A theory of evaluative comments in chess with a note on minimaxing

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Classical game theory partitions the set of legal chess positions into three evaluative categories: won, drawn and lost. Yet chess commentators employ a much larger repertoire of evaluative terms than this, distinguishing (for example) a 'drawn' from a 'balanced' position, a 'decisive' from a 'slight' advantage, an 'inaccuracy' from a 'mistake' and a 'mistake' from a 'blunder'. As an extension of the classical theory, a model of fallible play is developed. Using this, an additional quantity can in principle be associated with each position, so that we have not only its 'game-theoretic value' but also its 'expected utility'. A function of these two variables can be found which yields explications for many evaluative terms used by chess commentators. The same model can be used as the basis of computer play. It is shown to be easier to justify, and to adjust to realistic situations, than the minimax model on which state of the art chess programs are based.

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Introduction

The game tree of chess contains about 10^{46} positions (Good, 1968) a substantial proportion of which are terminal. The rules of the game assign a value to every terminal position, $+1$, 0 or -1 according to whether the position is won, drawn or lost for White. These values can be backed up the game tree using the minimax rule, so that in principle every position can be given a value, including the initial position. This last is known as 'the value of the game', and is widely conjectured to be 0 for chess. If this conjecture is correct, and if both sides play faultlessly, i.e. only execute value-preserving moves (it follows from the 'back-up' method of assigning values that there is at least one such move available from every non-terminal position), then the game must end in a draw. A fragment of a hypothetical game tree is depicted in **Fig. 1.** In **Fig. 2** the method of attaching game-theoretic values to positions is illustrated.

An evaluation function could, in principle, map board positions into a larger set of values, making it possible to express a distinction between positions which are 'marginally' won and positions which are 'overwhelmingly' or 'obviously' won, or between drawn positions in which White, or Black, 'has the edge' and drawn positions which are 'equally balanced', and so forth. Two circumstances suggest that a useful purpose might be served by multi-valued functions.

(i) Chess Masters and commentators have developed a rich descriptive language for the expression of such distinctions.

(ii) Computer chess programs employ real-valued functions for evaluating terminal positions, not of the game tree which is too large, but of the lookahead tree. Values backed up from the lookahead horizon are used to select the next move. We lack a formal basis for assigning definite interpretations to such values.

There is thus a need for a stronger theory of position-evaluation. This paper discusses chess, but the treatment is general and covers all two-person zero-sum games of perfect information without chance moves.

Requirements of a theory

A good theory should explicate a variety of commentators' concepts. **Table 1** is a representative list. Where a conventional symbol is available it precedes the verbal comment.

Main features of the theory

The game-theoretic model presupposes perfect play, whereas in the real-life game of chess (whether human or computer) both sides are susceptible to error. Our theory is based on this distinction, and presents the following main features:

(1) We follow Good (1968) and interpret the values of terminal positions as *utilities* as though the game were played for a unit stake. Values for preterminal positions are then calculated as *expected utilities*. In order to avoid confusion we shall refer to these throughout as 'expected utilities' or 'scores', never as 'values', reserving the latter term for game-theoretic values.

(2) A model of imperfect but skilled play is developed. Chess skill appears in this model as an adjustable parameter running from 0 (random play) to ∞ (perfect play).

(3) In the new model the classical game-theoretic treatment appears as a special case.

The calculation of expected utilities

Consider a state, s_0, from which transitions to successor states $s_1, s_2, s_3, \ldots s_n$ can occur with respective probabilities $p_1, p_2, p_3, \ldots p_n$. Let us suppose that these successor states have associated utilities $u_1, u_2, u_3, \ldots u_n$. Then the expected utility associated with s_0 is

$$\sum_{i=1}^{n} p_i u_i$$

It follows trivially that if we interpret as utilities the values attached by the rules of chess to the terminal positions then the values assigned to the non-terminal positions by minimaxing can be interpreted as expected utilities. In this special case the *p*s associated with those arcs of the game tree which carry a change of game-theoretic value are all 0. Consequently, the evaluation of $\sum_{i=1}^{n} p_i u_i$ at each node reduces to obtaining the 'min' or the 'max' of the successor-values according to whether White or Black has the move. The above specification is ambiguous in the case when two or more of the moves applicable to a given board position are value-preserving. We can either select one of these at random and assign a probability of unity to it and zero probabilities to the rest, or we can divide the unit probability equally among them. In the case of error-free play, calculation of expected utilities according to either procedure leads to the same result. As the basis of a model of actual play we shall adopt the second alternative, which is illustrated in Fig. 2.

We now relax the game-theoretic condition that at each choice-point on the tree there is a probability of unity that a value-preserving move ('sound' or 'correct' move) is chosen,

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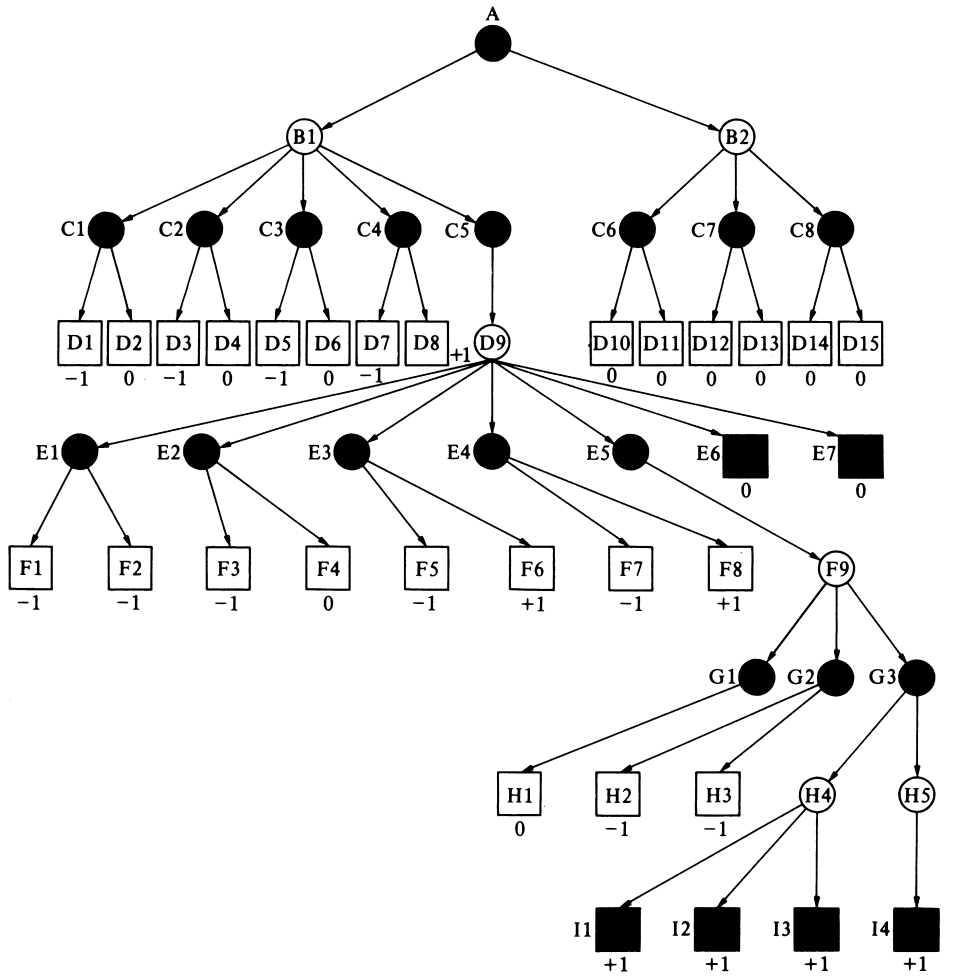

Fig. 1 A game tree with its terminal nodes (shown as squares) labelled with outcome values from the set {+1, 0, −1}. Shading of the remaining nodes (circles) indicates which player has the move

and we introduce the possibility of error. In constructing a model of error, we express the relative probabilities of making alternative moves from a given position as a monotonic increasing function (decreasing function for Black, since all utilities are expressed from White's standpoint) of the expected utilities of the corresponding successor positions. Thus the move leading to the highest expected utility will be chosen with highest probability (but not with probability 1 as in the game-theoretic error-free model), the move leading to the next highest expected utility with next highest probability and so on. We thus envisage an idealised player whose statistical behaviour reflects the rank-ordering of the expected utilities of chess positions. Using such a model it is again possible to label all the nodes of the tree, working upwards from the terminal nodes, but by a procedure which differs from the minimax method.

The notion of discernibility

In order to carry out some illustrative computations based on this idea, we now choose an actual monotonic function. No significance is claimed for the particular choice, since the points which we seek to establish are qualitative rather than quantitative. Certain ideas must, however, be reflected in any such function. A central one is that of *discernibility*. We conceive the player as standing upon a given node of the game-tree and looking towards its successors. These are labelled with their expected utilities, but the labels are not fully discernible to him. Discernibility is directly related to the strength of the player (the labels are fully discernible to an infinitely strong player) and inversely related to the number of moves separating the node from the end of the game: next-move mates and stalemates are fully discernible even to the beginner, but next-move expected utilities obtained by backing up are less so. Reflecting these considerations, we shall define the discernibility from a board state s_0 of the expected utility of a given successor state s_j as:

$$d = (M + 1)^{[3(r_j+3)/(r_j+\epsilon)]} \quad (1)$$

where M is the merit of the player in kilopoints of the US Chess Federation scale, so that $0 \leqslant M$, and r_j is the number of moves that the value associated with s_j has been backed up. The symbol ϵ denotes an arbitrarily small quantity introduced to avoid the expression becoming infinite for $r_j = 0$.

The expected utilities themselves are real numbers lying in the range from −1 through 0 to +1. They are interpreted as being in logarithmic measure, to base d. Using this base, we take the antilogarithms of the expected utilities associated with the n successors of a given position as giving the *relative probabilities* with which a player of merit M who has reached s_0 selects the corresponding moves. Thus, for the transition $s_0 \rightarrow s_j$,

$$p_j \propto d^{u_j} \quad (2)$$

Normalising these so as to obtain actual probabilities, p_1, p_2, . . . p_n, the expected utility of a position is evaluated as $\sum_{i=1}^{n} p_i u_i$, where u_i is the expected utility of the position generated by the ith member of the set of available moves. Starting at the

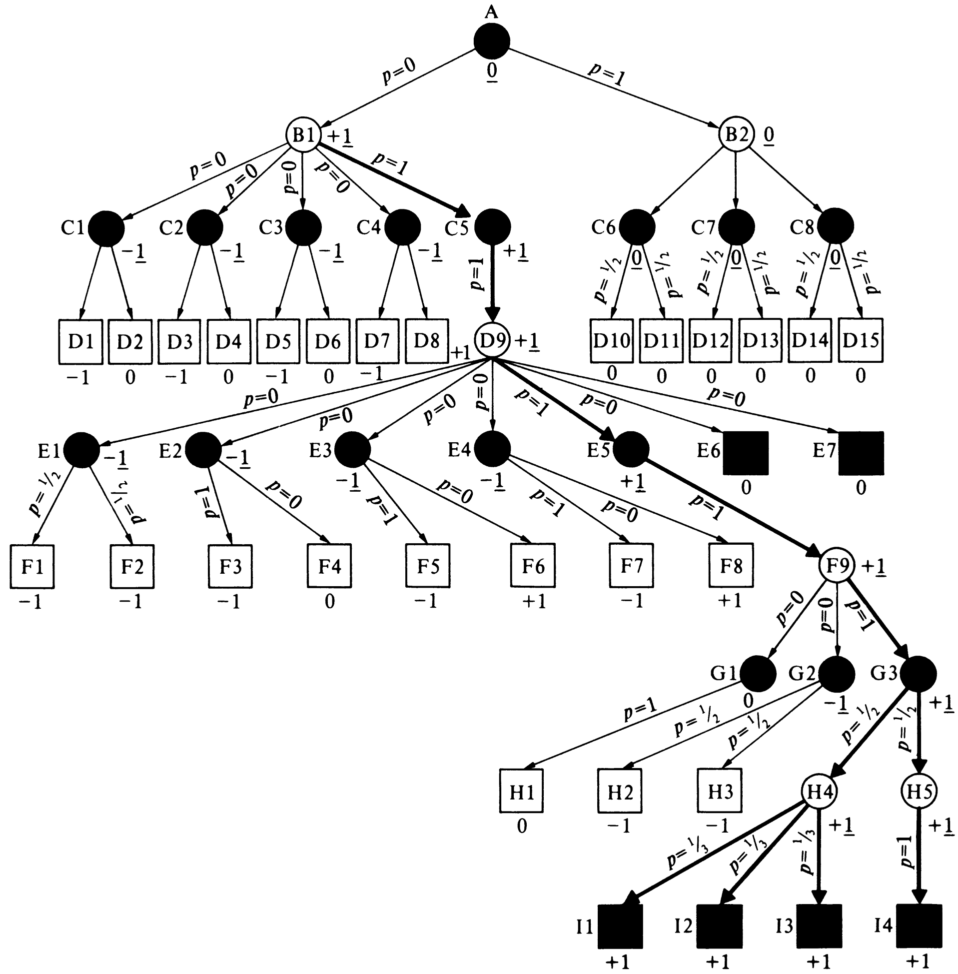

Fig. 2 The game tree of Figure 1 with its non-terminal nodes labelled (underlined values) by minimax back-up. White's best strategy from B1 is drawn with a heavy line. Arcs are marked with the conditional move-probabilities corresponding to perfect play: since the game-theoretic value of B1 is +1, Black chooses with probability 1 to move to B2

Table 1 A representative list of commentators' comments

(1)	A dead draw (nothing that either players can do can avert a draw)
(2)	A complicated position
(3) =	A balanced position
(4) ±	White has a slight advantage
(5) ±	White has a clear advantage
(6) + −	White has a decisive advantage
(7)	A certain win for White
(8)	A difficult position for White
(9)	A losing move
(10)	An inaccurate move: White weakens his position
(11)	White strengthens his position
(12) ?	A mistake
(13) ??	A blunder
(14) !	A strong move
(15) !!	A very strong or brilliant move
(16) !?	A brilliant but unsound move
(17)	Best move
(18) (!)	Best move in difficult circumstances
(19)	A safe move
(20)	White should press home his advantage
(21)	Black should play for time

terminal positions, this gives a method for assigning expected utilities to successively higher levels of the game tree until every position has been labelled.

A sample computation

Consider the terminal fragment of game-tree shown in Fig. 1. We shall illustiate step by step the calculation of expected utilities so as to label every node in the diagram. First we make assumptions for the playing strengths M_W and M_B of White and Black respectively. If we are to extract examples of the broad range of evaluative concepts from so ultra-simplified a game tree we must set these strengths very low. Let us set $M_W = 0{\cdot}2$ and $M_B = 1{\cdot}4$: White is thus an abject beginner and Black a weak tournament player. In our model $M = 0$ implies random play. The notation $u(s)$ denotes the expected utility of position s.

H4: All successors have the same value, $+1$: $u(\text{H4}) = +1$.
H5: There is only one successor, so the move-probability is unity: $u(\text{H5}) = +1$.
G1: Unique successor: $u(\text{G1}) = 0$.
G2: Equivalued successors: $u(\text{G2}) = -1$.
G3: Equivalued successors: $u(\text{G3}) = +1$.
F9: From proportionality (2) we have
Move to G1: $d^0 = 1$ = relative probability.

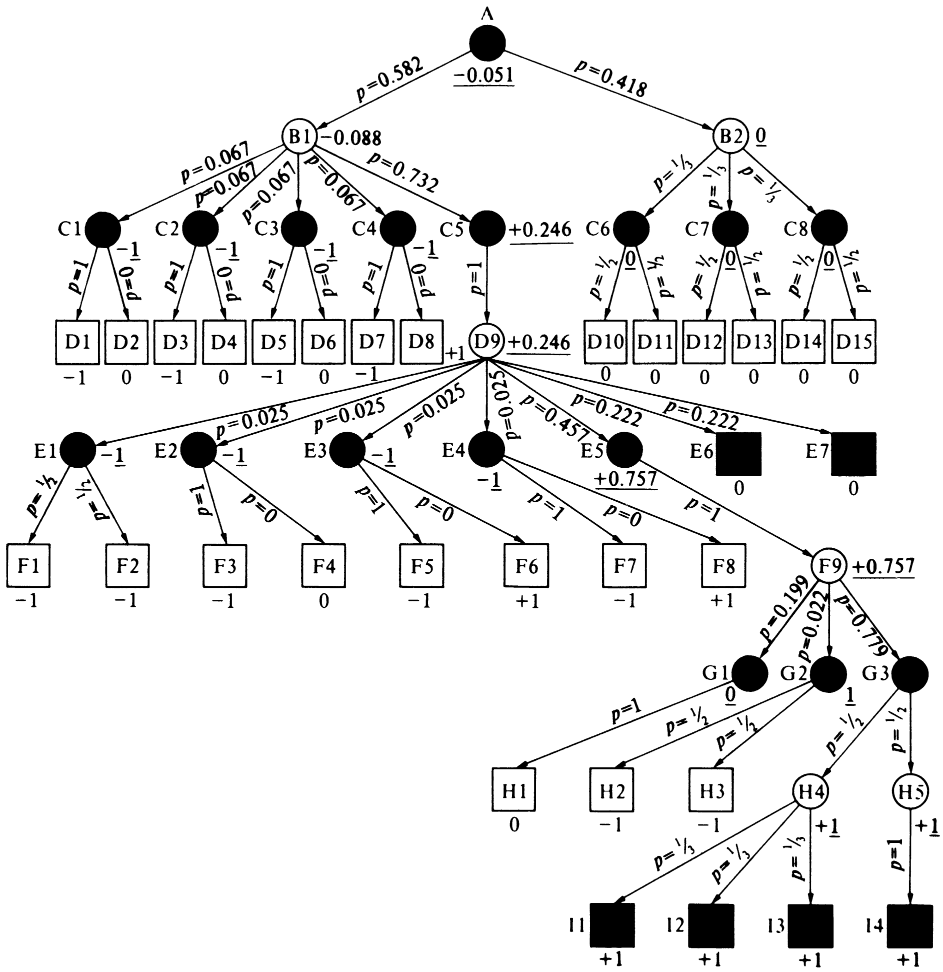

Fig. 3 The game tree of Figures 1 and 2 labelled with expected utilities calculated from a model of fallible play. White has been credited with playing strength M_W = 0·2 and Black has M_B = 1·4. Conditional move-probabilities generated by this model are entered against the corresponding arcs and are used to 'back up' expected utilities to successively higher levels. As before, backed up values are underlined

Move to G2: $r = 1$, so, from Eqn (1), $d = 1{\cdot}2^{12} = 8{\cdot}915$. Relative probability = 1/8·915 = 0·1121.

Move to G3: $r = 2$, so $d = 1{\cdot}2^{7\cdot 5} = 3{\cdot}925$ = relative probability.

Normalised probabilities: G1, 0·1985; G2, 0·0222; G3, 0·7792.

$u(\mathrm{F9}) = (0{\cdot}1985 \times 0) + (0{\cdot}0222 \times -1) + (0{\cdot}7792 \times +1) = +0{\cdot}757$.

E1: Equivalued successors. $u(\mathrm{E1}) = -1$.

E2: $r = 0$. $u(\mathrm{E2}) = -1$, and similarly for $u(\mathrm{E3})$ and $u(\mathrm{E4})$.

E5: Unique successor. $u(\mathrm{E5}) = 0{\cdot}757$.

D9: **Move to E1**: $r = 1$. $d = 1{\cdot}2^{12}$. Relative probability = 1/8·915 = 0·112 and similarly for moves to E2, E3, and E4.

Move to E6: Relative probability = 1, and similarly for move to E7.

Move to E5: $r = 4$. $d = 1{\cdot}2^{5\cdot 25} = 2{\cdot}604$. Relative probability = 2·0640.

Normalised probabilities: E1, 0·025; E2, 0·025; E3, 0·025; E4, 0·025; E5, 0·457; E6, 0·222; E7, 0·222 (total 1·001).

$u(\mathrm{D9}) = (0{\cdot}457 \times 0{\cdot}757) - 0{\cdot}100 = 0{\cdot}246$.

C1: $r=0$. $u(\mathrm{C1}) = -1$, and similarly for $u(\mathrm{C2})$, $u(\mathrm{C3})$ and $u(\mathrm{C4})$.

C5: Unique successor. $u(\mathrm{C5}) = 0{\cdot}246$.

C6: Equivalued successors. $u(\mathrm{C6}) = 0$, and similarly for $u(\mathrm{C7})$ and $u(\mathrm{C8})$.

B1: **Move to C1**: $r = 1$. $d = 1{\cdot}2^{12}$. Relative probability = 1/8·915 = 0·112 and similarly for moves to C2, C3 and C4.

Move to C5: $r = 6$. $d = 1{\cdot}2^{4\cdot 5} = 2{\cdot}272$. Relative probability = 1·2240.

Normalised probabilities: C1, 0·06703; C2, 0·06703; C3, 0·06703; C4, 0·06703; C5, 0·73190 (total 1·00002).

$u(\mathrm{B1}) = (0{\cdot}7319 \times 0{\cdot}246) - 0{\cdot}2681 = -0{\cdot}088$.

B2: Equivalued successors. $u(\mathrm{B2}) = 0$.

A: **Move to B1**: $r = 7$. $d = 2{\cdot}4^{4\cdot 286}$. Relative probability = 1·391.

Move to B2: Relative probability = $d^0 = 1$.

Normalised probabilities: B1, 0·582; B2, 0·418.

$u(\mathrm{A}) = (0{\cdot}582 \times -0{\cdot}088) + (0{\cdot}418 \times 0) = -0{\cdot}051$.

In **Fig. 3** the tree of Fig. 1 is shown with expected utilities, calculated as above, attached to the nodes. The expected utility of the root node, A, turns out to be one twentieth of a unit in Black's favour—a 'slight plus' for Black. The analysis of Black's 'plus' is worth pursuing, for it illustrates certain fundamental concepts to which our theory is directed, in particular the idea that a *losing move* (in the game-theoretic sense of a transition for White to value −1 or for Black to value +1) can also be the 'best' move against a fallible opponent.

Note that Black can secure a certain draw by moving to B2. Note also that the move to B1 is a losing move in the game-

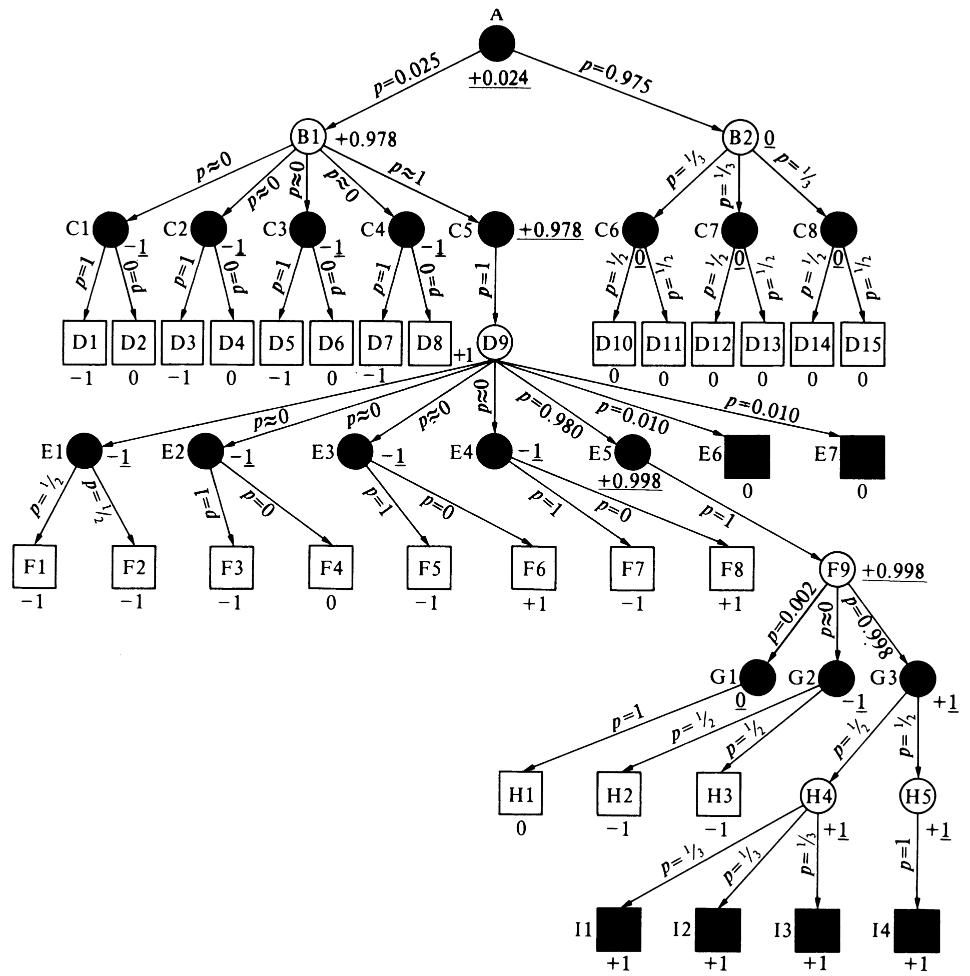

Fig. 4 Expected utilities backed up the game tree using a different assumption about the strengths of the players, namely $M_W = M_B = 1{\cdot}4$; i.e. both players are of weak club standard. The expected utility associated with the root node now favours White, and the model of Black's play shows a 40:1 preference at this choice-point for the 'safe draw'

theoretic sense, for White can then win by the sequence B1 → C5 → D9 → E5 → F9 → G3, as shown by the heavy line in Fig. 2. Yet the *expected utility* of the move, $-0{\cdot}088$, is marginally better for Black than that of the 'correct' move (expected utility zero), and our model of Black, possessed of a weak tournament player's discernment, shows a 58% preference for the move. The statistical advantage arises, as can be seen by inspecting the diagram, from the fact that play is switched into a subtree where the error-prone White has numerous opportunities for error presented to him. He has to find the needle of sound play in a haystack of hazards. In such a situation we sometimes say that Black sets 'traps' for his opponent. If the aesthetic features of the move to B1 appeal to the commentator, he may even use the annotation '!?', which we take to mean 'brilliant but unsound'. A sufficient increase in the strength of White could give cause to remove the '!' or even to convert it into a second '?'. To illustrate this point we have recalculated the entire diagram after setting $M_W = M_B = 1{\cdot}4$, shown in **Fig. 4.** Here the move to B1 does not appear as 'best', nor even as a mistake, but as a blunder, and correspondingly our model of Black shows a preference of approximately 40:1 for B2.

Returning to the list of specimen evaluative comments in Table 1, we can now derive explications for them **(Table 2).** Wherever possible, an explication is expressed in terms of two functions of a board position, namely its game-theoretic value v and its expected utility u. Where a move, rather than a position, is described, we use the notation Δv and Δu to denote the changes in the corresponding quantities affected by the move. We denote by s_1 the position from which the move is made and by s_2 the position which it generates. Some items of the original list have for completeness been differentiated into sub-concepts. Some of these would never appear in a chess book although under assumptions of very low playing strength they are generated by our model. Case 2 of (6) is an example of this: a 'decisive advantage' of this kind would characterise, for example, the initial position if Bobby Fischer gave Queen odds to a beginner.

We exhibit systematically in **Table 3** various combinations of u and v, entering in each case the evaluative comment which seems most appropriate.

'Tension'

The minimax value of s can be regarded as in some sense summarising the values of the terminal nodes of the tree rooted in s. More obviously, the expected utility of s, which has the form of a weighted mean, constitutes a summary of a different kind of this same set of quantities. It seems natural to proceed to statistics of higher order, i.e. from representative values and means to variances. Might such second-moment statistics also possess recognisable meaning in terms of the chess commentator's vocabulary?

Good (1968) discusses a property of chess positions which he

Table 2 Explication of the evaluative comments of Table 1

	Comment	Explication
(1)	A dead draw	$v = 0$ for all terminal descendants of s
(2)	s is complicated	The first few levels of the tree rooted in s have high branching ratios
(3) =,	s is balanced	$v = 0$ *and* $u \simeq 0$
	Case 1: s is lifeless	var $(v_t) \simeq 0$ see text
	Case 2: s has high tension	var $(v_t) \gg 0$ see text
(4) $\pm$,	White has a slight advantage	$v = 0$ *and* $u > 0$
(5) $\pm$,	White has a clear advantage (good winning chances)	$v = 0$ *and* $u \gg 0$
(6) + −,	White has a decisive advantage	$u \simeq +1$
	Case 1: White has excellent winning chances	$v = 0$ *and* $u \simeq +1$
	Case 2: Although White's game is theoretically lost, he is almost bound to win	$v = -1$ *and* $u \simeq +1$
	Case 3: An easy win for White	$v = +1$ *and* $u \simeq +1$
(7)	A certain win for White	$v = +1$ *and* $u = +1$
(8)	s is difficult	$v \gg u$
	Case 1: White needs accuracy to secure the draw	$v = 0$ *and* $u \ll 0$
	Case 2: White needs accuracy to secure the win	$v = +1$ *and* $0 < u \ll 1$
	Case 3: Although theoretically won, White's position is so difficult for him that he should offer a draw	$v = +1$ *and* $u < 0$
(9)	A losing move	$v(s_2) = -1$ *and* $v(s_1) > -1$
(10)	An inaccuracy: White's move weakens his position	$\Delta v = 0$ *and* $\Delta u < 0$
(11)	White's move strengthens his position	$\Delta v = 0$ *and* $\Delta u > 0$
(12) ?,	A mistake	$\Delta v = -1$ *and* not $(\Delta u \ll 0)$
(13) ??,	A blunder	$\Delta v < 0$ *and* $\Delta u \ll 0$
(14) !,	A strong move	$\Delta v = 0$ *and* $\Delta u > 0$ *and* s_1 is difficult
(15) !!,	A very strong or brilliant move	$\Delta v = 0$ *and* $\Delta u \gg 0$
(16) !?,	A brilliant but unsound move	$\Delta v < 0$ *and* $\Delta u \gg 0$
(17)	Best move	Δu is max
(18) (!),	Best move in difficult circumstances	Δu is max *and* s_1 is difficult
(19)	A safe move	$\Delta v = 0$ *and* s_2 is lifeless
(20)	'White should press home his advantage.' The rationale for trying to shorten the game when ahead can be understood by noting in Fig. 3 how the advantage decays as we move backwards from the terminal positions. In **Fig. 5** White, in moving from B1, has been given an additional option in the form of a move to C5·1, from which Black is forced to move directly to F9 (S-shaped arc in Fig. 5). Game-theoretically the choice between moving to C5 and moving to C5·1 is equally balanced since they are both 'won' positions for White. But the expected utilities, +0·246 against +0·757, tell the true story, that if he incurs needless delay in a won position, especially if it is a *complicated* position (high branching ratio of immediately dependent tree), he multiplies his chances of error. Our model selects the move to C5·1 with 1·7 times the frequency of C5, with a corresponding increase of u(B1) (see Fig. 5).	
(21)	'Black should play for time' is the complementary advice one should give to the *other* player in the foregoing situation. If our hypothetical node C5·1 had a second branch leading to D9 (shown as a broken line in Fig. 5), then Black should prefer it to F9.	

calls 'agitation'. He defines it by considering how sharply the estimated utility of a position is changed by investing a further unit of work in deepening the forward analysis. This quantity will necessarily be positively related to the variance of the distribution of u values over the dependent sub-tree, and hence to the measure which we develop below for the 'tension' of a position. The former British Champion, Alexander, uses this term in an introductory chapter to *Fischer v. Spassky, Reykjavik 1972*. Alexander (1972) writes (see **Fig. 6**),

'Let me illustrate (a little crudely) this question of tension by comparing two openings:

A. (Giuoco Pianissimo) 1. P–K4, P–K4; 2. Kt–KB3, Kt–QB3; 3. B–B4, B–B4; 4. P–Q3, P–Q3; 5. Kt–B3, Kt–B3.

B. (Gruenfeld Defence: see the Siegen game Spassky v. Fischer) 1. P–Q4, Kt–KB3; 2. P–QB4, P–KKt3; 3. Kt–QB3, P–Q4; 4. P × P, Kt × P; 5. P–K4, Kt × Kt; 6. P × Kt, B–Kt2; 7. B–QB4, P–QB4. The moves in example A are perfectly correct—but after five moves the game is as dead as mutton; it is too simple, too balanced, and is almost certain to lead to an early and dull draw. The moves in example B are objectively no better—but the position is full of tension; White has a powerful Pawn centre but Black can exert pressure on it and, if he survives the middle game, may stand better in the ending—the players are already committed to a

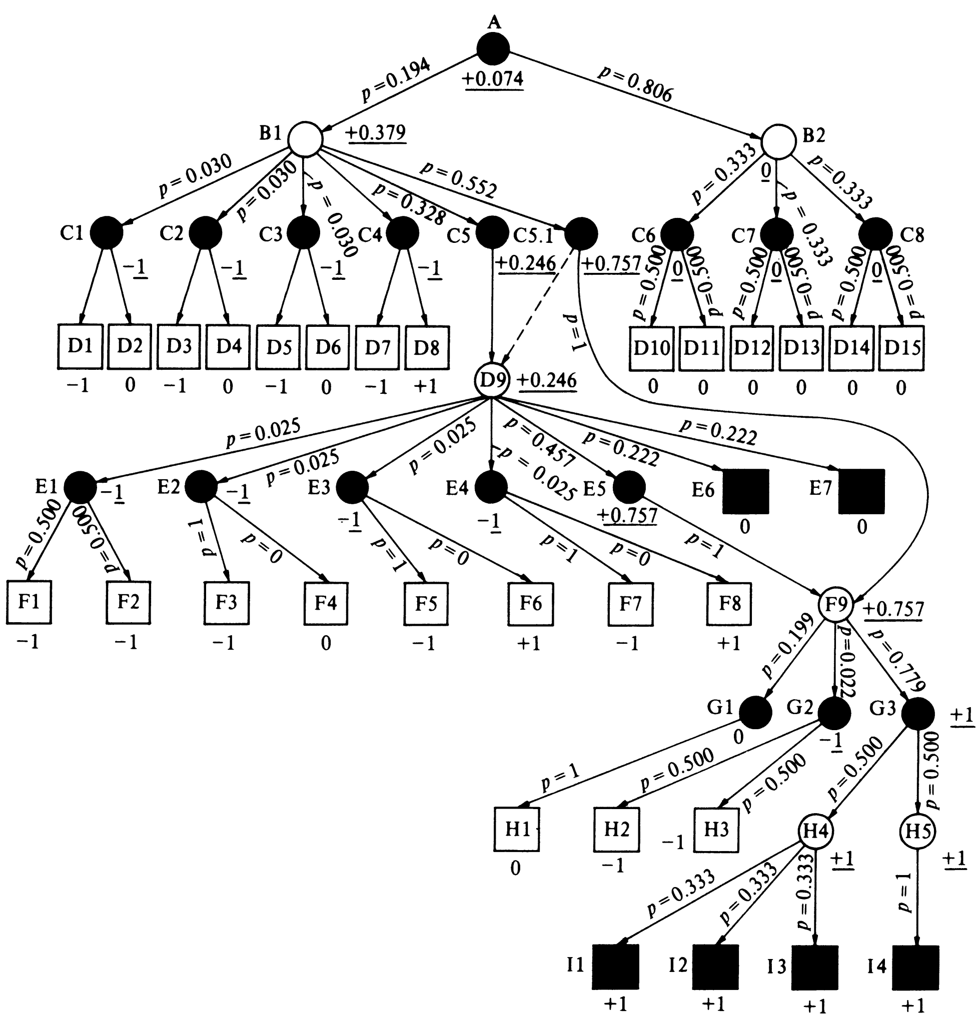

Fig. 5 A modified version of Figure 3 in which a new node, C5·1, has been added leading to F9 (the broken line represents a hypothetical delaying move for Black, see text). Although without effect on the game-theoretic values of nodes lying above it in the tree, interpolation of this short-cut option tips the balance of expected utilities, so that at the root the move to B2 becomes 'best'

Giuoco Pianissimo

Gruenfeld defence

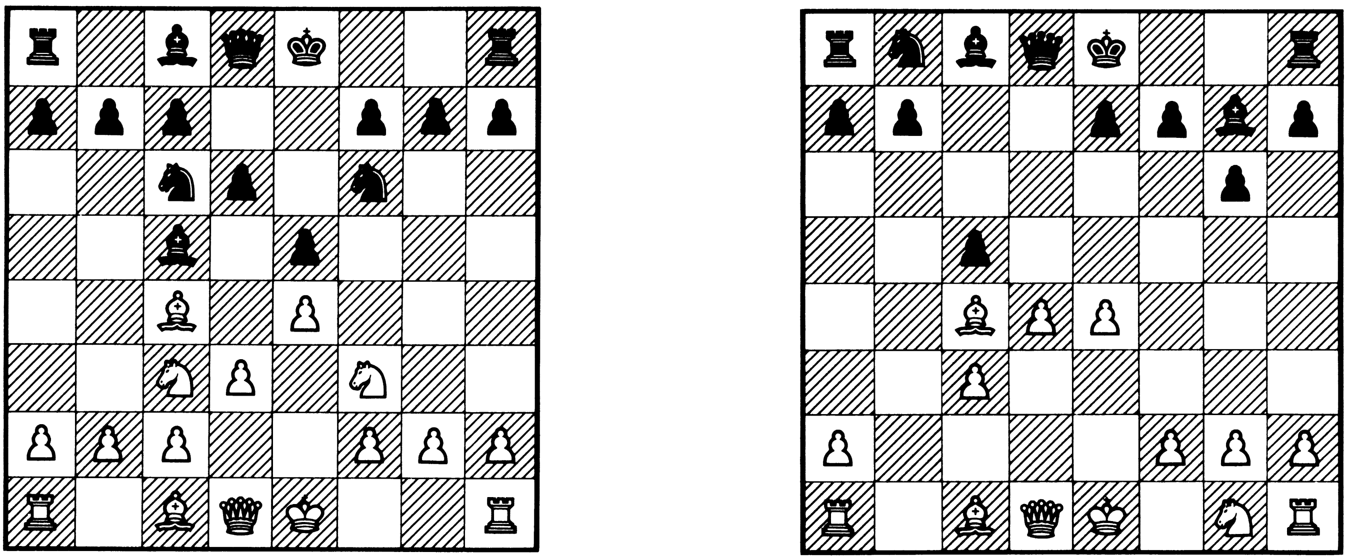

Fig. 6 Positions of low and high 'tension' (from Alexander, 1972)

Table 3 Evaluative comments on positions (comments on moves are not shown here) corresponding to various combinations of expected utility, u, and game theoretic value, v

	$v = -1$	$v = 0$	$v = +1$
1. $u = 0$	s is virtually impossible (because of the unlikelihood that u should be identically zero).	s is a certain draw ('dead draw').	s is virtually impossible (because of the unlikelihood that u should be identically zero).
2. $u = -1$	s is a certain win for Black.	s is impossible.	s is impossible.
3. $u = +1$	s is impossible.	s is impossible.	s is a certain win for White.
4. $u \approx 0$	White has excellent drawing chances. Black needs accuracy to ensure his win.	s is a balanced position.	Black has excellent drawing chances. White needs accuracy to ensure his win.
5. $u \approx -1$	An easy win for Black (decisive advantage).	Black has excellent winning chances. White needs accuracy to make sure of the draw.	White has a theoretical win but is almost bound to lose.
6. $u \approx +1$	Black has a theoretical win but is almost bound to lose.	White has excellent winning chances. Black needs great accuracy to make sure of the draw.	An easy win for White (decisive advantage).
7. $-1 \ll u < 0$	Black has a mildly difficult win.	Black has a slight advantage. White needs care to make sure of the draw.	White needs extreme accuracy to make sure of his win (a very difficult win for White).
8. $+1 \gg u > 0$	Black needs extreme accuracy to make sure of his win (a very difficult win for Black).	White has a slight advantage. Black needs care to make sure of the draw.	White has a mildly difficult win.
9. $-1 < u \ll 0$	Black has a clear advantage.	Black has good winning chances. White needs accuracy to make sure of the draw.	White has a theoretical win but is likely to lose.
10. $+1 > u \gg 0$	Black has a theoretical win but is likely to lose.	White has good winning chances. Black needs accuracy to make sure of the draw.	White has a clear advantage.

difficult and complex struggle in which a draw is not very likely.'

A simple way of capturing the spirit of Alexander's definition within the framework of our theory is to use the *weighted mean square* of the terminal values of the tree rooted in s, i.e.

$$\mathrm{var}(v_t) = \sum_{t \in T} p_t v_t^2$$

where T is the set of terminal positions and p_t is the probability of arriving at the tth member of this set starting at s. A value of unity corresponds to maximal tension and a zero value to minimal tension (the latter can only be attained by a 'dead draw'). The tension of the root node of Fig. 3 is estimated by this method at 0·559. Referring to comment No. (3) above we assign this root node to Case 2 rather than to Case 1 of the category 'balanced'. Note that although 'tension' is calculated from game-theoretic values, v_t, use is made of the u_ts in the calculation of the probabilities, p_t, and hence the measure is affected by variation of the merit parameters M_W and M_B. As soon as we postulate greater playing strength on the part of White some of the tension of the position is reduced. The tension of node A in Fig. 4 is only 0·024, reflecting the fact that the Black is almost certain to steer play into the 'dead draw' sub-tree.

Note that $\sum_{t \in T} p_t v_t^2$ is equal simply to the probability of a non-drawn outcome. But we have preferred to formulate the expression explicitly as a variance, since in realistic cases game-theoretic values are not likely to be available, or calculable in practice. The approximating formula $\sum_{t \in U} p_t y_t^2$ may then prove useful, where the y_ts have been assigned by some evaluation function (or by human intuition) to the members of U, the set of states on the lookahead horizon.

Summary of ideas so far

We have extended the strict game-theoretic model of chess, which assigns to board positions only three values: +1, 0 and −1. A good model should do justice to the profusion of chess commentators' evaluations. Specimen evaluative comments have been displayed as benchmarks against which to assess the extended theory. We have illustrated with worked examples a simple model based on the notions of utility and statistical expectation. Our model finds no particular difficulty in explicating the specimen evaluative comments. It also reduces to the game-theoretic model in the special case of error-free play.

Application to computer chess

A worthwhile study would be to explore parts of a non-trivial sub-game of chess of which complete game-theoretic knowledge exists, as in K + N versus K + R (Bratko and Michie, 1980; Kopec and Niblett, 1980). The program's own comments on sample end-game play could be compared with the intuitions of experienced players.

A more satisfying use of the model would be for generating computer play. The procedure exhibited earlier for calculating scores by backwards iteration from the terminal nodes of the game tree was derived from classical decision theory. State of the art tournament programs also use 'backed-up' scores and they base move-selection on them. But they follow the minimax model. Might not such programs benefit from using expected utilities rather than minimax? After all, the near-universal adoption of the minimax rule in computer game-playing rests on no theoretical foundation.*

Minimaxing for purposes of computer play originated from the intuitions of Shannon (1950) and Turing (1953), neither of whom offered explicit justification. The rule's empirical record has done the rest.

When lookahead is conducted to the end of the game, the validity of minimaxing rests on its built-in guarantee against selecting a game-theoretically 'losing move'. The reader can remind himself of this by inspecting Fig. 2: the constant-value subtree rooted in a given node defines a value-preserving strategy for all play ensuing from that node, provided that we

*Beale and Bratho have, however, recently established a sufficient condition (in *Advances in Computer Chess*, Vol. 3, forthcoming. Pergamon).

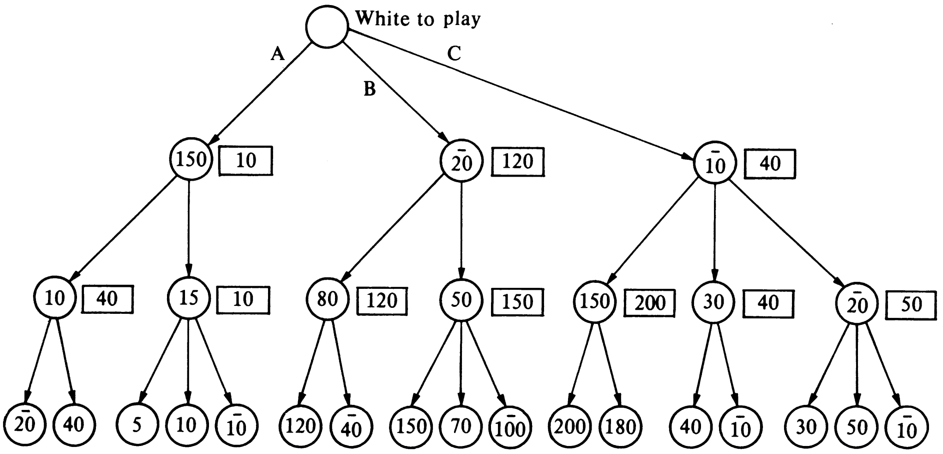

Fig. 7. Terminal positions are shown as boxes in this lookahead tree, in which the nodes are marked with 'face scores' (bars over negative). Boxed figures are values backed up from the lookahead horizon. If move-selection were decided by face scores then move A would be chosen, but if backed-up scores then move B. What is the rationale for B?

have some rule for tie-breaking among a node's equivalued successors. But Fig. 3 shows that against a *fallible* opponent, this concept of 'validity' is harmful, for here a 'losing move' is Black's decision-theoretically best choice.

A further difficulty arises when computational resources do not permit complete lookahead. For this Shannon and Turing independently prescribed that the program should look ahead to some limited depth, and then assign to the terminal nodes of the lookahead tree estimates of their game-theoretic values supplied by an 'evaluation function'—typically a linear combination of terms corresponding to measurable features of the position (piece advantage, mobility etc.). These scores are then backed up by the minimax rule to the current position's immediate successors, in place of the desired but inaccessible game-theoretic values. The rule of play selects the successor with the most favourable backed-up score (move B in **Fig. 7**).

Except in the (unrealistic and uninteresting) case that the evaluation function approximates the game-theoretic value so closely that the decisions given by the rule are invariant with respect to the depth of lookahead, this rule has lacked formal justification.* We are thus free to attribute its empirical success to the fact that it can be regarded as an approximation to a decision-theoretically correct rule of the kind developed earlier. Note that the larger are the values of M_W and M_B, the closer is the approximation; in the limit the two models coincide.

The new model raises a point of particular relevance to the present situation in computer chess. Fast, partly parallel, special-purpose chess machines have recently been developed and interfaced to powerful computers (see for example Moussouris *et al.*, 1979). Chess programs of conventional type interfaced to such machines become capable of searching to an average depth in excess of 9-ply, almost twice that attained by chess masters (see de Groot, 1965; note that we are speaking of the average length of the longest branch of the lookahead tree). To give such a machine the best chances it should be endowed with a 'hunger for complexity'. The idea must be continually to drive for high tension positions avoiding simplifying exchanges where possible. In this way cognitive strain on the human player is intensified by the need for vigilance against tactical traps which may lie ⩾9-ply deep. Such a policy calls for a model incorporating opponent fallibility.

Concluding remarks

An objection to the theory here developed is that the opponent model is arbitrary. Two comments are in order.

(1) It is of no theoretical consequence what particular opponent model is used for illustration, provided only that it has the right overall properties. The reader is free to use the theory with any opponent model he pleases.

(2) No choice of opponent model is as arbitrary, or as inflexible, as minimax. Moreover, even on the basis of complete lookahead to the end of the game, minimax back-up does not yield the best strategy against a fallible opponent.

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References

Alexander, C. H. O'D. (1972). *Fischer v. Spassky, Reykjavik* 1972. Penguin, Harmondsworth.

Bratko, I. and Michie, D. (1980). A representation for pattern-knowledge in chess endgames, in *Advances in Computer Chess*, edited by M. R. B. Clarke, Vol. 2, pp. 31–54. Edinburgh University Press, Edinburgh.

Good, I. J. (1968). A five-year plan for automatic chess, in *Machine Intelligence*, edited by E. Dale and D. Michie, Vol. 2, pp. 89–118. Edinburgh University Press, Edinburgh.

Good, I. J. (1977). Dynamic probability, computers and the measurement of knowledge, *Machine Intelligence*, edited by E. W. Elcock and D. Michie, Vol. 8, pp. 139–150. Edinburgh University Press, Edinburgh.

Groot, A. de (1965). *Thought and Choice in Chess*, edited by G. W. Baylor. Mouton, The Hague. [English translation with additions of the Dutch 1946 version.]

Kopec, D. and Niblett, T. (1980). How hard is the play of the King–Rook–King–Knight ending?, in *Advances in Computer Chess*, edited by M. R. B. Clarke, Vol. 2, pp. 57–81. Edinburgh University Press, Edinburgh.

Moussouris, J., Holloway, J. and Greenblatt, J. (1979). CHEOPS: A chess-oriented processing system, in *Machine Intelligence*, edited by J. E. Hayes, D. Michie and L. I. Mikulich, Vol. 9, pp. 351–360. Horwood, Chichester; Halsted, New York.

Shannon, C. E. (1950). Programming a computer for playing chess, *Philosophical Magazine*, Vol. 41, pp. 256–275.

Turing, A. M. (1953). Digital computers applied to games, in *Faster than Thought*, edited by B. W. Bowden, pp. 286–310. Pitman, London.

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