Correspondence

The Tower of Hanoi – Again

Sir,

M. C. Er's reply[1] to my letter[2] is a mixture of quibbles and personal abuse – a sure sign that someone has lost an argument.

The points he makes are easy to answer. Let us take them in order:

(1) The conclusion to be drawn from the work of Luger[3] is that the Tower of Hanoi is an easy problem when looked at the right way – how else can Er explain the 10.3% of subjects who solved the problem with a minimum number of steps on the **first** try? Like a good cryptic crossword clue it can seem extremely obscure until the right approach is found, and then seem extremely easy. It is this very property which makes the Tower of Hanoi a successful puzzle.

(2) Er claims that I constantly confused 'whether a problem is a trivial problem or whether a representation makes it so'. In the case of the Tower of Hanoi there is a representation – the tabulation of the states of the discs – which is trivially derivable and which makes the problem trivial. The two cases merge. Note that Er has nothing to say about the tabulation approach: it would destroy his case.

I agree with Er that representation approaches are valuable, but I say that not all representations are equally good. In choosing to represent moves rather than status, Er has ignored a simple representation, and one which has such close analogies with modular arithmetic and positional notation that deductions from it are easy and crisp. He has concentrated on moves and represented them as bit strings, for which mathematical apparatus is less well developed.

(3) Er's defence of the space he took to develop simple properties of the Tower of Hanoi is that other writers have taken even more space. This is a pretty poor defence, especially since most of the papers he cites were not attempting to solve the Tower of Hanoi problem, but were using it as an *example* in studying either human or machine problem solving.

(4) Since 0, 1 and 2 are merely tags given to the pegs in an arbitrary way, I lose no generality in always calling the source peg '0' and the peg to which the first move is made '1'. Assigning values to variables is so much the stuff of everyday Computer Science I thought no reader would have trouble with the concept. However, since Er has affected difficulty with renumbering, it is easy to transform the congruence to describe moves from any peg to any other.

If the source peg is s and the target peg is t, we can transform the original Pj to $P'j$ using the linear congruence:

$$P'_j \equiv s+(-1)^{t+1} P_j \pmod 3$$

I prefer the simplicity of renumbering.

(5) In criticising my proof, Er complains that I introduce peg Q and peg R 'without telling readers which one was peg 1 and which one was peg 2'. At that stage of the proof **it does not matter**. I need only to show how an optimal solution for discs 0, 1,..., n can be derived from an optimal solution for discs 0, 1,..., $n-1$.

Er's other point is answered by my remarks in section 4.

(6) Er attacks my derivation of his property 5, firstly by arbitrarily disallowing renumbering, and secondly by stating that the 'set of inequalities does not converge to 2^n-1; another valid solution is 2^n'. Since I said (and he quoted) the optimal solution occurs the *first* time the inequalities have a solution we can rule out 2^n as a solution.

(7) Er attacks my character and attitude because I state that his theorem connecting the Tower of Hanoi with the binary numerals follows immediately from the congruence. He states: 'Heard did not see it when he wrote his first correspondence or he would have mentioned it.' Er's theorem states that with 1-based disc numbering the number of 'the disc to be moved in step x is precisely the position index of the rightmost 1 in the binary numeral x'.

But the position index of the rightmost 1 in the binary numeral x is the position index of the digit which changes between the Gray code representation of $x-1$ and of x. Hence Er's theorem is exactly equivalent to the rule given by Martin Gardiner in 1972,[4] and quoted by Er in his original paper.

Why would I quote something already well known? And further, just what did Er think was new in his theorem?

I thank Er for his example of division using Roman numerals. It would be possible to write such an algorithm – it might be a good student exercise in string manipulation – but, because of the existence of arabic numerals, few would bother to write it up as a paper.

Given the existence of recursive and congruence solutions to the Tower of Hanoi problem, one cannot but reach a similar conclusion about Er's original paper.

Yours faithfully

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References

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2. R. J. Heard, Further remarks on the trivial nature of the Tower of Hanoi problem. *The Computer Journal* **28** (5), 543 (1985).
3. G. F. Luger, The use of the state space to record the behavioral effects of subproblems and symmetries in the Tower of Hanoi problem. *International Journal of Man–Machine Studies* **8**, 411–412 (1976).
4. M. Gardiner, Mathematical games: the curious properties of the Gray code and how it can be used to solve puzzles. *Scientific American*, Vol. 227(2) 106–109 (1972).

A Biological Model Solution to the Towers of Hanoi Problem

Dear Sir,

Recent correspondence,[1,2] while mainly concerned with notions of complexity/simplicity specifically focused on the Towers of Hanoi Problem, at least emphasises the surprisingly many facets (solutions) to the problem. We say 'surprisingly many' because, since the solution is unique, only the mode of deriving it may vary. Further, both a recursive solution and a suggested iterative algorithm[3] were given at the very outset of the appearance of the problem (hereafter TH) in the literature (1892). Perhaps we shouldn't say 'surprisingly many' because in our mind we know very well that there is an infinite number of algorithms to solve any problem, yet in our heart we feel that all but a finite number are 'paddings' of the others. In any case, most of the solutions to TH have been variants on deriving iterative algorithms.[4,5,6,7] In this note we observe that the recursive solution is realised in a natural way by one of the simplest of the idealised biological growth models.

A DOL-system can be thought of as a finite sequence of cells each of which may 'divide', synchronously with the others, into a finite sequence of cells, according to certain rules. There is a collection S of states available to each cell: at each division, the daughter cells (and the states into which they are born) depend only upon the state of the mother cell at division. It is convenient to identify cells with their states, so that an array of cells is just a string over S, and the division rules are just productions $\{s \to W_s : s \in S\}$, where $W_s \in S^*$ (the set of finite strings over S). This is the simplest of the L-systems originated by A. Lindenmayer.[8] (A thoroughly mathematical treatment of L-systems is presented in Ref. 9.)

We choose some positive integer N and display a DOL-system which grows into the solution for TH for n rings ($n \leqslant N$). Let the pegs in TH be labelled A, B, C, and let the movement of a ring from X and Y be denoted by the doublet XY. Let $\mathscr{P}(A,B,C,n)$ be the procedure which transfers n rings from A to B, using C as an aid. The recursive solution for TH is just

$$\begin{aligned}\mathscr{P}(A,B,C,n) = {} & \mathscr{P}(A,C,B,n-1)\\ & \mathscr{P}(A,B,C,1)\\ & \mathscr{P}(C,B,A,n-1)\end{aligned}$$

Let $\mathscr{L}$ be a DOL-system in which the states are tuples $XYZk$ $(0<k<N)$ and the production rules accord to the recursion above. That is, $XYZk \to XZY(k-1).XYZ1.ZYX(k-1)$ for $k>1$. (We use the period to indicate cell boundaries for the convenience of the reader.) For $k=1$, we write just XY in place of $\mathscr{P}(X,Y,Z,1)$, since the procedure for a one-ring system is just the movement of the ring from A to B; the corresponding production is the identity $XY \to XY$ (at the next time-step the cell neither divides nor changes state). Let $\Rightarrow$ denote the transition of a cellular array from one time-step to the next. Note that each such transition represents the unfolding of a stage of the recursive solution to TH, and that, for an n-ring system, after n steps the cellular array is immobolised as the sequence of moves generated by $\mathscr{P}(A,B,C,n)$, where the cellular array begins as a single cell.

As an example, the growth of the system which represents the 3-ring TH is:

$$\begin{aligned}ABC3 & \Rightarrow ACB2.AB.CBA2\\ & \Rightarrow ABC1.AC.BCA1.AB.CAB1.CB.ABC1\\ & \Rightarrow AB.AC.BC.AB.CA.CB.AB\end{aligned}$$

where the last configuration is invariant.

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