 1} form, and then instead of performing a normal matrix multiplication and summation procedure do an {(agreement) − (disagreement)} count along the matrix row and vector column entries to determine the resultant coefficient values of S]. The reader may readily confirm the following; the final coefficient r_{123}, for example, represents five agreements and three disagreements.

$$\begin{matrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_1 \oplus x_2 \\ x_1 \oplus x_3 \\ x_2 \oplus x_3 \\ x_1 \oplus x_2 \oplus x_3 \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ +2 \\ +2 \\ +2 \\ +6 \\ -2 \\ -2 \\ 2 \end{bmatrix} \begin{matrix} r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_{12} \\ r_{13} \\ r_{23} \\ r_{123} \end{matrix}$$

Hence the supreme significance of the {+1, −1} mathematics is that it gives a readily-executed arithmetic correlation procedure between inputs and output of a binary function, giving us a final numerical measure of what is 'important' in determining the function output.

1.3 *Uses of spectral information*

The principal uses which have been considered for binary spectral data are as follows. The possible availability of such techniques in the ternary area is clearly an incentive to pursue a parallel line of mathematical development for the analysis and synthesis of ternary functions.

Table 4 Tabulation of the principal characteristics of established binary transforms, and suggested corresponding characteristics for the ternary area

	Established binary characteristics	*Suggested corresponding ternary characteristics*
1. Size of transform matrix for n independent variables	$2^n \times 2^n$	$3^n \times 3^n$
2. No. of spectral coefficients to define a given function of n independent variables	2^n	3^n
3. Orthogonality construction of a transform matrix	The summation of the products of corresponding entries of any two rows sums to zero	The summation of the products of corresponding entries of any two rows sums to zero
4. Transform matrix row and column entries	Two values, viz $\{+1, -1\}$	Three values, $\{?, ?, ?\}$
5. Meaning of the rows of the transform matrix	A constant x_0, the independent binary inputs x_1 to x_n, and all different Exclusive-OR (Mod_2 additions) of the inputs	A constant X_0, the independent ternary inputs X_1 to X_n, and all different Mod_3 additions of the inputs
6. Meaning of the spectral coefficient values	Correlation coefficients of all inputs and Exclusive-OR's of all inputs with the output $f(x)$	Correlation coefficients of all inputs and Mod_3 additions of all inputs with the output $f(X)$
7. Maximum magnitude of any spectral coefficient	2^n	3^n

(*a*) The classification of Boolean functions into compact canonical classifications; use of the canonic classes for the specification of universal logic gates—'ULG's' (Lechner, 1971; Dertouzos, 1965; Hurst, 1978; Edwards, 1975a; Tzafestas *et al.*, 1976; Edwards, 1975b)

(*b*) Direct use of the spectral coefficient values in logic synthesis procedures (Lechner, 1971; Coleman, 1971; Dertouzos, 1965; Karpovsky, 1976; Hurst, 1978; Edwards, 1975a; 1977; University of Bath, 1977)

(*c*) Determination of simple and complex symmetries in Boolean functions by equal-magnitude coefficient pairing; use of such symmetries in logic synthesis (Hurst, 1977; 1978; Edwards and Hurst, 1976)

(*d*) Optimisation of state assignments for arbitrary sequential machines (Edwards and Hurst, 1976; 1978)

(*e*) Use of spectral data for fault diagnosis and the synthesis of easily tested logic networks (Edwards, 1977; Hurst, 1974)

Details of these areas as far as they have been currently reported may be found in the cited references.

Having surveyed this binary area, let us now consider the problems of pursuing some parallel line of development in the ternary field.

2. A provisional consideration of ternary transforms

The fundamental requirement from a ternary transform is that it shall transform the three-valued ternary data into some alternative and useful spectral data, as in Fig. 1, where each coefficient value in the resulting spectral domain has some global information content.

It is tempting to compile the tabulation of **Table 4**, to mirror in the ternary area the characteristics which we have seen to be present in the established binary area. Whether we can achieve the ternary listing remains to be considered later.

It is evident that for the ternary spectral coefficients, however derived, to be fully explicit the coefficient values should be unique for any given function $f(X)$. This implies that a reverse transform from the ternary spectral domain back to the three-valued domain must exist, although whether the inverse transform can have such a simple relationship to the forward transform as exists in the binary case remains to be considered. However, the generation of the inverse transform must take second place to the initial formulation of a useful forward transform, which gives some or all of the characteristics suggested in Table 4.

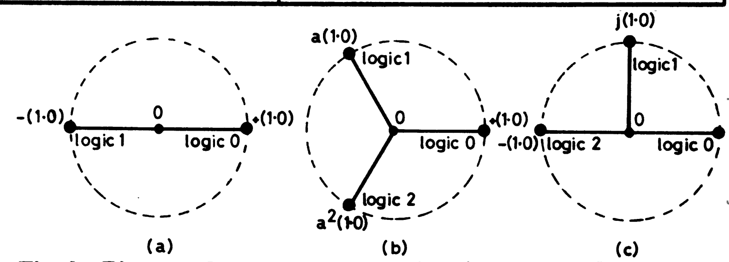

Fig. 2 Binary and ternary representations in complex planes, representing a recoding of binary values $\{0, 1\}$ and ternary values $\{0, 1, 2\}$,
(*a*) **binary, real-positive and real-negative, $\{+(1\cdot0), -(1\cdot0)\}$ or merely $\{+1, -1\}$**
(*b*) **ternary, symmetrical, $\{+(1\cdot0), a(1\cdot0), a^2(1\cdot0)\}$ or merely $\{1, a, a^2\}$**
(*c*) **ternary, quadrature, $\{+(1\cdot0), j(1\cdot0), -(1\cdot0)\}$ or merely $\{1, j, -1\}$**

2.1 '*Correlation*' *for the ternary case*

Still looking at the desirata listed in Table 4, let us first consider how we might numerically indicate the correlation of a particular ternary input condition with a ternary output $f(X)$, that is details 6 and 7 of Table 4. The meaning in the binary case was simple and explicit, a high positive value spectral coefficient, say r_2, indicating a high agreement of $f(x)$ with x_2, with a high negative value coefficient indicating a high agreement with $\bar{x}_2$ (disagreement with x_2) (Hurst, 1978; University of Bath, 1977; Hurst, 1974). Real positive and real negative values therefore were adequate to define the binary correlations

What is mathematically achieved by simple multiplication in the binary transform is the positive value of $+1{\cdot}0$ when *either* of the two binary values of $+1$ or -1 are multiplied together, i.e. multiplication of matrix row and function column vector entries of the same value (agreement) yields $+1{\cdot}0$. Unfortunately as there are only two and not three square roots of any constant, it is impossible to use any three real or complex numbers in our proposed ternary transforms to give a fixed (maximum) value when any one of three values is multiplied by itself. Thus fundamentally we cannot have identically encoded matrix row and column vector entries for the ternary case.

However, consider **Fig. 2.** In Fig. 2(*b*) a symmetrical complex-number recoding for the three ternary logic values is suggested. The '*a*' operator is a $+120°$ rotational operator, $= (-\frac{1}{2} + j\,0{\cdot}866)$.*

Now in order to achieve a maximum positive value coefficient in a domain transform, the matrix row entries may be recoded $\{0, 1, 2\} \rightarrow \{1, a, a^2\}$, but the function column vector must be recoded $\{0, 1, 2\} \rightarrow \{1, a^2, a\}$. Continuing, if we now consider say a matrix row corresponding to input X_2 in a 9×9 transform matrix, and a function $f(X)$ which is merely $f(X) = X_2$, we have the part-transform matrix operation:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & a & a & a & a^2 & a^2 & a^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ a^2 \\ a^2 \\ a^2 \\ a \\ a \\ a \end{bmatrix} = \begin{bmatrix} \cdot \\ R_2 \\ \cdot \\ \cdot \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

$$[\mathrm{T}_t] \qquad\qquad \mathrm{F}_t] = \mathrm{S}_t]$$

whence

$$\begin{aligned} R_2 &= 1 + 1 + 1 + a^3 + a^3 + a^3 + a^3 + a^3 + a^3, \\ &= 9, \\ &= \text{a maximum correlation of } f(X) \text{ with the matrix row entries.} \end{aligned}$$

Should we have $f(X)$ some other function of X_2, say $\bar{X}_2$ or $\bar{\bar{X}}_2$, where $\bar{X}_2$ and $\bar{\bar{X}}_2$ are defined by

X_i	$\bar{X}_i$	$\bar{\bar{X}}_i$
0	1	2
1	2	0
2	0	1

then this part-transform may be readily seen to yield the following spectral coefficient values:

(*a*) $\underline{f(X) = \bar{X}_2} : R_2 = a^2 + a^2 + a^2 + a^2 + a^2 + a^2 + a^2 + a^2 + a^2 = 9a^2$

(*b*) $\underline{f(x) = \bar{\bar{X}}_2} : R_2 = a + a + a + a + a + a + a + a + a = 9a$

Hence useful and distinguishable spectral coefficient values seem to result; by suitable completion of the remaining rows of the transform matrix remaining spectral coefficients giving other correlation measurements should be possible.

If instead of the 120° '*a*' operator of Fig. 2(*b*) we choose the 90° '*j*' operator of Fig. 2(*c*), where $j \triangleq \sqrt{-1}$, we may proceed as previously. If we now recode the matrix row entries $\{0, 1, 2\}$ by $\{1, j, -1\}$, and the function column vector entries $\{0, 1, 2\}$ by $\{1, -j, -1\}$, the spectral coefficient value for R_2 in the above cases becomes:

(*a*) $\underline{f(X) = X_2 : R_2} = 1 + 1 + 1 - j^2 - j^2 - j^2 + 1 + 1 + 1, = 9$

(*b*) $\underline{f(X) = \bar{X}_2 : R_2} = -j - j - j - j - j - j - 1 - 1 - 1, = -3 - 6j$

(*c*) $\underline{f(X) = \bar{\bar{X}}_2 : R_2} = -1 - 1 - 1 + j + j + j + j + j + j = -3 + 6j$

Whilst these three results are unique and distinctive for the three cases, the previous correlation coefficient values which all involve a '9' for $f(X)$ superficially appear preferable. We will, therefore, continue our discussions here with the '*a*' operator as an integral part of our transform procedure.

2.2 *The* $\{1, a, a^2\}$ *transform matrices*

Let us build up our ternary transform matrices in a similar manner to that illustrated in Section 1.1 for the binary Hadamard matrix. Therefore:

$$\Delta_0 \triangleq [1];\ \Delta_{n+1} \triangleq \begin{bmatrix} \Delta_n & \Delta_n & \Delta_n \\ \Delta_n & a\Delta_n & a^2\Delta_n \\ \Delta_n & a^2\Delta_n & a\Delta_n \end{bmatrix} \qquad (4)$$

c.f. equation (1), where Δ temporarily represents the ternary matrix $[\mathrm{T}_t]$. Hence for the first two useful-dimensioned matrices we have:

$$\left.\begin{aligned} \Delta_1 &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}, \\ \Delta_2 &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a & a^2 & 1 & a & a^2 & 1 & a & a^2 \\ 1 & a^2 & a & 1 & a^2 & a & 1 & a^2 & a \\ 1 & 1 & 1 & a & a & a & a^2 & a^2 & a^2 \\ 1 & a & a^2 & a & a^2 & 1 & a^2 & 1 & a \\ 1 & a^2 & a & a & 1 & a^2 & a^2 & a & 1 \\ 1 & 1 & 1 & a^2 & a^2 & a^2 & a & a & a \\ 1 & a & a^2 & a^2 & 1 & a & a & a^2 & 1 \\ 1 & a^2 & a & a^2 & a & 1 & a & 1 & a^2 \end{bmatrix} \end{aligned}\right\} \qquad (5)$$

The Hadamard matrices of Section 1 are particular cases of these ternary matrices, with one of the three entries deleted and recoded $\{+1, -1\}$.

Considering these ternary matrices, we may observe:

(*a*) the product of the entries of any row multiplied by the complex conjugates of the entries always sums to a constant $k = 3^n$, and also the entries of any row multiplied by the complex conjugates of any other row always sums to zero.* The matrix therefore is an orthogonal matrix (Birkhoff and MacLane, 1965; Hohn, 1964; Bell, 1975)

(*b*) except for the first row, each row contains an equal number of 1's, *a*'s and a^2's, and therefore sums to zero**

(*c*) the product of the corresponding entries in any two rows of the matrix or the entries of any single row multiplied by themselves, always yields a valid row in the matrix—indeed all rows may be generated by the products of other appropriate rows.

*This '*a*' operator may be recognised as the same complex operator employed by power engineers in the analysis of three-phase ac systems. A more detailed study of all uses of this operator in power engineering may possibly be of significance for our ternary logic research.

*Recall the complex conjugate of $(x \pm jy)$ is $(x \mp jy)$, where x and y are real numbers and $j = \sqrt{-1}$.

**Recall the complex additions and multiplications: $a + a = 2a$; $a^2 + a^2 = 2a^2$; $a + a^2 = -1$; $1 + a + a^2 = 0$; $aa = a^2$; $aa^2 = a^3 = 1$; $a^2a^2 = a^4 = a$.

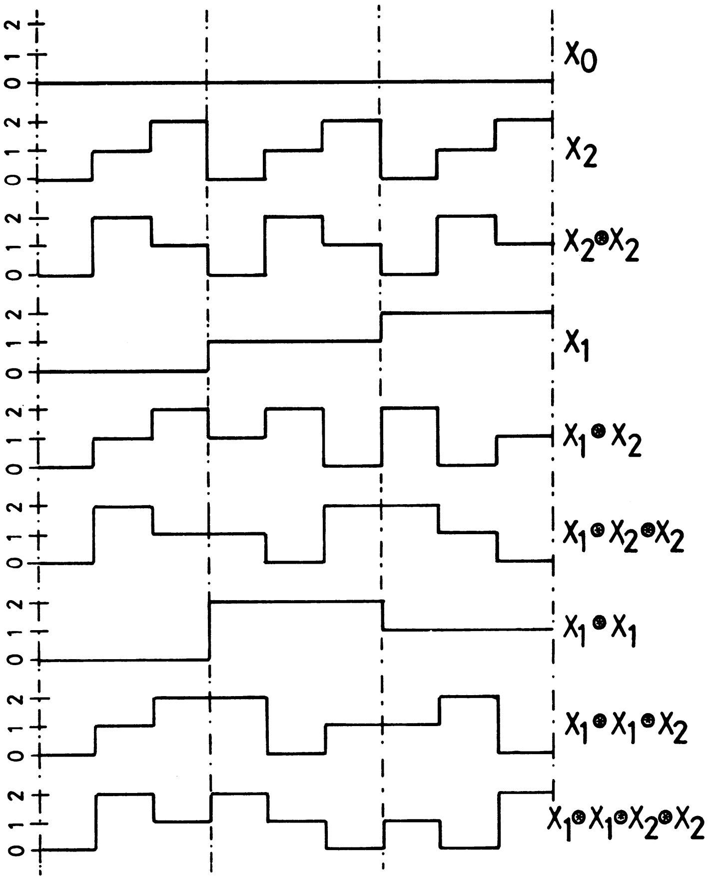

Fig. 3 **The $n = 2$ (9 × 9) ternary matrix functions, plotted with the matrix $\{1, a, a^2\}$ row entries recoded as $\{0, 1, 2\}$**

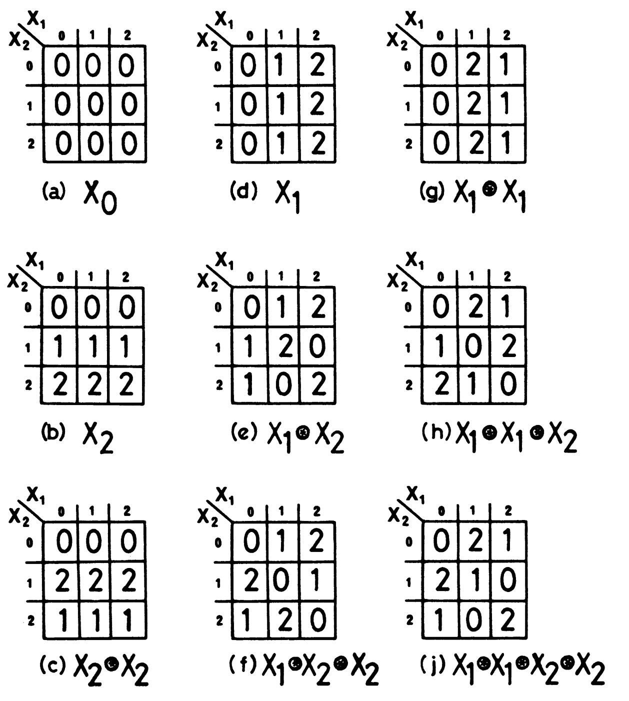

Fig. 4 **Karnaugh-map plots of the nine $n = 2$ matrix functions of Fig. 3, recoded $\{0, 1, 2\}$**

The last two properties are evident from the definition of Δ_n in Equation (4).

Paralleling the development with the binary transforms, we may ascribe an appropriate logical property to each row of these ternary matrices. Taking for example the 9 × 9 matrix and recalling that the matrix entries $\{1, a, a^2\}$ are a recoding of the more conventional logic values $\{0, 1, 2\}$ respectively, we may identify the matrix rows as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a & a^2 & 1 & a & a^2 & 1 & a & a^2 \\ 1 & a^2 & a & 1 & a^2 & a & 1 & a^2 & a \\ 1 & 1 & 1 & a & a & a & a^2 & a^2 & a^2 \\ 1 & a & a^2 & a & a^2 & 1 & a^2 & 1 & a \\ 1 & a^2 & a & a & 1 & a^2 & a^2 & a & 1 \\ 1 & 1 & 1 & a^2 & a^2 & a^2 & a & a & a \\ 1 & a & a^2 & a^2 & 1 & a & a & a^2 & 1 \\ 1 & a^2 & a & a^2 & a & 1 & a & 1 & a^2 \end{bmatrix} \qquad \begin{array}{l} \textit{Logical significance} \\ X_0 \\ X_2 \\ X_2 \circledast X_2 \\ X_1 \\ X_1 \circledast X_2 \\ X_1 \circledast X_2 \circledast X_2 \\ X_1 \circledast X_1 \\ X_1 \circledast X_1 \circledast X_2 \\ X_1 \circledast X_1 \circledast X_2 \circledast X_2 \end{array}$$

where $\circledast$ is the logical operation of modulo-3 addition (c.f. Mod_2 addition, = Exclusive-OR in the binary case). Notice that whilst in the binary transforms the entries of any row multiplied by themselves reconstruct the first (null) row of the matrix, in the ternary case each row (other than the first) has to be multiplied by itself three times to reconstruct the first row of the matrix. This is consistent, since

$$[f(X) \circledast f(X) \circledast f(X)] = \text{constant for any } f(X) \ .$$

Equally, the entries of any row multiplied by their respective complex conjugates reconstruct the first row.

What is not obvious, however, is how to allocate any 'sequency' meaning to the rows of these ternary transforms, corresponding to the binary situation illustrated in Section 1. The $n = 2$ ternary matrix entries may be graphically illustrated as in **Fig. 3**, where clearly there is a dissimilar 'waveform' associated with each transform row, but precisely how to quantify this, or indeed if there is any practical significance in attempting such a quantification, is not apparent at this stage of development.

An alternative representation of these $n = 2$ entries is shown in **Fig. 4**, using a Karnaugh-map format similar to that employed in the binary area. We will refer to these mappings again in Section 3.3.

2.3 *The inverse transform*

The inverse transform $[T]^{-1}$ of any given transform $[T]$ is defined by:

$$[T][T]^{-1} \triangleq [I]$$

where $[I]$ = the unit matrix, that is a matrix with all 1's on the main diagonal and zeros elsewhere.

Now for any square orthogonal matrix $[T]$ we have that $[T]^{-1}$ is given by the hermitian conjugate matrix $[T]^+$, times an appropriate scaling factor, where $[T]^+$ is obtained by taking the complex conjugate of all the individual entries of $[T]$ and transposing corresponding rows and columns, that is:

$$[T]^{-1} = \frac{1}{k}[T]^+ \ , \ = \frac{1}{k}[T^*]^t \ ,$$

where $[T^*]$ represents the complex conjugate entries of $[T]$ and $[\ ]^t$ represents the row and column transpose (Birkhoff and MacLane, 1965; Hohn, 1964; Bell, 1975).*

This is illustrated by the following development. Using for the moment the same notation as in Equations (4) and (5), we have:

$$[\Delta_{n+1}]\,[\Delta_{n+1}]^{-1} = [I] \ ,$$

$$\begin{bmatrix} \Delta_n & \Delta_n & \Delta_n \\ \Delta_n & a\Delta_n & a^2\Delta_n \\ \Delta_n & a^2\Delta_n & a\Delta_n \end{bmatrix} \begin{bmatrix} \\ \Delta_{n+1} \\ \\ \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

whence the following result readily follows:

*Note, in the binary case, with only real positive and real negative and not complex values in $[T]$, then $[T^*] \equiv [T]$ and hence $[T]^{-1}$ is merely $\frac{1}{k}[T]^t$.

Table 5 The spectral coefficients for all twentyseven functions of a single ternary variable

	X_i			$S_t]$		
	0	1	2	R_0	R_1	R_{11}
f_0	0	0	0	3	0	0
f_1	0	0	1	$2+a^2$	$1+2a$	$2+a^2$
f_2	0	0	2	$2+a$	$2+a$	$1+2a^2$
f_3	0	1	0	$2+a^2$	$2+a^2$	$1+2a$
f_4	0	1	1	$1+2a^2$	$2+a$	$2+a$
f_5	0	1	2	0	3	0
f_6	0	2	0	$2+a$	$1+2a^2$	$2+a$
f_7	0	2	1	0	0	3
f_8	0	2	2	$1+2a$	$2+a^2$	$2+a^2$
f_9	1	0	0	$2+a^2$	$a+2a^2$	$a+2a^2$
f_{10}	1	0	1	$1+2a^2$	$2a+a^2$	$1+2a^2$
f_{11}	1	0	2	0	0	$3a^2$
f_{12}	1	1	0	$1+2a^2$	$1+2a^2$	$2a+a^2$
f_{13}	1	1	1	$3a^2$	0	0
f_{14}	1	1	2	$a+2a^2$	$2+a^2$	$a+2a^2$
f_{15}	1	2	0	0	$3a^2$	0
f_{16}	1	2	1	$a+2a^2$	$a+2a^2$	$2+a^2$
f_{17}	1	2	2	$2a+a^2$	$1+2a^2$	$1+2a^2$
f_{18}	2	0	0	$2+a$	$2a+a^2$	$2a+a^2$
f_{19}	2	0	1	0	$3a$	0
f_{20}	2	0	2	$1+2a$	$1+2a$	$a+2a^2$
f_{21}	2	1	0	0	0	$3a$
f_{22}	2	1	1	$a+2a^2$	$1+2a$	$1+2a$
f_{23}	2	1	2	$2a+a^2$	$2+a$	$2a+a^2$
f_{24}	2	2	0	$1+2a$	$a+2a^2$	$1+2a$
f_{25}	2	2	1	$2a+a^2$	$2a+a^2$	$2+a$
f_{26}	2	2	2	$3a$	0	0

$$\begin{bmatrix} \Delta_n & \Delta_n & \Delta_n \\ \Delta_n & a\Delta_n & a^2\Delta_n \\ \Delta_n & a^2\Delta_n & a\Delta_n \end{bmatrix} \begin{bmatrix} \Delta_n & \Delta_n & \Delta_n \\ \Delta_n & a^2\Delta_n & a\Delta_n \\ \Delta_n & a\Delta_n & a^2\Delta_n \end{bmatrix} = k \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (6)$$

Note that the complex conjugate of $(0 + a\Delta_n)$ is $(0 - a\Delta_n)$, $= a^2\Delta_n$, and that of $(0 + a^2\Delta_n)$ is $(0 - a^2\Delta_n)$, $= a\Delta_n$. Hence the inverse of the forward ternary transform $[T_t]$ is obtained by interchanging a and a^2 in the forward transform, with a constant scaling factor $\frac{1}{k}$, k in this case being 3^n. For $n = 1$ and $n = 2$ we therefore have:

$$\left.\begin{aligned} [\Delta_1]^{-1} &= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \\ [\Delta_2]^{-1} &= \frac{1}{3^2}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a^2 & a & 1 & a^2 & a & 1 & a^2 & a \\ 1 & a & a^2 & 1 & a & a^2 & 1 & a & a^2 \\ 1 & 1 & 1 & a^2 & a^2 & a^2 & a & a & a \\ 1 & a^2 & a & a^2 & a & 1 & a & 1 & a^2 \\ 1 & a & a^2 & a^2 & 1 & a & a & a^2 & 1 \\ 1 & 1 & 1 & a & a & a & a^2 & a^2 & a^2 \\ 1 & a^2 & a & a & 1 & a^2 & a^2 & a & 1 \\ 1 & a & a^2 & a & a^2 & 1 & a^2 & 1 & a \end{bmatrix} \end{aligned}\right\} \quad (7)$$

From these developments it appears that we have a possible transformation procedure to convert ternary data into a spectral domain, and vice versa should we so require. Let us therefore examine some simple results of applying these domain transforms.

3. Execution of the proposed ternary transforms

The execution of the matrix transformations considered in the previous section will hopefully enable ternary functions to be transformed into the spectral domain with useful results. We are therefore in a position to mirror Fig. 1 for the ternary area, as shown in **Fig. 5**.

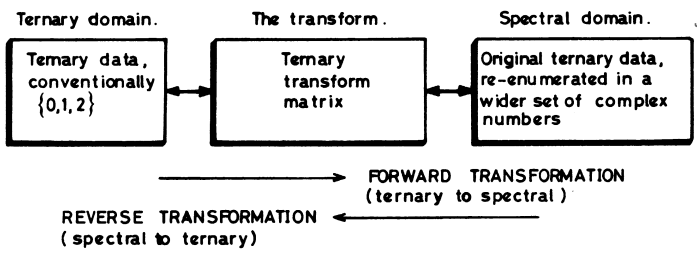

Fig. 5 The spectral transformation of ternary data, corresponding to the binary situation of Fig. 1

As the amount of data and size of the ternary transform matrix rapidly increases with n, we will confine our discussions here to $n \leqslant 2$ cases. The transform matrices for $n > 2$ can readily be compiled from the development of Equations (5) and (7).

3.1 *Functions of one variable* X_1

Using the $n = 1$ transform

$$[T_t] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix},$$

the spectral coefficients for all twentyseven possible functions of one ternary variable may be computed. The results are listed in **Table 5.** Note, the designations R_0, R_1, R_{11} for the spectral coefficients correspond to the matrix rows X_0, X_1, $X_1 \circledast X_1$, respectively.

To demonstrate the inverse transform, take say function f_{22}, $f(X) = 2, 1, 1$:

$$\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} a + 2a^2 \\ 1 + 2a \\ 1 + 2a \end{bmatrix}$$

$$[\mathrm{T}_t]^{-1} \qquad \mathrm{S}_t]$$

$$= \begin{bmatrix} (a + 2a^2) + (1 + 2a) + (1 + 2a) \\ (a + 2a^2) + (a^2 + 2) + (a + 2a^2) \\ (a + 2a^2) + (a + 2a^2) + (a^2 + 2) \end{bmatrix} \times \frac{1}{3},$$

$$= \mathrm{F}_t]$$

$$= \begin{bmatrix} 2 + 5a + 2a^2 \\ 2 + 2a + 5a^2 \\ 2 + 2a + 5a^2 \end{bmatrix} \times \frac{1}{3},$$

$$= \begin{bmatrix} 2 + 2a + 2a^2 + 3a \\ 2 + 2a + 2a^2 + 3a^2 \\ 2 + 2a + 2a^2 + 3a^2 \end{bmatrix} \times \frac{1}{3},$$

$$= \begin{bmatrix} 3a \\ 3a^2 \\ 3a^2 \end{bmatrix} \times \frac{1}{3},$$

$$= \begin{bmatrix} a \\ a^2 \\ a^2 \end{bmatrix},$$

$$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{recoded in normal } \{0, 1, 2\} \text{ values}$$

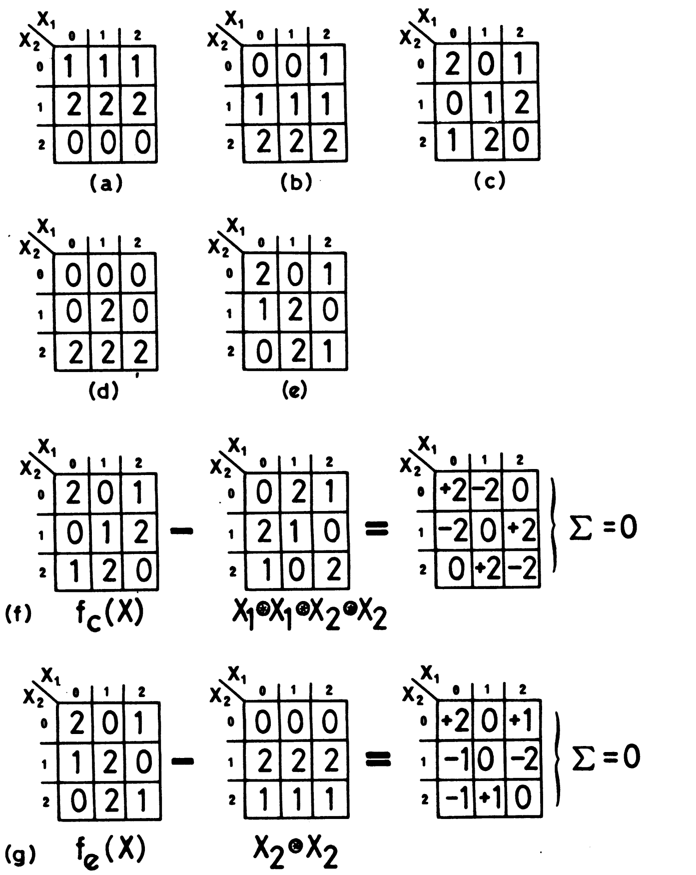

Fig. 6 Example two-variable ternary functions and their correlation, all encoded {0, 1, 2}
(*a*) to (*e*) the five example functions
(*f*) correlation of function (*c*) with transform matrix row function [$X_1 \circledast X_1 \circledast X_2 \circledast X_2$]
(*g*) correlation of function (*e*) with transform matrix row function [$X_2 \circledast X_2$]

Study of these spectral coefficients shows certain clear correlations, namely:

(*a*) function $f_0, f(X) = 0, 0, 0$: max real correlation with R_0, as expected; zero correlation with R_1 and R_{11}.

(*b*) function $f_5, f(X) = 0, 1, 2$: max real correlation with R_1, as expected; zero other correlations.

(*c*) function $f_7, f(X) = 0, 2, 1$: max real correlation with $R_{11}, = X_1 \circledast X_1$, as expected; zero other correlations.

(*d*) function $f_{11}, f(X) = 1, 0, 2$: max shifted correlation with R_{11}, zero other correlations; function f_{11} is $\overline{[X_1 \circledast X_1]}$, and hence this correlation is explicit.

(*e*) function $f_{13}, f(X) = 1, 1, 1$: max shifted correlation with R_0, zero other correlations; function is $\bar{R}_0$, and hence correlation is explicit.

(*f*) function $f_{15}, f(X) = 1, 2, 0$: max shifted correlation with R_1, zero other correlations—function is $\bar{X}_1$, and hence correlation is explicit.

(*g*) function $f_{19}, f(X) = 2, 0, 1$: max shifted correlation with R_1, zero other correlations—function is $\bar{\bar{X}}_1$, and hence correlation is explicit.

(*h*) function $f_{21}, f(X) = 2, 1, 0$: max shifted correlation with R_{11}, zero other correlations; function is $\overline{\overline{[X_1 \circledast X_1]}}$, and hence correlation is explicit.

Table 6 Five example ternary functions of two variables and their spectral coefficient values

Function f(X)	*Spectral coefficient values* R_0	R_2	R_{22}	R_1	R_{12}	R_{122}	R_{11}	R_{112}	R_{1122}
(*a*) $f(X) = 1, 2, 0, 1, 2, 0, 1, 2, 0$	0	$9a^2$	0	0	0	0	0	0	0
(*b*) $f(X) = 0, 1, 2, 0, 1, 2, 1, 1, 2$	$a + 2a^2$	$8 + a^2$	$a + 2a^2$	$1 + 2a$	$1 + 2a$	$1 + 2a$	$2 + a^2$	$2 + a^2$	$2 + a^2$
(*c*) $f(X) = 2, 0, 1, 0, 1, 2, 1, 2, 0$	0	0	0	0	$9a$	0	0	0	0
(*d*) $f(X) = 0, 0, 2, 0, 2, 2, 0, 0, 2$	$5 + a$	$5 + a$	$4 + 2a^2$	$1 + 2a^2$	$2 + a$	$2a + a^2$	$2 + a$	$2a + a^2$	$1 + 2a^2$
(*e*) $f(X) = 2, 1, 0, 0, 2, 2, 1, 0, 1$	0	0	0	$3a + 3a^2$	$3 + 3a$	$6a$	3	$3a^2$	$3a$

(*i*) function $f_{26}, f(X) = 2, 2, 2$: max shifted correlation with R_0, zero correlation elsewhere; function is $\bar{R}_0$, and hence correlation is explicit.

3.2 *Functions of two variables* X_1, X_2

Consider the five example functions illustrated in **Fig. 6**(*a*)–(*e*). Evaluating the spectrum for each gives the values listed in **Table 6**; the coefficient designations $R_0, R_2, R_{22}, R_1, R_{12}$, etc. are the coefficients given by the descending rows of the 'Hadamard-ordered' transform of Equation (5), and therefore represent correlations with $X_0, X_2, X_2 \circledast X_2, X_1, X_1 \circledast X_2$, etc. respectively.

Although we have not here attempted any formal mathematical proof of the completeness of the ternary transform matrices, relying rather upon extension of the known properties of the complete orthogonal binary transforms, nevertheless from using these matrices it appears that the spectral coefficients for any given function $f(X)$ are indeed unique, and hence explicitly define $f(X)$. Also, although again not formally proven, it is evident that if the spectrum for any given function $f(X_1, \ldots, X_i, \ldots, X_n)$ is evaluated, and *all* coefficients containing i in their subscript identification are zero-valued, then input X_i is redundant. This mirrors the situation in the binary area.

However, if we examine the particular spectral details in Table 6 we may comment:

(*a*) function (*a*) is merely $\bar{X}_2$, as revealed by the single non-zero coefficient of $R_2 = 9a^2$; similarly function (*c*) is merely $\overline{[X_1 \circledast X_2]}$

(*b*) a zero value for the R_0 coefficient always indicates the same number of minterms of value 0, 1 and 2 in the given function—confirm this also in Table 5

(*c*) similarly, a zero value elsewhere confirms that the minterm values of the given function $f(X)$ are 'equally spaced' in value either side of the minterm values of the matrix row function X_m to which they are compared, that is $\sum^{3^n}$ {minterm value of $f(X)$ − corresponding minterm value in X_m} = 0, where the minterm values are coded {0, 1, 2}.

For example, consider the zero-valued coefficient R_{1122} in function (*c*); if we compare this function with the matrix row function $[X_1 \circledast X_1 \circledast X_2 \circledast X_2]$ given in Fig. 3, then the greater-than and less-than value of each minterm in comparison with the matrix function values are as shown in Fig. 6(*f*). Similarly Fig. 6(*g*) shows the comparison of function (*e*) with matrix function $[X_2 \circledast X_2]$, again summing to zero. However it must be emphasised that whilst this plus/minus numerical summation to zero always holds for all zero-valued spectral coefficients, the converse is *not* always true.

What is not immediately obvious is the full significance of the spectral coefficients which are not zero or maximum-valued. The higher the number in a spectral coefficient then 'more like' the given function certainly is to a particular matrix row or Mod_3 additions to the row; for example $R_2 = 8 + a^2$ for function (*b*) represents a function which is only one minterm out of agreement with X_2, but the complex nature of the 120° rotational operators a and a^2 tends to prevent a full insight as is available with the real-positive, real-negative coefficients of the binary case. It can be reasoned that the coefficient $8 + a^2$ is one term removed from the maximum real value of 9, and similarly $1 + 2a$, $2 + a^2$, etc. are each one term removed from zero; this is sketched in **Fig. 7**, but this leaves coefficient values such as $4 + 2a^2$ as not readily meaningful.

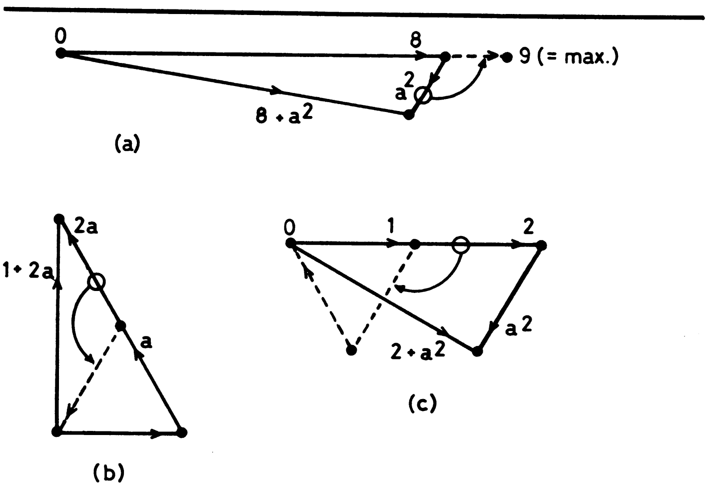

Fig. 7 **The geometric meaning of coefficients which are near-maximum or near-zero valued**
(*a*) **coefficient** $8 + a^2$, **modified to maximum with one 120° operation**
(*b*) **coefficient** $1 + 2a$, **similarly modified to zero**
(*c*) **coefficient** $2 + a^2$, **similarly modified to zero**

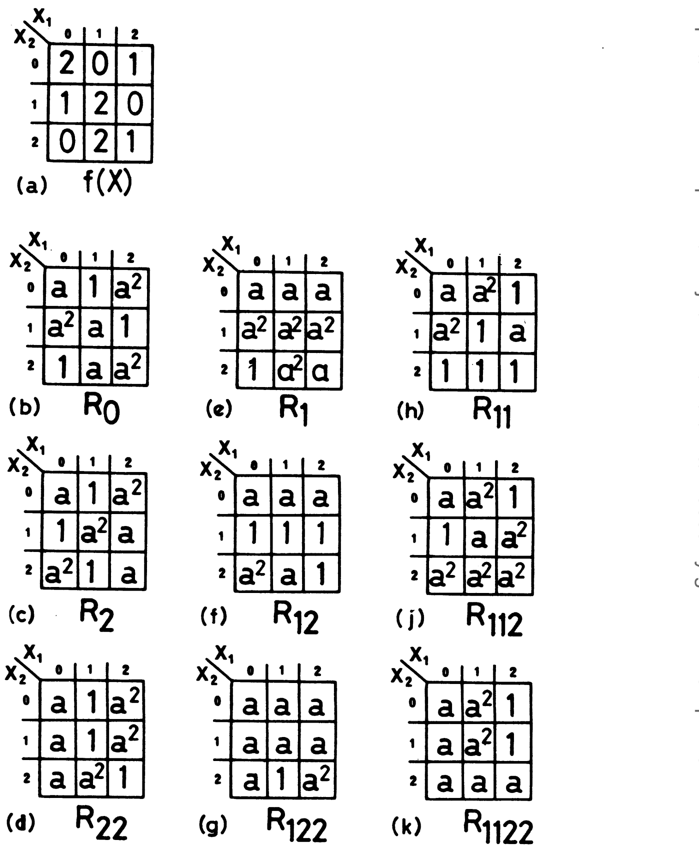

Fig. 8 **Mapping determination of spectral coefficient values**
(*a*) **given function** $f(X)$
(*b*) – (*k*) **comparison of** (*a*) **with the nine ternary matrix function maps shown in Figs. 3**(*a*) **to** (*j*) **respectively.**

3.3 *Geometric determination of the spectral coefficients*

We can obtain the same spectral coefficients by a mapping-and-comparison procedure (Hurst, 1978), rather than executing the mathematical transform. Take for example the function (*e*) of Fig. 6. Let us replot it again in **Fig. 8**(*a*), and then compare each minterm entry with the matrix transform maps previously given in Fig. 4. The results of each minterm comparison will be recorded as:

(*a*) identical 0, 1, or 2 minterm values:		record as 1 (zero phase shift)
(*b*) function minterm 0,	matrix minterm 1:	record as a (+120° phase shift)
ditto ,	matrix minterm 2:	record as a^2 (+240° phase shift)
(*c*) function minterm 1,	matrix minterm 0:	record as a^2 (+240° phase shift)
ditto ,	matrix minterm 2:	record as a (+120° phase shift)
(*d*) function minterm 2,	matrix minterm 0:	record as a (+120° phase shift)
ditto ,	matrix minterm 1:	record as a^2 (+240° phase shift)

This procedure gives the set of comparison mappings of Fig. 8(*b*)–(*k*). Reading off these results we have:

$$\begin{aligned} R_0 &= 3 + 3a + 3a^2, = 0 \\ R_2 &= 3 + 3a + 3a^2, = 0 \\ R_{22} &= 3 + 3a + 3a^2, = 0 \\ R_1 &= 1 + 4a + 4a^2, = 3a + 3a^2 \\ R_{12} &= 4 + 4a + a^2\ , = 3 + 3a \\ R_{122} &= 1 + 7a + a^2\ , = 6a \\ R_{11} &= 5 + 2a + 2a^2, = 3 \\ R_{112} &= 2 + 2a + 5a^2, = 3a^2 \\ R_{1122} &= 2 + 5a + 2a^2, = 3a \end{aligned}$$

which is as previously determined.

With this comparison procedure we are performing the identical steps as involved in the detailed matrix row and column vector multiplications. However, it must be concluded that performing the spectral coefficient evaluation by this hand method does not greatly enhance appreciation of the meaning of the values as correlation coefficients.

4. Further considerations

The previous section has considered the possible ternary transforms as a direct extension of the existing binary area. Possibly this may not be the best approach and a completely separate attack should be made. However, the few mathematical papers on multi-real-valued transforms which have appeared to date do generalise to the same form of transform as considered here when applied to the ternary case (Wallis, 1972; Kitahasi and Tanaka, 1972; Chrestenson, 1955). A more rigorous mathematical basis for the transforms discussed here may be found in certain of those publications, including tests of completeness for the matrices.

4.1 *Interpretation of resulting spectra*

However the problem of interpretation of the spectral coefficient values must be solved if useful engineering applications are to be proposed. The 120° geometric operator, whilst obviously relevant for ternary transforms, has been seen to produce results which appear at this stage of research to be difficult to visualise and interpret.

Possibly we should recode or expand the resultant spectral coefficients in some further manner, so as to give us, say, real-number values as in the binary case. For example, it may be useful to produce three separate real-positive-number coefficients to indicate directly 'how like' the output is to, say, X_i, $\bar{X}_i$, $\bar{\bar{X}}_i$, rather than rely upon the complex relationship to X_i given by the one complex spectral coefficient R_i considered here; further, two and not three separate coefficients may be adequate, similar in principle to that used in colour TV transmissions to convey three-colour information on two colour-data highways. The three separated coefficients would involve a considerable expansion of data, with a high redundancy, but this ultimately may be useful. In its turn this may lead to alternative and preferable transform matrices.

However, yet another aspect may be considered. In the binary area, the Hadamard, Walsh and Rademacher-Walsh transforms each relate to the binary inputs and Exclusive-OR's of the inputs. It is, however, possible to construct yet another $\{+1, -1\}$ transform matrices which do not have these Exclusive-OR relationships in their structure, and hence the spectra which result from such alternative transforms no longer directly include Exclusive correlation meanings (Bell, 1975; Liebler and Roesser, 1971). Therefore, in the ternary case it may equally be possible to depart from the Mod_3 addition relationships in the ternary transforms, as developed here through Equations (4) and (5), and produce further transforms which include possibly Maximum (ternary OR) or Minimum (ternary AND) rather than Mod_3 relationships. Further, one can raise the question how important is completeness and the existence of the inverse transform for particular logic design purposes, if one only requires some correlation meaning from the execution of a forward matrix transformation.

4.2 *Fast ternary ('Walsh') transform procedures*

As far as execution of the present ternary transform matrices is concerned, computation of the simple examples considered above shows that there are many identical row-and-column multiplications present in the full transformation. This may also be discerned in the structures of Fig. 8. Hence, it is possible to formulate a fast transform procedure, comparable with the butterfly diagrams which define the fast Walsh transform procedures of the binary matrices (Karpovsky, 1976; Andrews and Caspari, 1969), although perforce the number of computations necessary rises sharply with the increase in radix from the binary case.

4.3 *Quasi-binary ternary recoding*

A further possible alternative approach may be made to recode the ternary logic values in a quasi-binary mode for transformation purposes, recoding $\{0, 1, 2\}$ as, say, $\{1, 1; 1, -1; -1, 1\}$. This is clearly wasteful of information content, as the $3^n \times 3^n$ ternary matrix would now (presumably) become a $2^{2n} \times 2^{2n}$ binary matrix. However it would be interesting to investigate the resulting spectral coefficient values, and see what economies or rationalisation could be made in the transform and/or the resulting spectral coefficients. It would eliminate the problems with the 120° geometric operator, but may in its turn introduce other complications in interpretation of the resulting spectral coefficient values.

5. Conclusions

The basic simplicity and beauty of the binary spectral domain cannot, it seems, be maintained in the ternary area. The higher-valued radix fundamentally introduces more complex definitions and relationships to quantify 'how like' a function is to its various inputs and combinations of these inputs, and yet the practical significance of spectral data hinges upon the recognition of such relationships.

If research into ternary transforms continues, then hopefully we may reach the same goals as recently have been reached in the binary area, including

(*a*) classification of ternary functions by their spectral coefficients into canonical classifications, and the use of such canonic classes for the specification of ternary universal logic gates

(b) detection of simple and complex symmetry patterns in ternary functions, and their exploitation for synthesis purposes and

(c) spectral translation techniques for synthesis purposes.

These and other fields of application should be kept firmly in mind in this continuing area of research.

However, it could well be that further research confirms that odd-valued radices prove inconvenient, and in many respects even-valued radices provide more amenable and explicit results. This may suggest that quaternary (4-valued) rather than ternary may be the most convenient radix higher than binary to investigate and develop.

It is hoped that publication of this largely tutorial paper may encourage research in this whole area.

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(a) Final Report NATO Award No 67, July-December 1976, University of Manitoba (Part 6); and

(b) CS.77001-R 'Concerning Transforms for Three-Valued Systems', Virginia Polytechnic Institute, February 1977.

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