

The Number of States of a Dynamic Storage Allocation System

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The numbers of occupancy states of linear or circular arenas for dynamic storage allocation of immovable blocks of arbitrary size are expressed simply in terms of Fibonacci numbers, as are the numbers of equivalence classes of states induced by reflective and rotational symmetries.

1. TERMINOLOGY

In the standard model for dynamic storage allocation with immovable blocks,¹ a *state* is a partition of an *arena* of length n into *blocks* of positive integral lengths, where each block is marked either *busy* or *idle*. Betteridge² and Benes³ considered the state model in analysing the behavior of allocation policies under Poisson inputs and were able to get exact results for small arenas; Robson's minimax results⁴ provide qualitative insight about the effectiveness of certain allocation policies in large arenas. Knuth¹ and Reeves⁵ have made preliminary progress in the statistical mechanics of large arenas. This paper considers the sizes of various pertinent state spaces.

It has proved convenient to study circular arenas, where all cells are inherently similar.⁵ Indeed, the artificial construct of a small circular arena may, by avoiding end effects, model the behavior of a large arena better than would a similarly small linear arena. We may reduce the size of models still further by assuming homogeneity and treating circular states that are rotated images of each other as identical. In the same way, Benes profitably combined mirror-image states of linear arenas.³ Thus it will be useful to understand equivalence classes of states under various automorphism groups—reflection, rotation, or both.

The internal layout of a contiguous run of idle blocks does not matter; for convenience we shall let all idle blocks have length 1. In mathematical terms, a state is an ordered partition of n in which two species of 1 may occur.

States invariant under an automorphism—in particular reflection—will be called *symmetric*. States not known to be symmetric under any but the identity automorphism will be called *unrestricted*.

In making proofs, it is helpful to recognize certain special states called *closed*. The closed states of a linear arena are those states that have a busy block at a given end. In a circular arena, some cell boundary may be chosen as a *cut* or origin for numbering the cells. Then the closed states of the circular arena are those states in which a block spans the cut. The spanning block, which must be busy, is called the *cut block*.

Any state of an arena of length n will be called an n -state for short, and a block of length k will be called a k -block. The term n -state may be further qualified as *linear* or *circular* according to the kind of arena. A rotation of a circular arena through j cell positions will be

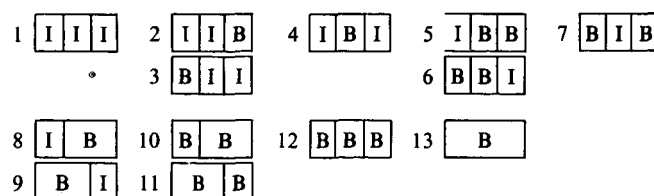


Figure 1. The 13 unrestricted linear 3-states. Busy blocks are labelled B, idle blocks I. Unpaired states are the five states symmetric under reflection. Pairings show the nine equivalence classes under reflection. States 2, 5, 7, 8, 10, 11, 12, 13 are the eight states closed on the right.

called a j -cell rotation. See Fig. 1 for examples of much of this terminology.

The Fibonacci numbers, 0, 1, 1, 2, 3, ..., which appear as a pleasant surprise in this study, are denoted $F_0, F_1, F_2, F_3, F_4, \dots$.

2. NUMBERS OF STATES

The number of n -states of linear and circular arenas will be designated L_n and C_n . Restriction to closed states will be signified by a superscript c : L_n^c and C_n^c ; symmetry reflection by an overbar: \bar{L}_n , \bar{L}_n^c , \bar{C}_n , and \bar{C}_n^c ; and symmetry under j -cell rotation by a second subscript: $C_{n,j}$ and $\bar{C}_{n,j}$. Most of these quantities satisfy simple difference equations, for which the initial conditions can be read off from Table 1.

Table 1. Unrestricted and reflectively symmetric states of 1-cell and 2-cell arenas. A cut 2-block is shown as two open-ended busy blocks. From this table may be read the initial values for the difference equations of section 2. For example, the four boxes under $n = 2$ in rows under the heading linear states contain 5 states; hence $L_2 = 5$

	$n = 1$	$n = 2$
	Unrestricted 1-states	Unrestricted 2-states
	Symmetric 1-states	Symmetric 2-states
Circular states C_n		
Linear states L_n	I	I I B I
Closed linear L_n^c	B	B B B I B
Closed circular C_n^c		B B

2.1. Unrestricted states

Consider the number L_n of linear n -states and the number L_n^c of closed linear n -states. A closed n -state may be made either by appending a busy 1-block to an unrestricted $n-1$ -state or by extending the closing busy block of a closed $n-1$ -state:

$$L_n^c = L_{n-1} + L_{n-1}^c \quad (1)$$

An n -state may be closed or may be made by appending an idle block to an unrestricted $n-1$ -state:

$$L_n = L_n^c + L_{n-1} \quad (2)$$

Recalling the Fibonacci difference equation,

$$F_n = F_{n-1} + F_{n-2}$$

we see that Eqns (1) and (2) are satisfied by

$$L_n = F_{2n+1} \quad (3)$$

$$L_n^c = F_{2n} \quad (4)$$

These solutions satisfy the initial conditions, $L_1 = 2 = F_3$ and $L_1^c = 1 = F_2$. Equation (3) was first given by Betteridge.²

Turning next to circular arenas, observe that a closed circular n -state may be made either by extending the closing block of a closed linear $n-1$ -state into a new cell added at the other end of the arena to make a cut block, or by inserting a new cell into the cut block of a closed circular $n-1$ -state:

$$C_n^c = L_{n-1}^c + C_{n-1}^c$$

With the appropriate initial condition, $C_2^c = 1$, this difference equation is satisfied by

$$C_n^c = F_{2n-1} - 1 \quad (5)$$

A circular n -state is either a closed circular n -state or is identical to a linear n -state:

$$C_n = C_n^c + L_n$$

Substituting from Eqns (3) and (5), we find that

$$C_n = F_{2n+1} + F_{2n-1} - 1 \quad (6)$$

2.2. Reflectively symmetric states

A closed symmetric n -state of a linear arena may be made by extending the closing block(s) of a closed symmetric linear $n-2$ -state by one cell at each end or by adding a busy 1-block at each end of a symmetric linear n -state.

$$\bar{L}_n^c = \bar{L}_{n-2}^c + \bar{L}_{n-2} \quad (7)$$

A symmetric linear n -state either is a closed symmetric linear n -state or may be made by adding an idle block at each end of a symmetric linear $n-2$ -state:

$$\bar{L}_n = \bar{L}_n^c + \bar{L}_{n-2} \quad (8)$$

Recurrences (7) and (8) are satisfied by

$$\bar{L}_n^c = F_{n+1} \quad (9)$$

$$\bar{L}_n = F_{n+2} \quad (10)$$

Since $\bar{L}_1^c = 1 = F_2$, $\bar{L}_2^c = 2 = F_3$, $\bar{L}_1 = 2 = F_3$, and $\bar{L}_2 = 3 = F_4$, these solutions start right. According to Eqn (10), we have $\bar{L}_0 = 1$, a convention that will be useful later on.

The interesting automorphisms of circular arenas are rotations (cyclic groups), reflections, and both together (dihedral groups). We first consider reflection of a cut circular arena. A closed symmetric circular n -state may be made either by adding a cut 2-block to the ends of a symmetric linear $n-2$ state, or by extending the cut block of a closed symmetric circular $n-2$ -state with two new cells:

$$\bar{C}_n^c = \bar{L}_{n-2} + \bar{C}_{n-2}^c$$

The solution that satisfies the initial conditions, $\bar{C}_1^c = 0$ and $\bar{C}_2^c = 1$, is

$$\bar{C}_n^c = F_{n+1} - 1 \quad (11)$$

A symmetric circular n -state is identical either to a symmetric linear n -state or to a closed symmetric circular n -state:

$$\bar{C}_n = \bar{L}_n + \bar{C}_n^c$$

whence, from Eqn (10) and (11),

$$\bar{C}_n = F_{n+3} - 1 \quad (12)$$

This solution extends correctly down to $n=1$ and $n=2$.

2.3. Circular states under rotation

Under repeated j -cell rotations of a circular n -state, with $1 \leq j \leq n$, any given cell visits sites spaced at integer multiples of (n, j) , where (n, j) denotes the greatest common divisor of j and n . Thus the number of circular n -states invariant under such a rotation is

$$C_{n,j} = C_{(n,j)} \quad (13)$$

Finally, consider the combined effect of a reflection followed by a j -cell rotation, $1 \leq j < n$, with this permutation scheme:

$$\begin{array}{cccccccccc} 1 & 2 & \dots & j-1 & j & j+1 & j+2 & \dots & n-1 & n \\ j & j-1 & \dots & 2 & 1 & n & n-1 & \dots & j+2 & j+1 \end{array}$$

An n -state invariant under such a permutation partitions into two reflectively symmetric substates of j and $n-j$ cells. If the n -state is closed, then the substates must be also, and each closed j -substate must either be one of the \bar{C}_j^c reflectively symmetric closed circular j -states or be wholly contained within a larger busy block. Thus

$$\bar{C}_{n,j}^c = (\bar{C}_j^c + 1)(\bar{C}_{n-j}^c + 1) - 1$$

where the term -1 accounts for the impossibility of both substates being spanned by one endless busy block. Non-closed n -states symmetric under reflection and j -cell rotation must be the product of reflectively symmetric linear j - and $n-j$ -states. Adding these to the closed configurations, we finally find the number of n -states symmetric under reflection and j -cell rotation to be

$$\bar{C}_{n,j} = (\bar{C}_j^c + 1)(\bar{C}_{n-j}^c + 1) - 1 + \bar{L}_j \bar{L}_{n-j}$$

which becomes, upon substitution from Eqns (10) and (11),

$$\bar{C}_{n,j} = F_{j+1} F_{n-j+1} + F_{j+2} F_{n-j+2} - 1 \quad (14)$$

Though Eqn (14) has been derived only for $1 \leq j < n$, it holds for $j=n$ as well, in which case it reduces to Eqn (12).

2.4. Numbers of states: summary

The following theorem summarizes the results of Section 2.

Theorem 1. The numbers of n -states of linear and circular arenas invariant under (1) identity, (2) reflection, (3) j -cell rotation, and (4) reflection then j -cell rotation are as given in Table 2.

Table 2. Numbers of n -states invariant under various automorphisms. F_1, F_2, \dots are Fibonacci numbers, 1, 1, 2, 3, ...

Arena type	Auto-morphism	Symbol and number	
Linear	Identity	$L_n = F_{2n+1}$	(3)
	Reflection	$\bar{L}_n = F_{n+2}$	(10)
Circular	Identity	$C_n = F_{2n+1} + F_{2n-1} - 1$	(6)
	Reflection	$\bar{C}_n = F_{n+3} - 1$	(12)
	Rotation	$C_{n,j} = C_{(n,j)}, 1 \leq j \leq n$	(13)
	Reflection and rotation	$\bar{C}_{n,j} = F_{j+1}F_{n-j+1} + F_{j+2}F_{n-j+2} - 1$	(14)

Some numerical values of these quantities may be found in Table 4.

3. EQUIVALENCE CLASSES

All the results of this section depend on the following.

Counting theorem.⁶ The number of equivalence classes induced on a finite set of points by an automorphism group G is

$$\frac{1}{|G|} \sum_{g \in G} f(g)$$

where $|G|$ is the order of G and $f(g)$ is the number of points fixed under the action of group element g .

In the present instance, the points are n -states and the group G comprises reflections, rotations, or both.

Script letters designate the number of equivalence classes under the various automorphism groups:

- \mathcal{L}_n linear arena under identity
- $\bar{\mathcal{L}}_n$ linear arena under reflection
- \mathcal{C}_n circular arena under identity
- $\bar{\mathcal{C}}_n$ circular arena under reflection
- \mathcal{C}_n^o circular arena under rotation
- $\bar{\mathcal{C}}_n^o$ circular arena under reflection and rotation

Trivially,

$$\mathcal{L}_n = L_n \quad (15)$$

$$\mathcal{C}_n = C_n \quad (16)$$

3.1. Equivalence classes under reflection

Each of the L_n states of a linear arena has a distinct mirror image, except for the \bar{L}_n symmetric states. Thus,

by counting each state once and each symmetric state a second time, we count two representatives of each equivalence class. Hence the number of equivalence classes is

$$\bar{\mathcal{L}}_n = \frac{1}{2}(L_n + \bar{L}_n) = \frac{1}{2}(F_{2n+1} + F_{n+2}) \quad (17)$$

This formula illustrates the counting theorem: there are two group elements, identity and reflection, under which L_n and \bar{L}_n states are invariant, respectively.

By the same reasoning it follows that the number of equivalence classes of states of a circular arena under reflection is

$$\bar{\mathcal{C}}_n = \frac{1}{2}(C_n + \bar{C}_n)$$

By Theorem 1,

$$\bar{\mathcal{C}}_n = \frac{1}{2}(F_{2n+1} + F_{2n-1} + F_{n+3} - 2) \quad (18)$$

3.2. Equivalence classes under rotation

Again from the counting theorem, together with Eqn (13), we obtain the number of equivalence classes under the group of all j -cell rotations, $1 \leq j \leq n$:

$$\mathcal{C}_n^o = \frac{1}{n} \sum_{j=1}^n C_{n,j} = \frac{1}{n} \sum_{j=1}^n C_{(n,j)} = \frac{1}{n} \sum_{d|n} \phi(n/d) C_d \quad (19)$$

Here $\phi(n)$ is Euler's function: $\phi(1) = 1$, otherwise $\phi(n)$ is the number of positive integers j , $1 \leq j < n$, such that $(n, j) = 1$. The author knows of no closed form for Eqn (19).

The counting theorem gives the number of equivalence classes under the full dihedral group of rotations and reflections as

$$\bar{\mathcal{C}}_n^o = \frac{1}{2n} \left[\sum_{j=1}^n C_{n,j} + \sum_{j=1}^n \bar{C}_{n,j} \right]$$

Here the first sum covers the nonreflecting elements of the dihedral group and the second covers the reflecting elements. Substituting from Eqns (14) and (19) we obtain

$$\bar{\mathcal{C}}_n^o = \frac{1}{2n} \left[\sum_{d|n} \phi(n/d) C_d + \sum_{j=1}^n F_{j+1} F_{n-j+1} + \sum_{j=1}^n F_{j+2} F_{n-j+2} - n \right] \quad (20)$$

It may be shown (see Appendix) that

$$G_n = \sum_{k=0}^n F_k F_{n-k} = \frac{1}{2}(n F_{n+1} + n F_{n-1} - F_n) \quad (21)$$

Simple changes of variable in the second and third sums in Eqn (20), which we call V_n , yield

$$\begin{aligned} V_n &= \sum_{j=1}^n F_{j+1} F_{n-j+1} + \sum_{j=1}^n F_{j+2} F_{n-j+2} \\ &= G_{n+2} - F_0 F_{n+2} - F_1 F_{n+1} - F_{n+2} F_0 \\ &\quad + G_{n+4} - F_0 F_{n+4} - F_1 F_{n+3} \\ &\quad - F_2 F_{n+2} - F_{n+3} F_1 - F_{n+4} F_0 \end{aligned}$$

Since $F_0 = 0$, $F_1 = 1$, and $F_{n+1} + F_{n+2} = F_{n+3}$,

$$V_n = G_{n+2} + G_{n+4} - 3F_{n+3}$$

which becomes, upon substituting for G_n from Eqn (21) and simplifying with Fibonacci identities,

$$V_n = n F_{n+3}$$

Finally, substituting V_n back for the sums in Eqn (20) yields

$$\bar{C}_n^o = \frac{1}{2n} \left[\sum_{d|n} \phi(n/d) C_d + n(F_{n+3} - 1) \right] \quad (22)$$

3.3. Equivalence classes: summary

Theorem 2. The numbers of equivalence classes induced upon n -states of linear and circular arenas by (1) identity, (2) reflection, (3) rotation, and (4) reflection and/or rotation are as given in Table 3.

Table 4 gives some numerical values for these quantities. The asymptotic formulas there follow from linear relations among various solutions of the Fibonacci recurrence,

$$F_n = \frac{u^n - v^n}{\sqrt{5}}$$

$$F_{n+1} + F_{n-1} = u^n + v^n$$

$$u_n = \left(\frac{1 + \sqrt{5}}{2} \right)^n, \quad v_n = \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

and from noticing that the $d = n$ terms dominate the sums in Eqns (19 and (22).

Table 3. Numbers of equivalence classes of n -states under various automorphism groups

Arena type	Auto-morphisms	Symbol and number	
Linear	Identity	$\bar{L}_n = L_n = F_{2n+1}$	(15)
	Reflection	$\bar{E}_n = \frac{1}{2}(F_{2n+1} + F_{n+2})$	(17)
Circular	Identity	$C_n = C_n = F_{2n+1} + F_{2n-1} - 1$	(16)
	Reflection	$\bar{C}_n = \frac{1}{2}(F_{2n+1} + F_{2n-1} + F_{n+3} - 2)$	(18)
	Rotation	$C_n^o = \frac{1}{n} \sum_{d n} \phi(n/d) C_d$	(19)
	Reflection and rotation	$\bar{C}_n^o = \frac{1}{2n} \left[\sum_{d n} \phi(n/d) C_d + n(F_{n+3} - 1) \right]$	(22)

Table 4. Numbers of states and equivalence classes, with asymptotic formulas. In the asymptotic formulas $u = (1 + \sqrt{5})/2$

n	$L_n = \mathcal{L}_n$	\mathcal{L}_n	\mathcal{E}_n	$C_n = \mathcal{C}_n$	\mathcal{C}_n	\mathcal{C}_n^o	\mathcal{C}_n^o	
1	2	2	2	2	2	2	2	
2	5	3	4	6	4	5	4	
3	13	5	9	17	7	12	7	
4	34	8	21	46	12	29	13	
5	89	13	51	122	20	71	23	
6	233	21	127	321	33	177	46	
7	610	34	322	842	54	448	88	
8	1597	55	826	2206	88	1147	186	
9	4181	89	2135	5777	143	2960	395	
10	10946	144	5545	15126	232	7679	880	
11	28657	233	14445	39602	376	19989	1989	
12	75025	377	37701	103681	609	52145	4644	
13	196418	610	98514	271442	986	136214	10934	
14	514229	987	257608	710646	1596	356121	26210	
15	1346269	1597	673933	1860497	2583	931540	63319	
16	3524578	2584	1763581	4870846	4180	2437513	154377	
17	9227465	4181	4615823	12752042	6764	6379403	378443	
18	24157817	6765	12082291	33385281	10945	16698113	933022	
19	63245986	10946	31628466	87403802	17710	43710756	2308956	
20	165580141	17711	82798926	228826126	28656	114427391	5735371	
$n \rightarrow \infty$	$u^{2n+1}/\sqrt{5}$	$u^{n+2}/\sqrt{5}$	$u^{2n+1}/(2\sqrt{5})$	u^{2n}	$u^{n+3}/\sqrt{5}$	$u^{2n}/2$	u^{2n}/n	$u^{2n}/2n$

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APPENDIX

Generating functions may be used to prove Eqn (21). Let

$$F(x) = \sum_{n=0}^{\infty} F_n x^n.$$

From the definition of the Fibonacci numbers it follows that

$$F(x)(1 - x - x^2) = x.$$

Evidently

$$G(x) = \sum_{n=0}^{\infty} G_n x^n = \sum_{n=0}^{\infty} x^n \sum_{k=0}^n F_k F_{n-k} = F(x)^2,$$

whence

$$G(x)(1 - x - x^2)^2 = x^2.$$

As its characteristic equation has the same roots as that for F_n , but repeated, the implied recurrence for G_n must have homogenous solutions of the form

$$G_n = A_1 F_n + A_2 F_{n+1} + A_3 n F_n + A_4 n F_{n+1}.$$

Equation (21) follows by equating coefficients.