Computers and Change-Ringing

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This paper describes some problems of change-ringing on church bells, and gives details of their solution on a Pegasus.

Introduction

The basic problem of change-ringing may be stated mathematically as follows: Given the n numbers 1, 2, . . . n (which will, in this context, be understood to represent church bells with different musical notes), find rules for generating, in some order, all $n!$ possible permutations ("changes"), or sub-sequences thereof, when this order is subject to the following restrictions.

(1) The first and last permutation of any sequence or sub-sequence must be 1, 2, . . . n, which is known as *rounds*, but otherwise no two rows may be the same.
(2) In any row no number may occupy a position more than one place removed from its position in the preceding row, i.e. consecutive rows may differ only by the exchange of adjacent numbers.
(3) No number may remain in the same position in more than two consecutive rows.

There are other less exacting restrictions which need not concern us here (see Ref. 1, pp. 16 and 21).

Any set of rules satisfying these restrictions defines a *method in change-ringing*. To obtain greater variety of methods rule (3) is not always observed, especially when n is an odd number.

A row $\nu_1, \nu_2, \ldots \nu_n$ is said to be *in-course* if it is an even permutation of 1, 2, . . . n, and *out-of-course* if it is an odd permutation.

We consider only the special case $n = 8$ since this presents the most interesting problems theoretically, as well as being a case often met in practice. Changes on eight bells are known as *Major*.

Plain Bob

This is the simplest method in change-ringing. In it, consecutive rows differ by the greatest number of exchanges possible, subject to the restrictions mentioned above. The construction of the method will be described in some detail, since it is typical of the way in which all the more sophisticated methods are built up.

Denote by ν_r the number occupying the rth position in any row. Starting with rounds (row R_1, Table 1), exchange the *four* pairs of numbers ν_1 and ν_2, ν_3 and ν_4, ν_5 and ν_6, and ν_7 and ν_8. This is the greatest number of exchanges possible, and the result is R_2. A similar operation on R_2 would result in R_1 again, which is not permissible. Exchange, therefore, only the *three* inner pairs of numbers, namely ν_2 and ν_3, ν_4 and ν_5, ν_6 and ν_7, and hence obtain R_3. This row may now be altered in

TABLE 1

R_1	12345678	IN
R_2	21436587	IN
R_3	24163857	OUT
R_4	42618375	OUT
R_5	46281735	IN
R_6	64827153	IN
R_7	68472513	OUT
R_8	86745231	OUT
R_9	87654321	IN
R_{10}	78563412	IN
R_{11}	75836142	OUT
R_{12}	57381624	OUT
R_{13}	53718264	IN
R_{14}	35172846	IN
R_{15}	31527486	OUT
R_{16}	13254768	OUT
	13527486	
	L	

the same way as R_1, and the result (R_4) is then altered in the same way as R_2; and so on. This process continues until $\nu_1 = 1$ again, when we have arrived at R_{16}. The same process at this point would require the exchange of ν_2 and ν_3, ν_4 and ν_5, and ν_6 and ν_7, but it is easily seen that this would bring up rounds again, which is not permissible. Instead, therefore, exchange ν_3 and ν_4, ν_5 and ν_6, ν_7 and ν_8, to bring up the row 13527486. This row is now used as the starting-point for another sequence of 16 rows, and so on. Continuing in this way, rounds finally recur after 7 blocks of 16 rows each—112 rows in all. These 112 rows are said to constitute the *plain course* of Plain Bob Major, and each block of 16 rows, as typified by the rows R_1–R_{16} inclusive, is known as a *treble lead*, or *lead*. The first row of each lead is called a *lead-end*.

The courses of the rows of the first lead are as shown. The rows of the other leads are of the same course as those in the corresponding positions of the first lead. Note, in particular, that each lead-end is an in-course row. It is convenient to notice here, too, the "patterns" traced by each number through the other numbers. If we write out the plain course in full, we see that 1 follows throughout a simple diagonal path through the other numbers. Such a path is known as a *plain hunting path*, or a *plain hunt*. 2 also plain hunts until $\nu_1 = 1$, when it "dodges" back a step before resuming its plain hunting path again. All the other numbers, moreover, have a similar path to 2, starting with different phases.

Bobs and Singles

The rules so far described generate only 112 of the total 40,320 permutations possible with eight numbers. To obtain the rest two other devices are used, namely "bobs" and "singles." Normally the transition from the last row of one lead to the first row of the next is effected by exchanging the three pairs of numbers ν_3 and ν_4, ν_5 and ν_6, and ν_7 and ν_8. A *bob* consists in exchanging, instead, ν_2 and ν_3, ν_5 and ν_6, and ν_7 and ν_8. To distinguish the resulting "bobbed lead-end" from a "plain lead-end" (i.e. one produced by the normal process) a mark (–) is made to the left. The new lead-end is still an in-course row, which means that all rows in the subsequent lead are of the same course as they would have been had no bob been used.

At a *single*, on the other hand, only two pairs of numbers are exchanged, namely ν_5 and v_6, and v_7 and v_8; ν_1, ν_2, ν_3 and ν_4 are not moved at all. A letter S at the left indicates that the new lead-end is the result of a single. This lead-end is an out-of-course row, and hence after a single all rows will be of the opposite course to their counterparts immediately before the single.

Proof of Plain Bob Major

It can be shown that these two devices, bobs and singles, are indeed sufficient to generate all 40,320 permutations possible with eight numbers. Usually, however, we are only interested in shorter sequences of permutations than this. Since every sequence must begin and end with rounds, it must comprise a whole number of treble leads of 16 permutations each. The problem of *composition* is that of determining at which lead-ends to introduce bobs and singles so as to obtain a sequence of the required length. First, however, we describe how to *prove* a composition of P.B.M. Apart from rounds at the beginning and end, a composition may contain no repetition of rows, and we need some means of testing for this. Obviously to compare all the permutations would be very tedious, besides involving much risk of error. Fortunately a short cut is available—it depends on the fact that, whereas the positions of the numbers 2–8 in any row of a lead depend on the bobs and singles that have been used up to that point, the position of 1 will be completely unaffected by such bobs and singles; in fact, 1 plain hunts throughout the entire composition. This means that the position of 1 may be used as an "origin" to which may be referred the positions of the other seven numbers. Consider first any composition in which only bobs are used. We may take L (Table 1) as a typical lead of this composition. If we take any row of L, say R_5, another lead L′ could reproduce R_5 only in the same position as this row occurs in L; it could not reproduce R_5 in the position of R_{12}, since R_{12} is an out-of-course row and, in a composition containing bobs only, an in-course row such as R_5 could not possibly occur here; and it certainly could not reproduce R_5 in any other position because the positions of 1 would not correspond. But then L and L′ would be identical leads, and in particular their lead-ends would be the same. Hence, to prove any composition of P.B.M. in which bobs only are used, it is only necessary to compare the lead-ends; if these are all different then the composition is certainly true (i.e. contains no repetition of rows). Conversely, if a composition is true then its lead-ends are all different.

It now remains to consider what modifications to this result are necessary when singles are used as well as bobs. We have already noted that the effect of a single is to reverse the courses of all the rows that follow it. After a single, therefore, a lead may have out-of-course rows where previously only in-course rows were possible, and vice versa. Let us examine the effect of this on the lead L. It means that another lead, L′, could now reproduce R_5 in the position of R_{12}. Because of the special symmetry of the method the same lead would also reproduce R_1 in the position of R_{16}, R_2 in the position of R_{15}, and so on; that is, it would contain all the rows of L but in the reverse order. In particular its lead-end would be the last row of L, namely 13254768. 13254768 is known as the *false lead-end* (F.L.E.) corresponding to L.

Hence to prove any composition of P.B.M. in which both bobs and singles are used, it is necessary to compare each F.L.E., as well as each actual lead-end, with all the other actual lead-ends. The composition is true only if there is no repetition in either case.

In view of the importance of lead-ends in the proof of P.B.M., it is convenient to be able to transpose directly from one lead-end to the next without writing down all the intermediate rows. The scales for doing this are given below.

$$\text{Plain lead-end } \frac{2345678}{3527486}. \quad \text{Bob lead-end } \frac{2345678}{2357486}.$$

$$\text{Single lead-end } \frac{2345678}{3257486}$$

(since $\nu_1 = 1$ for each lead-end, this figure is omitted).

Proof of P.B.M. on Pegasus

The method used was quite straightforward. Depending on symbols read from paper tape (actually 0, 1 and 2 for plain, bob and single lead-ends respectively), each successive lead-end was calculated from the previous one by transposing according to one of the above scales. Each lead-end occupied 28 bits of a Pegasus word (4 bits for each decimal digit), and the necessary rearrangement of digits was accomplished using collate and logical-shift orders. If the composition contained singles, a false lead-end was also calculated by further transposing the result according to the scale:

$$\text{False lead-end } \frac{2345678}{3254768}.$$

The new lead-end (and its F.L.E. too, if necessary) was then compared with all the previous lead-ends and suitable warning given if repetition occurred.

Compositions of about 5,000 changes have been successfully proved by this program in 3 minutes (bobs only) or 4 minutes (singles as well as bobs). [Compositions of 5,000 changes or more are known as *peals*, and are of especial interest to change-ringers.]

Compositions

Compositions of P.B.M. of a given length are usually too numerous for a systematic enumeration to be possible, even on a high-speed automatic computer. Although individual composers may have developed short cuts to particular solutions, the method of composition remains basically one of trial and error, in which a sequence of lead-ends, of the required length, is written down and then proved or disproved in the way just described.

The composition program written for Pegasus uses pseudo-random numbers. Each successive lead-end of a composition may be either plain or bobbed (the present program ignores singles, although it could easily be modified to take these into account too) and a random number is generated to determine which. The resulting lead-end, having been transposed from the preceding lead-end by the appropriate scale, is then tested against all the previous lead-ends, and stored if there is no repetition. Otherwise it is altered to the lead-end of the opposite type, and again tested. If this lead-end, too, duplicates an earlier one the computer returns and alters the last lead-end not previously altered, tests, and continues the sequence from this point if the result of the test is satisfactory.

Compositions of about 5,000 changes, necessarily true, have been successfully obtained by this program in 8 or 9 minutes. The greatest difficulty is that of getting a composition of the required length ending with rounds. It is evident that a random sequence of permutations such as this program produces is unlikely to end naturally with rounds at a specified point, and an "alteration routine" must then be called in and used repeatedly until rounds are brought up at the point required. It may be necessary to return a distance of 10 lead-ends or more for this purpose, and this uses much computer time since going back ρ lead-ends may involve (in theory, at any rate) testing up to $2^{\rho+2} - \rho - 3$ new lead-ends.

Musical Quality of a Composition

An important practical consideration, neglected in the discussion so far, is the musical quality of a composition. This depends on what is known as the "coursing-order" of the numbers (Ref. 1). If the plain course of P.B.M. is written out in full, it is seen that the paths of alternate numbers are parallel, and the numbers themselves follow each other in their plain hunting and come to the first position in a row in a regular order which remains the same throughout, except that 1 is in a different relative position in each lead. This order, the *coursing-order* of the numbers, is 2468753. It is cyclical, and the numbers are said to *course each other* in this order. A bob alters the positions of ν_2, ν_3, and ν_4 at a lead-end, and hence alters also the order in which these numbers subsequently course each other and the other five numbers. Similarly a single alters the relative coursing order of ν_3 and ν_4.

Now the numbers 7 and 8 represent, in the actual practice of change-ringing, the two bells with the deepest musical notes, and it is generally agreed that the most musical compositions are those in which these two bells (the tenors) course each other in the plain course coursing-order (...87..) throughout. Thus it is desirable to use bobs and singles only at lead-ends where they will not alter the relative coursing-order of 7 and 8 ("will not part the tenors"). To find these lead-ends we need only to examine the lead-ends of the plain course, since obviously the lead-ends of a composition in which 7 and 8 are never parted can only contain these numbers in the same relative positions as they are to be found in the lead-ends of the plain course.

The lead-ends of the plain course are listed in Table 2.

TABLE 2

2345678		
3527486	—	W
5738264		
7856342		
8674523	—	B
6482735		
4263857	—	M
2345678	—	H

There are three at which bobs and singles could have been used without parting 7 and 8, marked W, M, and H (because the position of 8 at these lead-ends is said to be "wrong," "middle," and "home" respectively).

A bob could also be used without parting 7 and 8 at one other lead-end, namely B. If a bob were used at B, the resulting lead-end would be 17864523, which reproduces 7 and 8 in the same positions as in the preceding lead-end and hence does not alter their coursing-order relative to one another. A bob at B is known as a "bob before."

An abbreviated notation is often used to write down compositions in which the tenors are "kept together" throughout. In this scheme only the *course-ends* are written down; these are the lead-ends for which $\nu_8 = 8$. (A *course* comprises all rows between two consecutive course-ends, including the first course-end but not the second.) At the side, under the headings W, B, M, and H, are then indicated the positions of the bobs and singles of which these course-ends are the results. A typical example is given below (since, in each course-end, $\nu_1 = 1$, $\nu_7 = 7$, and $\nu_8 = 8$ these figures are omitted):

800 Plain Bob Major

23456	W	B	M	H
23564		1		—
52364				—
53264				S
25364				—
32564				—
35264				S
23456	—		—	

Note that bobs and singles can be called at W, M and H only once in a course. With bobs before, however, this is not the case as 2, 3 or 4 bobs may then be called at consecutive lead-ends, and the number must be indicated under the "B."

The composition program described earlier has been modified so as to "keep the tenors together" throughout and print out the results in the above notation. It is quicker than the old program since there is less difficulty in ending with rounds at a given point when 7 and 8 are never parted. [Actually a program has been written for Pegasus which will cause the computer to play sequences of changes on its loudspeaker. It is amusing to note that a party of change-ringers who heard this program were extremely impressed by what they called the "perfect striking" of the machine.]

Conclusion

This completes the work on P.B.M. that has been done on Pegasus. Other programs dealing with more complex methods in change-ringing have been developed, and several new peals have been produced. The application of a computer to this field is both amusing and interesting.

References

(1) SNOWDON, JASPER and WILLIAM. *Standard Methods in the Art of Change-Ringing* (two volumes, "Letterpress" and "Diagrams").

(2) SNOWDON, JASPER. *Ropesight.*

(3) SNOWDON, JASPER. *A Treatise on Treble Bob.*

All the above are published by Whitehead and Miller, Ltd., Leeds.

Simultaneous Equations and Linear Programming (*Continued from p.* 46)

and finally,

x_1	x_2	x	y_1	y_2	
1	0	2	0·244	−0·074	1
0	1	−3	0·171	0·050	1
0	0	−1	−1·000	−1·000	0

Thus,

$$x_1 = 2$$
$$x_2 = -3$$

And A^{-1} is, to three decimal places,

$$\begin{bmatrix} 0\cdot244 & -0\cdot074 \\ 0\cdot171 & 0\cdot050 \end{bmatrix}$$

or, exactly,

$$\frac{1}{41}\begin{bmatrix} 10 & -3 \\ 7 & 2 \end{bmatrix}.$$

This is the inverse of

$$\begin{bmatrix} 2 & 3 \\ -7 & 10 \end{bmatrix},$$

so,

$$\begin{bmatrix} 2 & 3 \\ 7 & -10 \end{bmatrix}^{-1} = \frac{1}{41}\begin{bmatrix} 10 & 3 \\ 7 & -2 \end{bmatrix},$$

which is the inverse of A^*.

Reference

DEUTSCH, M. L. (1959). Letter, *Comm. Assoc. Comp. Mach.*, Vol. 2, No. 3, p. 1.