Predicting Distributions of Staff

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The method of predicting the number and distribution of staff among various grades in future years is described. A mathematical model is obtained which can be solved to show the long-term consequences of the persistence of present patterns of recruitment and promotion.

Introduction

This investigation was made on behalf of an institution which has expanded very rapidly since the end of the war. By and large, the recruitment of additional staff followed the old pattern, newly created vacancies being filled by younger people, and although the expansion continues, comparatively many members of the staff are now simultaneously approaching the tops of their respective grades. Although there are proportionately more senior vacancies to fill, it has not been clear whether the pattern of promotion previously regarded as satisfactory could be continued, so we were asked to predict the probable numbers of staff in each of the various grades for the next few years.

In this paper we describe how we made the predictions and show how future trends can be determined mathematically when recruitment and promotion patterns are stable.

The Patterns of Recruitment, Promotion and Withdrawal

We distinguished six serving grades of staff which we subdivided by year of service within the grade. We called the subdivisions "statuses" to each of which we assigned a three-digit number, the first digit denoting the grade (1 to 6) and the second two the year of service in the grade. To complete the picture we gave recruits status 000 in the year prior to entry; we also gave statuses 700, 800 and 900 to those who withdrew during the year by resignation, retirement or death. In certain circumstances it might be better to equate status to salary, but in our investigations we did not do so, thus ignoring accelerated promotion within grades as well as promotions to higher grades at points on the salary scale above the minimum. In our notation, a hypothetical person who joined the staff in grade 3 in 1950, was promoted to grade 4 in 1953, and resigned at the end of 1956 would have a career recorded as

Year:	1949	1950	1951	1952	1953	1954	1955	1956	1957
Status:	000	301	302	303	401	402	403	404	700

We collated the careers of every member of the staff who was serving in 1939 or who had joined since, and used the records for the eleven years starting at 1949 to determine the probabilities of promotion, etc. The records of careers were punched on Hollerith cards which were then sorted and tabulated to give total numbers of persons in every status according to the status held in the following year. If the total in any status s moving to status r is C_{rs}, then the probability of making such a move is $C_{rs}/\sum_i C_{is}$. This categorization allowed recruits and withdrawals to be treated in the same way as serving members. Having obtained the totals C_{rs} we calculated the probabilities on DEUCE. At this stage we encountered a slight difficulty because of the occurrence of unit probabilities of withdrawal from some statuses of which we had only one or two recorded instances. For example, although we had instances of status 318 the only recorded 317 in one particular sub-group of the staff resigned; if this had been accepted as giving unit probability of resignation from status 317, no instances of status 318 would subsequently be predicted. To avoid this we resorted to grouping statuses where few recorded instances were available. In all we had some 30 distinct statuses after grouping.

Because promotions and recruitment were assumed to take place only to the bottom of the grades, moves to say status 401 were possible from anywhere in grades 0, 1, 2, and 3, but moves to 402 were only possible from 401—the latter type of move we termed "normal advancement." Demotions were so rarely recorded that we ignored them.

The Calculations of the Predicted Levels of Staff

Having obtained the probabilities of being recruited to various grades, being promoted, or withdrawing, two methods of proceeding with the predictions are possible. In the first, the number now serving but who will withdraw before the next year is calculated before the numbers who will proceed to each serving status in the next year. The number of new members who will serve next year is determined as the sum of those required to fill vacancies created by both withdrawals and new appointments due to expansion. These are distributed to various statuses according to the recruitment pattern established. In the second method, recruits and withdrawals are not considered separately, but only the overall increase in the staff from one year to the next. This can be done by noting that the people in any status in any year include those who were promoted to that status from a lower status in the previous year, and also those who were recruited direct to it; a matrix of probabilities can be constructed by taking account of the joint probabilities of vacancies caused by withdrawal being filled by recruitment.

The choice of method has an important bearing on the way in which the calculations can be done on DEUCE. Two DEUCE interpretive programs are particularly suitable—they are G.I.P. and T.I.P., both of which have been described previously by C. Robinson (1959). G.I.P. is particularly designed for use with matrices, and is the natural program to use for the second method. T.I.P., however, proved to be more suitable for the first method, particularly as the program can be written to take advantage of the fact that nearly half the probabilities are zero, and most promotions are in fact "normal advancements." In the event, we did use a T.I.P. program.

To test our methods and their applicability, we took the known distributions of staff in the years 1956, 1957 and 1958, and predicted from them the distributions in the following years. Statistical tests applied to the results showed that the differences between the actual and predicted distributions were surprisingly small and, in all but the case of one sub-group of the staff, we felt able to proceed with the predictions to future years with some confidence. In the one case where the results were not quite so satisfactory, we were able to trace the discrepancy to a change in the pattern of recruitment which had actually occurred during the years on which the probabilities were based. By taking the later pattern, we obtained satisfactorily improved results. We therefore completed our set task of predicting the levels of staff for the next five years in each sub-group.

In an institution of the size and nature of the one with which we were concerned, however, it is doubtful if long-range prediction would be wise except for the general indication of probable trends. This is because the probabilities are based on fairly small samples if detailed analysis is attempted, so that the probabilities are liable to rather large sampling errors. Further, the staff of the institution is highly specialized, and the institution is often recruiting in fields of scarcity so that recruitment patterns may be affected seriously by the "supply and demand" of specialist staff. For larger institutions the predictions should be more reliable for longer periods.

We have considered it worthwhile to examine the problem of prediction as a mathematical one, and have been able to obtain solutions in a number of likely circumstances.

Mathematical Equations

We use the following notation:

a_i $(i = 1, 2, \ldots, n)$	Number of persons serving in status i.
a_0	Number of persons whom it will be necessary to recruit for service in the following year.
$a_{n+1}, a_{n+2}, a_{n+3}$	Number of persons who will not serve in the following year because they have resigned, retired, or died.
T	Total number of staff in service.
δT	Increase in total serving in the following year.
p_{rs}	Probability that a person in status s will proceed to status r in the following year.
w_s	Probability that a person in status s will not serve in the following year.

We also define compound probabilities q_{rs} such that

$$q_{rs} = p_{rs} + w_s p_{r0}, \tag{1}$$

We distinguish quantities referring to the following year by accents. Clearly,

$$\sum_{r=1}^{n+3} p_{rs} = 1 \tag{2}$$

$$\sum_{r=n+1}^{n+3} p_{rs} = w_s \tag{3}$$

for all s.

Also, since the total number of persons who withdraw before the beginning of the following year is $\sum_{s=1}^{n} w_s a_s$, the number of recruits necessary to allow for both expansion and wastage is

$$a_0 = \delta T + \sum_{s=1}^{n} w_s a_s \tag{4}$$

We can write down the relation between the staff one year and the next in the form of the matrix equation

$$\begin{bmatrix} a'_1 \\ a'_2 \\ \cdot \\ \cdot \\ \cdot \\ a'_n \\ a'_{n+1} \\ a'_{n+2} \\ a'_{n+3} \end{bmatrix} = \begin{bmatrix} p_{10}, p_{11}, \ldots, p_{1n} \\ p_{20}, p_{21}, \ldots, p_{2n} \\ \cdot \\ \cdot \\ \cdot \\ p_{n0}, p_{n1}, \ldots, p_{nn} \\ 0, p_{n+1,1}, \ldots, p_{n+1,n} \\ 0, p_{n+2,2}, \ldots, p_{n+2,n} \\ 0, p_{n+3,1}, \ldots, p_{n+3,n} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \tag{5}$$

The probabilities $p_{n+1,0}$, $p_{n+2,0}$ and $p_{n+3,0}$ are each zero since no recruit can simultaneously withdraw. If demotion never occurs $p_{rs} = 0$ for all $r < s$.

The use of (4) and (5) effectively gives the first method of performing calculations described earlier. For theoretical purposes, these equations are inconvenient since the matrix is not square, nor are the vectors strictly comparable. However, eliminating a_0 from (4) and (5) and removing the elements a'_{n+1}, a'_{n+2} and a'_{n+3}, we obtain

$$\boldsymbol{a}' = \boldsymbol{Q}\boldsymbol{a} + \delta T \boldsymbol{p} \tag{6}$$

where $\boldsymbol{a}$ is the column vector $\{a_1, a_2, \ldots, a_n\}$, $\boldsymbol{p}$ is the column vector $\{p_{10}, p_{20}, \ldots, p_{n0}\}$, and $\boldsymbol{Q}$ is the $n \times n$ matrix whose ijth element is q_{ij} defined by (1). This equation represents a set of n simultaneous, first order, linear difference equations, the iterative use of which constitutes the second possible method of calculating the predicted values.

The equations are true whether the probabilities are constants or not, but we shall restrict our attention to the case where $\boldsymbol{Q}$ has constant elements. This case arises when we assume that rates of withdrawal, promotion, and recruitment are constant.

The Latent Roots of the Matrix Q

Before proceeding to solve (6) we require certain properties of the latent roots of $\boldsymbol{Q}$. Let $\boldsymbol{Q}$ have latent roots λ_i with associated latent column vectors $\boldsymbol{u}_i$. For simplicity we shall consider only the cases where the $\boldsymbol{u}_i$ are distinct—extension to the exceptional case is possible. We prove two theorems.

Theorem I. $\boldsymbol{Q}$ has at least one latent root equal to unity.

$$\sum_{r=1}^{n} q_{rs} = \sum_{r=1}^{n} p_{rs} + w_s \sum_{r=1}^{n} p_{r0}$$
$$= \sum_{r=1}^{n} p_{rs} + w_s$$
$$= \sum_{r=1}^{n+3} p_{rs} = 1 \qquad \text{[using (1), (2) and (3)].}$$

If, therefore, we add all the rows of the determinant $|\lambda \boldsymbol{I} - \boldsymbol{Q}|$, we obtain $(1 - \lambda)$ as a factor of the characteristic equation, so $\lambda = 1$ is a latent root of $\boldsymbol{Q}$.

Theorem II. Each latent root satisfies the inequality

$$\min_j |2q_{jj} - 1| \leqslant |\lambda| \leqslant 1.$$

By a theorem due to Frobenius,

$$|\lambda|_{max} \leqslant \max_j \sum_i |q_{ij}|$$

But all $q_{ij} \geqslant 0$, therefore $|\lambda|_{max} \leqslant \max_j \sum_i q_{ij} \leqslant 1$.

By the same theorem

$$|\lambda|_{min} \geqslant \min_j |(|q_{jj}| - \Sigma'|q_{ij}|)|$$
$$\geqslant \min_j |\{q_{jj} - (1 - q_{jj})\}|$$
$$\geqslant \min_j |2q_{jj} - 1|.$$

Hence, each $|\lambda|$ lies between $\min_j |2q_{jj} - 1|$ and 1.

From the second theorem we deduce that if the latent roots are real they have absolute values less than or equal to unity, and if they are complex, they cannot have real parts greater than unity.

Solution of the Difference Equation

Let $\boldsymbol{a}^{(t)}$ denote the value of $\boldsymbol{a}$ at time t, and let the matrix $\boldsymbol{E} = E\boldsymbol{I}$, where E is the displacement operator and $\boldsymbol{I}$ the unit matrix. Then $\boldsymbol{a}' = \boldsymbol{E}\boldsymbol{a}$ and (6) becomes

$$(\boldsymbol{E} - \boldsymbol{Q})\boldsymbol{a}^{(t)} = \delta T \boldsymbol{p}. \tag{7}$$

We know $\boldsymbol{a}^{(0)}$, the distribution of staff at $t = 0$. $\boldsymbol{a}^{(0)}$ and $\boldsymbol{p}$ may be expanded in terms of the latent vectors of $\boldsymbol{Q}$ to give

$$\boldsymbol{a}^{(0)} = \sum_{i=1}^{n} \alpha_i \boldsymbol{u}_i \tag{8}$$

$$\boldsymbol{p} = \sum_{i=1}^{n} \beta_i \boldsymbol{u}_i \tag{9}$$

say, where the α_i and β_i are known constants.

The solution* of (7) for $\delta T = 0$—corresponding to a non-expanding staff is:

$$\boldsymbol{a}^{(t)} = \sum_{i=1}^{n} \alpha_i \lambda_i^t \boldsymbol{u}_i. \tag{10}$$

(This is analogous to the complementary function of a set of linear differential equations with constant coefficients.)

Since by the second theorem $|\lambda_i| \leqslant 1$ for all i, and at least one $\lambda_i = 1$, $\boldsymbol{a}^{(t)}$ tends to the limit such that

$$\underset{t \to \infty}{Lt}\, \boldsymbol{a}^{(t)} = \alpha_1 \boldsymbol{u}_1 \tag{11}$$

where $\boldsymbol{u}_1$ is the latent vector corresponding to the latent root equal to unity. (We assume for simplicity it is a unique root.) Also, the complex latent roots lead to terms of the form

$$e^{-kt}(\alpha_1 \cos \theta t + \alpha_2 \sin \theta t)$$

with $k \geqslant 0$. Since $|\lambda| \geqslant \min_j |2q_{jj} - 1|$, we therefore have available a minimum estimate of the time taken by $\boldsymbol{a}^{(t)}$ to approach $\boldsymbol{a}^{(\infty)}$ to a given accuracy by putting $e^{-k} = \min_j |2q_{jj} - 1|$, since this gives a contribution which could not persist longer than any of those from actual λ's.

The solution of (7) when δT is not zero is formally

$$\boldsymbol{a}^{(t)} = \sum_{i=1}^{n} c_i \lambda_i^t \boldsymbol{u}_i + (\boldsymbol{E} - \boldsymbol{Q})^{-1} \delta T \boldsymbol{p}$$

where the c_i are arbitrary constants determined by $\boldsymbol{a}^{(0)}$. We have been able to find interpretations of the inverse matrix-difference operator $(\boldsymbol{E} - \boldsymbol{Q})^{-1}$, but in fact prefer to use methods based on the latent roots and vectors of $\boldsymbol{Q}$. We will give solutions in two cases.

Case (i). $\delta T = Ae^{-\mu t}$

Since $\boldsymbol{E}$ operates only on δT and $\boldsymbol{Q}$ on $\boldsymbol{p}$, and since $Ee^{-\mu t} = e^{-\mu} . e^{-\mu t}$, it follows that

$$\boldsymbol{a}^{(t)} = \sum_{i=1}^{n} \alpha_i \lambda_i^t \boldsymbol{u}_i + Ae^{-\mu t}\{1 + \boldsymbol{Q}e^{\mu} + (\boldsymbol{Q}e^{\mu})^2 + \ldots (\boldsymbol{Q}e^{\mu})^{t-1}\}\boldsymbol{p}$$

and using (9)

$$\boldsymbol{Q}^r \boldsymbol{p} = \sum_{i=1}^{n} \beta_i \lambda_i^r \boldsymbol{u}_i$$

* Strictly speaking, if t is a continuous variable, the solutions should contain arbitrary periodics with periods of one year if the **a**'s are required at yearly intervals. Also, the solutions do not give integral numbers of staff, but this does not matter when it is remembered that the probabilities are stochastic variables. The solutions are in the nature of statistical expectations.

Table 1

Example of probabilities of transfers between grades

To \ From	Recruits	1	2	3	4	5	6	7	8
1	0·7037	0·7643							
2	0·2963	0·0571	0·7664						
3			0·1241	0·7674					
4				0·1395	0·7838				
5					0·1622	0·6833			
6						0·1500	0·7571		
7							0·0991	0·7333	
8								0·1647	0·9058
Withdrawals		0·1786	0·1095	0·0931	0·0540	0·1667	0·1438	0·1020	0·0942

Table 2

Matrix arising from probabilities of Table 1

$$
\boldsymbol{Q} = \begin{bmatrix}
0\cdot8900 & 0\cdot0771 & 0\cdot0655 & 0\cdot0380 & 0\cdot1173 & 0\cdot1012 & 0\cdot0718 & 0\cdot0663 \\
0\cdot1100 & 0\cdot7988 & 0\cdot0276 & 0\cdot0160 & 0\cdot0494 & 0\cdot0426 & 0\cdot0302 & 0\cdot0279 \\
 & 0\cdot1241 & 0\cdot7674 & & & & & \\
 & & 0\cdot1395 & 0\cdot7838 & & & & \\
 & & & 0\cdot1622 & 0\cdot6833 & & & \\
 & & & & 0\cdot1500 & 0\cdot7571 & & \\
 & & & & & 0\cdot0991 & 0\cdot7333 & \\
 & & & & & & 0\cdot1647 & 0\cdot9058
\end{bmatrix}
$$

so

$$\boldsymbol{a}^{(t)} = \sum_{i=1}^{n} \left\{ \alpha_i \lambda_i^t + \frac{Ae^{-\mu}\beta_i(e^{-\mu t} - \lambda_i^t)}{(e^{-\mu} - \lambda_i)} \right\} \boldsymbol{u}_i. \qquad (12)$$

As t increases, provided $\mu > 0$,

$$\underset{t \to \infty}{Lt}\, \boldsymbol{a}^{(t)} = \left\{ \alpha_1 - \frac{Ae^{-\mu}\beta_1}{(e^{-\mu} - 1)} \right\} \boldsymbol{u}_1. \qquad (13)$$

Case (ii). $\delta T = C$, *constant*

In this case, the solution is found, in the same manner as in Case (i), to be

$$\boldsymbol{a}^{(t)} = (\alpha_1 + C\beta_1 t)\boldsymbol{u}_1 + \sum_{i=2}^{n} \left\{ \alpha_i \lambda_i^t + \frac{C\beta_i(1 - \lambda_i^t)}{(1 - \lambda_i)} \right\} \boldsymbol{u}_i \quad (14)$$

As t increases

$$\underset{t \to \infty}{Lt}\, \boldsymbol{a}^{(t)} = (\alpha_1 + C\beta_1 t)\boldsymbol{u}_1 + \sum_{i=2}^{n} \frac{C\beta_i}{1 - \lambda_i} \boldsymbol{u}_i \qquad (15)$$

The same method can be used to obtain solutions for other forms of δT, but the two cases given are probably the commonest in practice.

Note that in all cases considered, the distribution of staff tends to a scalar multiple—which may depend on t—of the dominant latent vector of $\boldsymbol{Q}$.

Illustrative Example

As an illustration we take a hypothetical stationary staff which falls into eight serving statuses. The probabilities of transfers are shown in Table 1.

These probabilities indicate that most people are staying in the same grade, some are advancing from one grade to the next higher, recruits enter one of the two lowest grades, and there are no demotions. From them we obtain the matrix **Q** shown in Table 2.

The latent roots of $\boldsymbol{Q}$ are 1, 0·8889, 0·7085, 0·6847, $0\cdot8143 \pm 0\cdot0969i$, $0\cdot7044 \pm 0\cdot0507i$. The complex roots can be put in the form $re^{\pm i\theta}$, where $r_1 = 0\cdot8200$, $\theta_1 = 6°\ 46'$, $r_2 = 0\cdot7062$, $\theta_2 = 4°\ 34'$, so that the oscillations of the corresponding contributions have periods of the order of 53 and 79 years, respectively. Before either contribution can have completed a full oscillation, the amplitude has been damped to less than 10^{-4} of the original amplitude so that neither oscillation will be important in the predicted values. The value of $\min_j |2q_{jj} - 1|$ is 0·3666, which is a rather poor lower bound for λ. This is due to the fact that nearly two-thirds the staff in grade 5 continue in that grade the next year. In the actual cases we have investigated, much better lower bounds were obtainable; because of the preponderance of normal advances the dominant terms in $\boldsymbol{Q}$ were on the super diagonal and the terms q_{jj} were small. The dominant latent root of $\boldsymbol{Q}$ in the example is

$$\boldsymbol{u}_1 = \{1{\cdot}0000,\ 0{\cdot}6690,\ 0{\cdot}3569,\ 0{\cdot}2303,\ 0{\cdot}1180,\ 0{\cdot}0728,\ 0{\cdot}0271,\ 0{\cdot}0473\}.$$

The distribution of staff, starting from

$$\boldsymbol{a}^{(0)} = \{800, 500, 200, 180, 75, 30, 20, 10\},$$

is illustrated in Fig. 1. The final distribution, $\boldsymbol{a}^{(\infty)}$, is calculated to be

$$\boldsymbol{a}^{(\infty)} = \{720, 482, 257, 166, 85, 52, 19, 34\}.$$

After only a short time the predicted values of all but the top two grades are almost the same as the final values, e.g. $\boldsymbol{a}^{(10)}$ is within 3% of $\boldsymbol{a}^{(\infty)}$ for all but the top two grades. The prediction would be useful here in pointing to the Parkinsonian expansion of grade 8 due to occur if the pattern of promotion and recruitment is continued.

Whilst it is unlikely that such patterns will persist for the several decades required to reach the ultimate distributions, analyses of the type described may be useful in that they can give advance warning of the necessity for changing staffing policy.

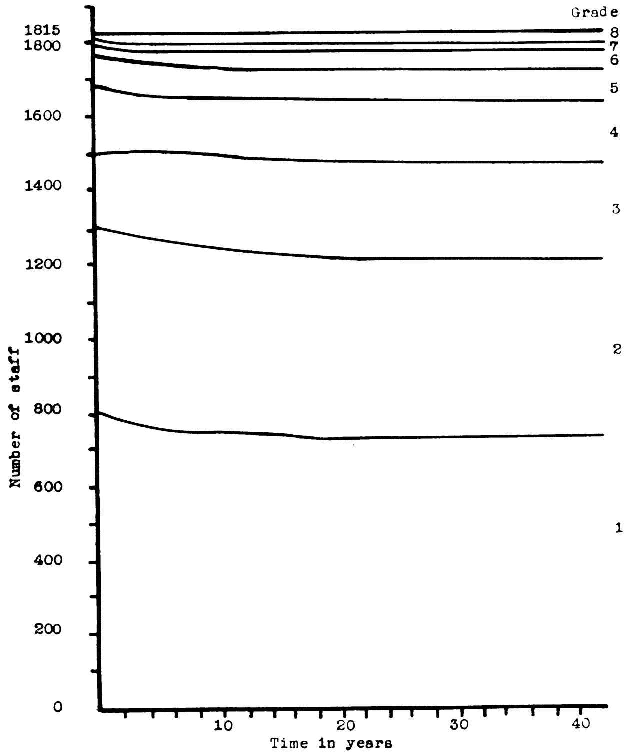

Fig. 1. Example of staff distribution

Reference

Robinson, C. (1959). "DEUCE Interpretive Programs," *The Computer Journal*, Vol. 1, p. 172.