Correspondence

Further Remarks on the Trivial Nature of the Tower of Hanoi Problem

Dear Sir,

M. C. Er's letter[1] in reply to mine[2] raises a number of points which call for further answers.

A brief history of the matter is this. When I read Er's original paper[3] I was surprised that anyone could spend six pages on such a problem. I wrote, pointing out how simple the problem is seen to be if we tabulate the *status* of the discs after each move. Indeed the whole problem can be summarised by one congruence: using zero-based disc and peg numbering, the peg upon which disc j resides after move x is given by:

$$P_j \equiv (-1)^j \left\lfloor \frac{x+2^j}{2^{j+1}} \right\rfloor \pmod 3$$

where $\lfloor\ \rfloor$ represents the floor function. It is a trivial problem, so I kept the letter brief. Now in his reply Er has gone on the attack, and has issued a number of challenges I wish to take up.

Some small points first. Er writes, 'The reason [Heard] says that the Tower of Hanoi is a trivial problem is because he has seen a solution to it in the literature.' No. I didn't have to – I knew that to solve the problem for discs 0, 1, ..., n we need to solve the problem for discs 0, 1, ..., $n-1$. (Indeed that is why the problem lends itself so well to a recursive solution.) I started with the one disc case and built up a tabulation from there – in fact it was on the back of a listing during a tea-break.

In an extraordinary statement, Er says: 'At any rate, [Heard's] equation cannot be used efficiently in an algorithm. His equation involves exponentiation and has to be used costly [*sic*] for calculating the status of each disc per cycle of move.' The congruence I quote was not intended to be used in an algorithm, but as a summary of the solution to the problem in well-known mathematical language. If an algorithm *were* based on it, presumably the constants for each disc would be set up before entering the per-move loop. This would involve successive multiplication by two, and not exponentiation. Multiplication by two is not known to be particularly inefficient.

A sketch of a proof. The rules of the Tower of Hanoi are such that we cannot move disc n until we have transferred discs 0, 1, ..., $n-1$ from peg 0 on to one of the other pegs – peg Q, say. We can then (and only then) move disc n on to the third peg – peg R. Lastly the discs 0, 1, ..., $n-1$ must be transferred from peg Q to peg R. We begin this last transfer with disc n on peg R and disc $n+1$ on peg 0, so these pegs can be used freely. We can therefore transfer the discs 0, 1, ..., $n-1$ from disc Q to disc R using the same moves that transferred them from peg 0 to peg Q, with appropriate renumbering of the pegs.

Thus, if we know the optimal solution for discs 0, 1, ..., $n-1$, we can find the optimal solution for discs 0, 1, ..., n. But the optimal solution for disc 0 is simply to move that disc from peg 0 to peg 1. Thus, by mathematical induction, we know all optimal solutions (apart from a possible renumbering of the pegs).

Observe first that the proof by mathematical induction is very similar to the recursive algorithm for the Tower of Hanoi problem. In fact, mathematical induction and recursion are in general closely related. Secondly, observe that the tabulation, the congruence, and the actual positions of the discs are equivalent: any one can be derived from any other.

Consequences

(1) The analogy with Gray code is clear. The congruence above becomes the congruence for Gray code if we take the modulus to base 2 instead of base 3. (The $(-1)^j$ becomes redundant.)

(2) Er says: 'It is not obvious that Property 5 follows from [Heard's] equation.' Er's Property 5 is that the optimal solution for discs 0, 1, ..., $n-1$ takes $2^n - 1$ steps.

The optimal solution occurs the first time that P_{n-1} is non-zero, and all the P_{n-1}, P_{n-2}, ..., P_0 have the same value. To find the x for which this occurs we examine the argument of the floor function and get the following set of inequalities:

$$2^{n-1} \leqslant x < 3.2^{n-1} \text{ (from disc } n-1)$$

$$3.2^{n-2} \leqslant x < 5.2^{n-2} \text{ (from disc } n-2)$$

$$7.2^{n-3} \leqslant x < 9.2^{n-3} \text{ etc.}$$

with solution converging to $2^n - 1$. This solution is even more obvious from the tabulation.

(3) The directions of the moves of the discs are obvious. x increases by 1 for each move, therefore the floor function can only either increase by 1 or remain constant. The $(-1)^j$ ensures that even discs go through the cycle 0, 1, 2, 0, 1, 2, ... and odd discs go through the cycle 0, 2, 1, 0, 2, 1, ...

(4) The much-vaunted theorem of Er's letter states that with 1-based disc numbering, the number of 'the disc to be moved in step x is precisely the position index of the rightmost 1 in the binary numeral of x'. This follows immediately from the congruence if we observe that the value of the floor function can change (and hence disc j can be moved) only if x is an odd multiple of 2^j.

Er says this theorem 'makes the Tower of Hanoi problem really trivial'. I can but agree, but I would add that there is no new insight here. The theorem itself can be derived trivially.

Yours faithfully

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The Tower of Hanoi Problem – A Further Reply

Dear Sir,

Heard's further letter, in response to my correspondence,[8] reveals many weaknesses in his arguments which I have an obligation to point out. I shall deal with them in what follows.

Heard's repeated tongue is that the Tower of Hanoi is a trivial problem because he has been able to solve the problem and derive a congruence equation for it. Although Heard has denied that there was any need to see a solution to it in the literature before deriving an equation for it, the very fact remains that he had seen at least my analysis of the problem.[7] In a long series of experiments conducted by Luger and his colleagues,[13–16] and also others,[9] their conclusions did not support Heard's subjective judgement that the Tower of Hanoi is a trivial problem. For example, of the 58 subjects, all of them college-educated adults, who participated in an experiment,[13] 89.66% of them did not solve the Tower of Hanoi problem in the minimum number of steps on the first attempt. If Heard wishes to challenge the validity of the conclusions, he is urged to repeat the experiments carried out by Luger and then submit his findings to a refereed journal for publication.

On close examination of Heard's arguments, I find that he is constantly confused with whether a problem is a trivial problem or a representation makes it so. Since Heard is proud of mathematical language, I shall illustrate the difference using elementary arithmetic. To calculate the quotient of dividing a number into another number, Heard would no doubt agree, it is a trivial exercise. But suppose these two numbers are represented in Roman numerals, it is doubtful that Heard would find it trivial without converting them to Arabic representation or other representations, especially when these two numbers are very big. The point is that the elementary arithmetic is not a trivial problem but Arabic representation of numbers makes it so. Similarly, the Tower of Hanoi is not a trivial problem but a suitable representation makes it so. This is precisely the theme of my paper,[7] and Heard has totally missed the point.

Perhaps what prompted Heard to initiate his attack[11] was that I had spent 6 pages[7] on dealing with the Tower of Hanoi Problem. A quick survey of previous publications on the topic reveals that Amarel[1] spent 41 pages, Barnard[3] 18 pages, Cohen[5] 13 pages, Gagne and Smith[9] 7 pages, Hormann[12] 20 pages, Luger and his colleagues[13–16] a total of 45 pages, Simon[22] 21 pages and Wood[24] 8 pages. If a paper of 6 pages on the Tower of Hanoi problem were too much, Heard should have complained some ten years ago when other longer papers on the same topic appeared.

Heard repeatedly claimed that the whole Tower of Hanoi problem could be summarised by his congruence equation and that all the properties of the Tower of Hanoi problem follow trivially from his equation. Before we are too carried away by his equation, it is worth pointing out that a similar equation has long since been discovered by Lunnon.[17] It is time that we look at the substance of the claims in detail.

According to his equation, Heard always moves all discs from peg 0 to peg 1 when the number of discs is odd, and from peg 0 to peg 2 when the number of discs is even. However, the original statement of the problem[7, 10] is to move all discs from the source peg to another target peg. There is no reason why one must restrict the source peg to peg 0, and the target peg to peg 1 or peg 2, depending on whether the number of discs is odd or even. Also, there is no reason why all even-numbered discs must move along the direction peg 0 → peg 1 → peg 2 → peg 0, and all odd-numbered discs along the counter-direction peg 0 → peg 2 (peg − 1) → peg 1 (peg − 2) → peg 0, irrespective of the number of discs. Clearly, it is necessary to introduce a scheme of renumbering the pegs, and a scheme of redefining the clockwise and the counter-clockwise directions, depending on whether the number of discs is odd or even. So far, both schemes have not been given by Heard, and cannot be derived from his equation. Therefore, his claim '... the congruence, and the actual positions of the discs are equivalent: any one can be derived from any other' simply does not stand. A good counter-example would be to move a tower of 5 discs from peg 1 to peg 0.

Faced with the lack of schemes of renumbering pegs and redefining moving directions, it is no wonder that Heard has great difficulty in proving his equation correct. His so-called 'sketch of proof' involves too many hand-wavings and cannot be regarded as a proof at all. For example, he introduced peg Q and peg R in the proof without telling the readers which one was peg 1 and which one was peg 2. How could he be sure that the resulting sequence of disc moves was optimal? Another example, on what ground could he assert: 'the optimal solution for disc 0 is simply to move that disc from peg 0 to peg 1'? Why then moving disc 0 from peg 0 to peg 2 cannot be the optimal solution! After all these hand-wavings, the correctness of his equation has not been proven!

In an attempt to show that Property 5[7] follows trivially from his equation,[11] Heard suggested: 'the optimal solution occurs the first time P_{n-1} is non-zero, and all the P_{n-1}, P_{n-2}, ..., P_0 have the same value'. If the purpose is to move a tower of 5 discs from peg 0 to peg 2, why does Heard believe that the optimal solution has been reached when all discs are sitting on peg 1? Moreover, his set of inequalities does not uniquely converge to $2^n - 1$; another valid solution is 2^n. His claim of 'converging' is simply not valid.

Having seen so many weaknesses associated with Heard's equation, it is indeed a wise move that Heard has retreated: 'the congruence... was not intended to be used in an algorithm'.

After seeing the theorem stating the relation between the Tower of Hanoi and the binary numerals,[8] Heard claimed: 'this [theorem] follows immediately from the congruence... the theorem itself can be derived trivially'. The very fact is that Heard did not see it when he wrote his first correspondence,[11] or else he would have mentioned it. The triviality arises because Heard has seen it in my correspondence.[8] Such an incidence, perhaps, tells us more about Heard's character and attitude than anything else. MacCallum[18] reported that he had managed to squeeze an implementation of the theorem into 50 steps for the original TR-57 calculator. I wonder whether or not Heard would remark that it was a trivial exercise.

Finally, it is worth pointing out that the Tower of Hanoi problem had attracted a large number of researchers[2–7, 9–10, 12–19, 21–24] to work on it, and is continuing to do so[1, 20] regardless of Heard's criticism.

Yours faithfully
M. C. ER

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