Convergence Properties of Gaussian Quadrature Formulae

By W. Barrett

An expression is found for the remainder of any Gaussian quadrature formula as a contour integral, and is then used to obtain estimates of the error in certain cases. Numerical examples are given comparing the actual and estimated errors.

A method of constructing more general quadrature formulae is described, and two examples are given of formulae so constructed.

Introduction

Interest in Gaussian quadrature formulae has increased considerably since the advent of electronic computers, because their immediate disadvantage—the practical computational difficulties in using them—no longer has the same force. However, a further disadvantage is that certain simple methods of error estimation are not available.

When using formulae based on finite differences, we can compare values obtained using different tabular intervals, but retaining differences of the same order; the behaviour of the error as the tabular interval is reduced does not then depend on the nature of the integrand. When using Gaussian formulae, on the other hand, improved accuracy is usually sought by using formulae with an increasing number of points, and when this is done, the rate of convergence of the process depends very materially on the nature of the integrand. The object of the present paper is to make a contribution to the understanding of this problem, and results are obtained which lead to estimates of the error when the integrand is an analytic function, and which indicate the rate of convergence to be expected under various conditions.

A generalization of the method used is applied to two formulae which are not Gaussian, but which can with certain advantages replace respectively the Gauss–Laguerre and Gauss–Hermite formulae. The second of these two formulae is in fact the trapezoidal rule, but the first is believed to be new.

The Remainder as a Contour Integral

The basic formula required is an expression for the remainder of a quadrature formula as a contour integral.

Let
$$\int_a^b f(x)w(x)dx = \sum_{r=1}^{n} \lambda_r f(x_r) + \mathscr{R}_n(f) \tag{1.1}$$

be a numerical quadrature formula. Here, $[a, b]$ is a specified range of integration and $w(x)$ is a given weight-function, non-negative in $[a, b]$; $\{x_r\}$ are n distinct specified "fixed points," λ_r are numerical coefficients, and $\mathscr{R}_n(f)$ is the remainder.

If we require the formula to be exact—that is, $\mathscr{R}_n(f) = 0$—whenever $f(x)$ is a polynomial of degree less than n, then it can be shown [see Szego (1939), 3.4.3] that

$$\lambda_r = \frac{1}{p_n'(x_r}\int_a^b \frac{p_n(t)w(t)}{t - x_r}dt,$$

where $p_n(t)$ is a polynomial of degree n with zeros $\{x_r\}$. It can further be shown that, replacing the symbol x by the symbol z representing a complex variable,

$$\mathscr{R}_n(f) = \frac{1}{2\pi i}\oint \frac{q_n(z)}{p_n(z)}f(z)dz, \tag{1.2}$$

where
$$q_n(z) = \int_a^b \frac{p_n(t)w(t)}{z - t}dt,$$

so that
$$\lambda_r = -q_n(x_r)/p_n'(x_r), \tag{1.3}$$

and the contour contains the interval $[a, b]$ and the fixed points $\{x_r\}$ in its interior, but no singularity of the function $f(z)$ lies on or within the contour.

In fact, if $a < t < b$,

$$f(t) = \frac{1}{2\pi i}\oint \frac{f(z)}{z - t}dz,$$

so that
$$\int_a^b f(t)w(t)dt = \frac{1}{2\pi i}\oint \left\{\int_a^b \frac{w(t)dt}{z - t}\right\} f(z)dz. \tag{1.4}$$

Again, since $p_n(x_r) = 0$, λ_r, as defined above, is the residue at $z = x_r$ of the analytic function

$$\frac{1}{p_n(z)}\int_a^b \frac{p_n(z) - p_n(t)}{z - t}w(t)dt,$$

so that

$$\sum_{r=1}^{n} \lambda_r f(x_r) = \frac{1}{2\pi i}\oint \left\{\int_a^b \frac{p_n(z) - p_n(t)}{z - t}w(t)dt\right\}\frac{f(z)}{p_n(z)}dz. \tag{1.5}$$

From (1.1), (1.4) and (1.5), (1.2) follows immediately.

It should be remarked that $q_n(z)$ is defined in the complex plane cut along the real interval $[a, b]$. However, when $a < x < b$, we can define functions $q_n(x + 0i)$ and $q_n(x - 0i)$; if $x = x_r$, these are equal, so that $q_n(x_r)$ in (1.3) is defined.

This method applies in particular to Gaussian formulae, for which $p_n(x)$ is one of a class of orthogonal polynomials with weight function $w(x)$ over the interval $[a, b]$; the formula is now exact if $f(x)$ is any polynomial of degree less than $2n$. The best known are those based on the classical systems of orthogonal polynomials:

SYSTEM	INTERVAL	WEIGHT-FUNCTION
Jacobi*	$[-1, 1]$	$(1-x)^{\alpha}(1+x)^{\beta}$ $(\alpha, \beta > -1)$
Laguerre	$[0, \infty]$	$x^{\alpha}e^{-x}$
Hermite	$[-\infty, \infty]$	e^{-x^2}

* Includes the Legendre polynomials, with $\alpha = \beta = 0$.

In these cases the polynomials $p_n(x)$ arise as solutions of certain second order linear differential equations, and it turns out that $q_n(z)/w(z)$ is in each case a second solution of the same differential equation. Reasonably simple asymptotic expressions can now be found, which represent $p_n(z)$, $q_n(z)$ well, at least for large values of n.

As an example, we find for the Jacobi case the following two formulae:

$$q_n(z)/p_n(z) \simeq 2\pi(z-1)^{\alpha}(z+1)^{\beta}[z+(z^2-1)^{\frac{1}{2}}]^{-2N}, \qquad (1.6)$$

where $N = n + \frac{1}{2}(\alpha + \beta + 1)$, which is valid except in an arbitrary but fixed neighbourhood of the real segment $[-1, 1]$;

$$q_n(z)/p_n(z) \simeq 2(z-1)^{\alpha}(z+1)^{\beta}K_{\alpha}(N\theta)/I_{\alpha}(N\theta), \qquad (1.7)$$

where $z = \cosh\theta$ and K, I represent the usual modified Bessel functions; this expression is valid except in a neighbourhood of the infinite real segment $[-\infty, -1]$; in particular, it is valid in the neighbourhood of the point $z = +1$.

A derivation of (1.6) is given in an appendix, and also a brief indication of the method of deriving (1.7).

Asymptotic Remainder Formulae

We are now in a position to derive two asymptotic expressions for the remainder of the Gauss–Jacobi quadrature formula, valid under different conditions on the function $f(z)$; these will be followed by a statement of the corresponding expressions for the Gauss–Laguerre and Gauss–Hermite formulae.

(i) Notice first that $|z + (z^2-1)^{\frac{1}{2}}|$ is constant (greater than unity if the sign of the square-root is properly chosen) along any ellipse with the points ± 1 as foci. Suppose, therefore, that $f(z)$ has no singularities within or on a particular such ellipse,

$$|z + (z^2-1)^{\frac{1}{2}}| = R, \qquad (2.1)$$

except for a pair of simple poles at z_0, $\bar{z}_0$, where

$$|z_0 + (z_0^2-1)^{\frac{1}{2}}| = R_0 < R,$$

and suppose that the function $f(z)(z-1)^{\alpha}(z+1)^{\beta}$ has residues ρ_0, $\bar{\rho}_0$ at these points. As contour in (1.2), we now take the ellipse (2.1), together with two small circuits, one enclosing each of the points z_0, $\bar{z}_0$. It can now be shown that, as $n \to \infty$, the contribution to the integral

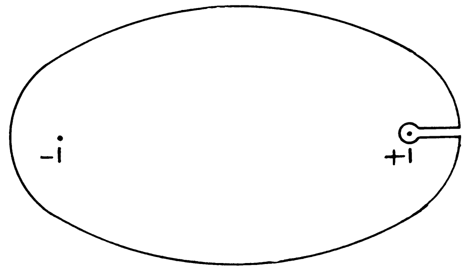

Fig. 1. Contour for proof of formula (2.3)

(1.2) from the ellipse tends to zero compared with that from either of the small circuits, and we obtain the formula

$$\mathscr{R}_n(f) \simeq -4\pi \,\mathrm{Rl}\,\{\rho_0[z_0 + (z_0^2-1)^{\frac{1}{2}}]^{-2N}\}. \qquad (2.2)$$

In interpreting this result, we notice that the quantity $z_0 + (z_0^2-1)^{\frac{1}{2}}$ is not in general real, so that (2.2) represents an exponentially damped oscillation, the damping factor being $1/R_0^2$ for each unit increase in n.

(ii) It is possible to relax the restrictions on $f(z)$ slightly, and to allow a singularity at an end of the range of integration, say at $x = 1$. The contour should now comprise an ellipse (2.1) on or within which there is no other singularity of $f(z)$, a small circuit enclosing the point $z = +1$, and also the real axis between this circuit and the ellipse, described once in each sense. Fig. 1 illustrates this contour, on and within which there is no singularity of $f(z)$.

Suppose, now, that $f(z)w(z) \simeq (1-z)^{\sigma}$, [9] $(\sigma > -1)$ as $z \to 1$; we may thus consider the effect of changing the weight-function on the computed value of the integral without changing the value of σ. We shall find that

$$\mathscr{R}_n(f) \simeq \frac{\sin\pi(\sigma-\alpha)}{2^{\sigma+1}\pi N^{2\sigma+2}}\int_0^{\infty}\frac{t^{2\sigma+1}K_{\alpha}(t)}{I_{\alpha}(t)}dt, \qquad (2.3)$$

provided that σ, $\sigma - \alpha > -1$ and that $\sigma - \alpha$ is not an integer.

From (1.7) and from the formula

$$f(z) \simeq (1-z)^{\sigma-\alpha}(1+z)^{-\beta},$$

we obtain, in fact,

$$f(z)q_n(z)/p_n(z) \simeq 2^{1-\sigma}\theta^{2\sigma}e^{\pm i\pi(\sigma-\alpha)}K_{\alpha}(N\theta)/I_{\alpha}(N\theta) \qquad (2.4)$$

as $n \to \infty$, $\theta \to 0$ (i.e. $z \to 1$); the sign is $-$ve in the upper $\frac{1}{2}$-plane and $+$ve in the lower.

In estimating the value of the contour integral (1.2), we first let the small circuit enclosing the point $z = 1$ tend to zero in size; from (2.4) it can then be shown that the contribution to (1.2) from the circuit tends to zero provided that $\sigma - \alpha > -1$. Using now the fact that $K_{\alpha}(N\theta)/I_{\alpha}(N\theta) \sim \exp(-2N\theta)$ as $N\theta \to \infty$, it follows that the contribution from the two portions of the contour which follow the real axis are together asymptotically equal to

$$\frac{2^{1-\sigma}\sin\pi(\sigma-\alpha)}{\pi}\int_0^\infty \frac{\theta^{2\sigma+1}K_\alpha(N\theta)}{I_\alpha(N\theta)}d\theta$$

as $N\to\infty$, which is reducible to the right-hand member of (2.3) by the substitution $t=N\theta$.

Finally, the contribution from the ellipse is

$$O(R^{-2N})=o(N^{-2\sigma-2}),$$

and the result (2.3) is established.

It should be observed that the rate of convergence, as determined by the factor $N^{-2\sigma-2}$, is lower than in (2.2). If $\sigma-\alpha$ is an integer, a slightly more refined argument leads to

$$\mathscr{R}_n(f)=o(N^{-2\sigma-2}),$$

so that, for greatest accuracy, $w(x)$ should be chosen so that $\sigma-\alpha$ is an integer, though there is little point in matching the weight function to the integrand to the extent of making $\alpha=\sigma$.

(iii) Corresponding results hold for the Gauss–Laguerre and Gauss–Hermite formulae.

(*a*) Gauss–Laguerre:

$$w(x)=x^\alpha e^{-x}.$$

If $f(z)w(z)\simeq z^\sigma$ as $z\to 0$, and $\sigma,\ \sigma-\alpha>-1$,

$$\mathscr{R}_n(f)\simeq\frac{4\sin\pi(\sigma-\alpha)}{\pi\kappa^{\sigma+1}}\int_0^\infty\frac{t^{2\sigma+1}K_\alpha(t)}{I_\alpha(t)}dt, \qquad (2.5)$$

where $\kappa=4n+2\alpha+2$.

If $f(z)$ has no singularities on or within the parabola $\mathrm{Rl}\sqrt{(-z)}=\log R$, except for a pair of simple poles at z_0, $\bar{z}_0$, the residues of $f(z)w(z)$ being ρ_0, $\bar{\rho}_0$, then

$$\mathscr{R}_n(f)\simeq -4\pi\,\mathrm{Rl}\,\{\rho_0 e^{-i\alpha\pi}[\exp\sqrt{(-z_0)}]^{-2\sqrt{\kappa}}\}. \qquad (2.6)$$

(*b*) Gauss–Hermite:

$$w(x)=e^{-x^2}.$$

If $f(z)$ has no singularities on or between the lines $\mathrm{Im}\,z=\pm\log R$, except for a pair of simple poles, and z_0, ρ_0 are similarly defined, $\mathrm{Im}\,z_0$ being positive,

$$\mathscr{R}_n(f)\simeq -4\pi\,\mathrm{Rl}\,\{\rho_0[\exp(-iz_0)]^{-2\sqrt{\kappa}}\}, \qquad (2.7)$$

where $\kappa=2(n+1)$.

In these formulae, certain conditions have to be imposed on the behaviour of $f(z)$ as $z\to\infty$ in the complex plane; further reference will be made to this in the next section. It will be noticed that the rate of convergence is in each case rather slower than for the Gauss–Jacobi formulae; effectively, a multiple of $n^{\frac{1}{2}}$ replaces a multiple of n, both in (2.2) and in (2.3).

Similar methods can be applied to a wider variety of functions $f(z)$, but these few results perhaps suffice to indicate the type of convergence to be expected. It should be added that, in numerical examples to which this theory has been applied, there is a very satisfactory agreement between predicted and calculated errors, even for quite modest values of n. A description of two examples appears in a later section.

A Generalization of the Gaussian Quadrature Formulae

We next consider two quadrature formulae which can be regarded as limiting forms of the Gauss–Laguerre and Gauss–Hermite formulae. One of these two formulae, with a disposable scale-factor, can replace a complete class of Gaussian formulae, and since its convergence properties as the scale-factor is increased are closely comparable with those of the corresponding Gaussian formula as the number of points is increased, there is a useful gain in simplicity in using such a formula.

The formulae in question are obtained by means of a generalization of (1.2), (1.3). Let $\phi(z)$, $\psi(z)$ be two analytic functions having the following properties, $w(x)$, and $[a, b]$ being defined as in (1.1):

(i) $\phi(z)$ is single-valued in the finite complex plane, and without singularities, except possibly at the real points a, b. Its zeros are to be all distinct, and at the points $\{x_r\}$ of the real interval (a, b); their number need not be finite, nor need the interval $[a, b]$.

(ii) $\psi(z)$ has no singularities in the plane cut along the real segment $[a, b]$.

(iii) $\psi(x-0i)-\psi(x+0i)=2\pi i\phi(x)w(x)$, when $a<x<b$; notice that $\psi(x_r)$ is defined uniquely, since $\phi(x_r)=0$.

Then in (1.1), if $\lambda_r=-\psi(x_r)/\phi'(x_r)$, $\mathscr{R}_n$ is given by

$$\mathscr{R}_n(f)=\frac{1}{2\pi i}\oint\frac{\psi(z)}{\phi(z)}f(z)dz, \qquad (3.1)$$

the contour being defined as in (1.2), with suitable changes if there are singularities of $f(z)$ at a or b.

To establish this, choose as contour the real interval $[a, b]$ described twice in opposite senses, with a small neighbourhood of each of the points a, b excluded by the contour, which is also indented at each of the points $\{x_r\}$. This contour is illustrated in Fig. 2.

As the excluded neighbourhoods and the indentations tend to zero in size, the limiting contribution from the rectilinear portions of the contour is

$$\int_a^b f(t)w(t)dt,$$

while that from the two indentations at x_r is $-\lambda_r f(x_r)$. The formula (3.1) follows immediately.

Now the rth zero of the Laguerre polynomial $L_n^{(\alpha)}(x)$ is asymptotically equal to that of the Bessel function $J_\alpha[\sqrt{(\kappa x)}]$, where κ is defined as in (2.5), as $\kappa\to\infty$, as

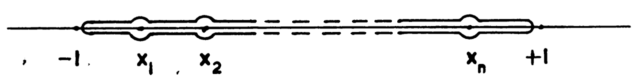

Fig. 2. Contour for proof of formula (3.1)

can be deduced from Szego (1939), 8.22.4. This suggests setting

$$\begin{aligned} w(x) &= x^{\alpha}, \qquad \text{with range } [0, \infty], \\ \phi(z) &= z^{-\alpha/2} J_{\alpha}[\sqrt{(\kappa z)}], \\ \psi(z) &= z^{\alpha/2} K_{\alpha}[\sqrt{(-\kappa z)}], \\ \lambda_r &= -x_r^{\alpha+\frac{1}{2}} K_{\alpha}[\sqrt{(-\kappa z_r)}]/\kappa^{\frac{1}{2}} J'_{\alpha}[\sqrt{(\kappa x_r)}] \\ &= \kappa^{-\frac{1}{2}} \pi^2 x_r^{\alpha+1} \{Y_{\alpha}[\sqrt{(\kappa x_r)}]\}^2. \end{aligned}$$

It is easily verified, using standard properties of Bessel functions, that the required conditions are satisfied.

In a similar manner, the Gauss–Hermite formula leads to

$$\begin{aligned} w(x) &= 1, \qquad \text{with range } [-\infty, \infty] \\ \phi(z) &= -\sin(z\sqrt{\kappa}) \\ \psi(z) &= \pi e^{\pm iz\sqrt{\kappa}} (+ \text{ in upper } \tfrac{1}{2}\text{-plane}, - \text{ in lower}). \\ \lambda_r &= \pi/\sqrt{\kappa}, \text{ with } x_r = \pi r/\sqrt{\kappa} \quad (r = 0, \pm 1, \pm 2, \ldots). \end{aligned}$$

The expressions (2.5), (2.6), (2.7) remain valid for these formulae, in which the quantity κ appears simply as a scale factor. Thus, to ensure sufficient accuracy, it is necessary simply to choose a large enough value for κ and then to take as many of the infinity of points in the quadrature formula as contribute significantly to the calculated value. Experience so far suggests that the number of points required is rarely materially more than for the corresponding Gaussian formula.

The conditions to be imposed on $f(z)$ as $z \to \infty$ are now weakened and simplified, so that they may conveniently be stated here. They are:

(i) $|\int f(z)w(z)dz|$ along the appropriate contour must converge;

(ii) $|z^{\frac{1}{4}}f(z)w(z)|$ for the first of the two quadrature formulae, or $|f(z)w(z)|$ for the second, must tend to zero as $z \to \infty$ in any way within the contour.

The first of the two formulae appears to be new; though it is not Gaussian, it might be called by the name "Gauss–Bessel." The second formula is merely the trapezoidal rule extended over an infinite number of equal intervals. So that what we find here is not a new formula, but the conclusion that a well-known and extremely simple formula is usually as good as the much less simple class of Gauss–Hermite formulae.

Numerical Examples

(i) $$\int_0^{\infty} \frac{e^{-x}dx}{1+\mu x^{-m}}, \qquad m = 4, \qquad \mu = \sqrt{10}.$$

This is a class of integrals arising in a theory of unimolecular reactions. The particular values of the parameters were chosen, not because they give better agreement with the theory, but because the form of the results is clearly displayed.

The integral was calculated by using the Gauss–Laguerre formula with $\alpha = 0$, and with the weight-function absorbed into the coefficients. To illustrate

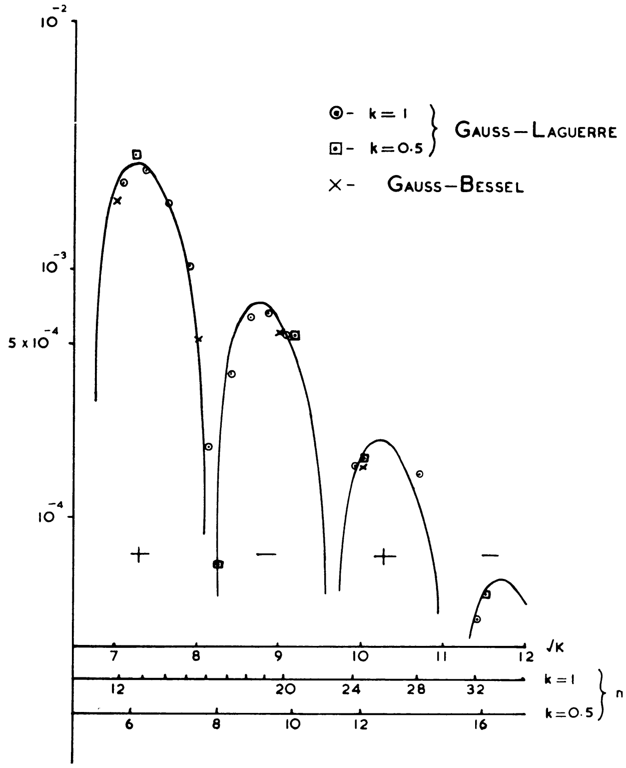

Fig. 3. Actual and estimated errors in the evaluation of

$$\int_0^{\infty} \frac{e^{-x}dx}{1+10^{\frac{1}{2}}x^{-4}} = 0{\cdot}275018$$

the fact that there is no advantage in matching the exponential behaviour of the weight-function as $x \to \infty$, to that of the integrand, another set of values was calculated with a scale-factor introduced into the variable of integration. That is, if λ_r are the coefficients in the Gauss–Laguerre formula, we write

$$\lambda_r^* = e^{x_r}\lambda_r$$

and $$\int_0^{\infty} f(x)dx \doteq k \sum_{r=0}^{n} \lambda_r^* f(kx_r),$$

where k is the scale-factor. The expressions (2.5) and (2.6) for the remainder now require $\kappa = (4n + 2\alpha + 2)/k$. The integral was also calculated using the Gauss–Bessel formula, with various values of κ.

The position of the poles of the integrand are

$$z = 10^{+1/8} \exp(\pm i\pi/4) \text{ and } 10^{+1/8} \exp(\pm 3i\pi/4),$$

of which the first two will dominate the remainder. The residues are readily calculable, and the expression (2.6) evaluated. Fig. 3 shows: (*a*) as a continuous curve, with several branches, the values of this expression plotted logarithmically against values of $\sqrt{\kappa}$, plotted linearly; (*b*) values of the error in the computed value for values of n and k, or of κ, as indicated in the legend.

Subsidiary scales give the values of these parameters for the different points plotted, and show that the use of a scale-factor can reduce the number of points required.

In the case of the Gauss–Bessel formula, the retention of a finite number only of points results in a truncation error additional to the theoretical error, and in obtaining the values plotted, enough points have been retained to make this truncation error negligible. However, in this example, if the number of points retained is not less than $\sqrt{\kappa}$, the truncation error will not be more than 10% of the theoretical error.

(ii) $$\int_0^\infty x^m e^{-x} dx, \text{ with } m = 0, 1, 2, 3.$$

A Gauss–Laguerre formula with α an integer and $\alpha \leqslant m$, $2n > m - \alpha$, will give an exact result. The errors with $\alpha = \frac{1}{2}$ are plotted logarithmically in Fig. 4, against values of κ, also plotted logarithmically; the straight lines represent the estimated error, as given by formula (2.5).

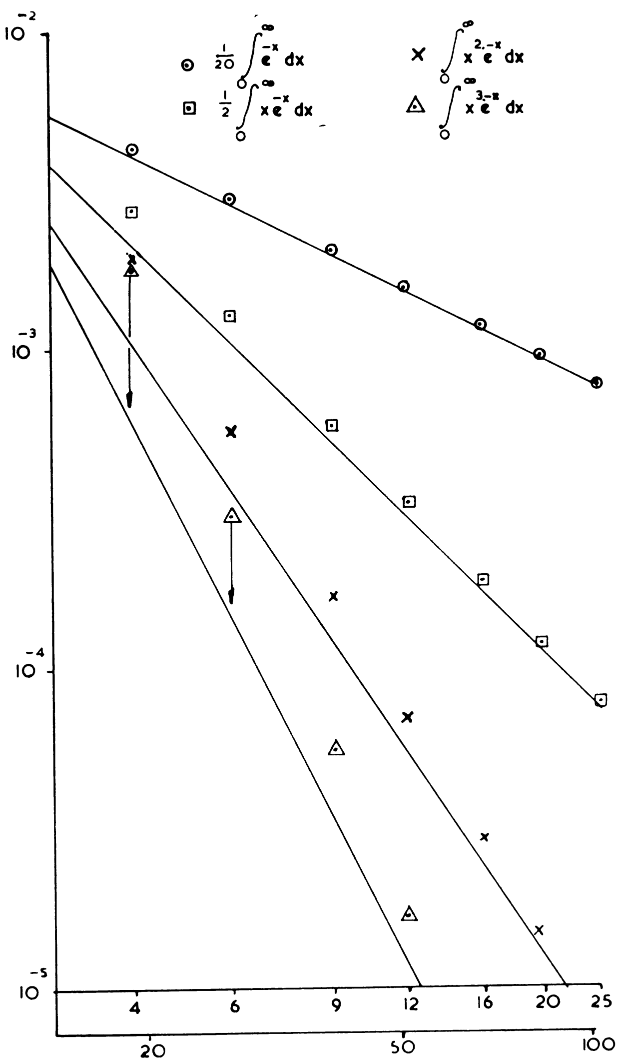

Fig. 4. Actual and estimated errors in the evaluation of $\int x^m e^{-x} dx$, with $\alpha = \frac{1}{2}$

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Appendix: A note on the derivation of the formulae (1.6) and (1.7)

In the case of the Gauss–Jacobi formulae, with range $[-1, 1]$ and weight-function $(1-x)^\alpha(1+x)^\beta$, $p_n(x)$ are the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, for which a definition may be found in *S.*, § 4.1*; they are solutions of the differential equation given in *S.*, 4.2.1.

The functions $q_n(z)$ are given by

$$q_n(z) = 2(z-1)^\alpha(z+1)^\beta Q_n^{(\alpha,\beta)}(z),$$

$$= 2^{-n}\int_{-1}^{1}(1-t)^{n+\alpha}(1+t)^{n+\beta}(z-t)^{-n-1}dt, \quad (5.1)$$

where $Q_n^{(\alpha,\beta)}(z)$ is a certain second solution (*S.*, 4.61.1) of the differential equation. This may readily be verified by integrating (5.1) by parts n times and comparing the result with the Rodrigues' formula (*S.*, 4.3.1)

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\}.$$

Now the formula given in *S.*, 8.21.9 may be rewritten

$$p_n(z) = A(n)(z-1)^{-\alpha/2-\frac{1}{4}}(z+1)^{-\beta/2-\frac{1}{4}} \times [z+\sqrt{(z^2-1)}]^{N}[1+0(1/n)], \quad (5.2)$$

where $A(n) = 2^{(\alpha+\beta-1)/2}/\sqrt{(n\pi)}$

and $N = n + (\alpha+\beta+1)/2$;

fractional powers are defined to be regular in the plane cut along $[-\infty, 1]$, and to be real and positive when z

* Numbers preceded by *S.* refer to Szego (1939).

is real and greater than unity; the formula is valid in the whole plane with a neighbourhood of the segment $[-1, 1]$ removed.

A similar formula for $Q_n^{(\alpha,\beta)}(z)$, *S.*, 8.71.19, gives

$$q_n(z) = B(n)(z-1)^{\alpha/2-\frac{1}{4}}(z+1)^{\beta/2-\frac{1}{4}} \times [z + \sqrt{(z^2-1)}]^{-N}[1 + 0(1/n)], \quad (5.3)$$

valid in the plane cut along $[-1, 1]$, with neighbourhoods of the two points ± 1 removed, $B(n)$ being a suitable function of n only.

If we can determine the function $B(n)$, the formula (1.6) will follow immediately. The simplest way to do this seems to be to make use of the relation

$$q_n(x-0i) - q_n(x+0i) = \oint \frac{p_n(z)w(z)}{z-x}dz = 2\pi i w(x) p_n(x), \quad (5.4)$$

where the contour is a small circuit enclosing the point x. If $x = \cos\vartheta$ $(-1 < x < 1;\ 0 < \vartheta < \pi)$, so that $x \pm \sqrt{(x^2-1)} = e^{\pm i\vartheta}$, it can be shown from (5.3) that

$$q_n(x-0i) - q_n(x+0i) = 2iB(n)(1-x)^{\alpha/2-\frac{1}{4}}(1+x)^{\beta/2+\frac{1}{4}} \times \{\cos(N\vartheta + \gamma) + 0(1/n)\},$$

where $\gamma = -(\alpha + \frac{1}{2})\pi/2$.

Now from *S.*, 8.21.10, we find

$$w(x)p_n(x) = 2A(n)(1-x)^{\alpha/2-\frac{1}{4}}(1+x)^{\beta/2-\frac{1}{4}} \times \{\cos(N\vartheta+\gamma) + 0(1/n)\}.$$

Hence, from (5.4), it suffices to put $B(n) = 2\pi A(n)$; substituting in (5.3) and dividing (5.3) and (5.2), the formula (1.6) is established.

We shall not consider in detail the derivation of (1.7); the starting point, however, is the formula, *S.*, 8.21.17,

$$(\sin\vartheta/2)^\alpha(\cos\vartheta/2)^\beta P_n^{(\alpha,\beta)}(\cos\vartheta) \simeq N^{-\alpha}\frac{\Gamma(n+\alpha+1)}{n!}(\vartheta/\sin\vartheta)^{\frac{1}{2}}J_\alpha(N\vartheta).$$

This is only stated to be true for real values of ϑ in $0 < \vartheta < \pi$, with a suitable remainder term; however, with a minor adjustment of the remainder term, it is also valid for complex values of ϑ. It is further possible to derive a corresponding formula for $q_n(z)$, and from these to deduce (1.7).

To derive and apply the corresponding expressions relating to the Gauss–Laguerre and Gauss–Hermite formulae would take too much space. The functions $p_n(z)$ and $q_n(z)$ are, however, expressible in terms of confluent hypergeometric functions, and the necessary properties of these functions, including asymptotic expressions, can be found in Erdelyi (1953), Vol. I, Chapter VI.

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